PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ

ESCUELA DE POSGRADO

VARIEDADES DE CONTACTO TÓRICAS

Tesis para optar el grado de

Magíster en Matemáticas que presenta

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Dirigido por

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San Miguel, 2017

Resumen

En este trabajo se presentará un estudio de las variedades de contacto obtenidas mediante el método de reducción de contacto, demostrado inicialmente por Geiges e impulsado por él mismo, E. Lerman entre otros. Dicho resultado tiene su esencia en el teorema de reducción simpléctica demostrado por K. R. Meyer en 1973 e independientemente por J. Marsden y A. Weinstein en 1974. Ambas contribuciones a la mecánica clasica impulsaron que en los últimos años se busque generalizar estos resultados al caso de contacto. Por ello, se pone mucha atención en el tipo de grupo de automorfismos que actuará en la variedad de estudio, con el objetivo de encontrar mayor información de la estructura de las variedades obtenidas luego de la reducción. La particularidad en los ejemplos que desarrollaremos será en que el grupo actuando en muchos casos será un toro de una cierta dimensión, lo cual nos generará las llamadas *variedades tóricas de contacto*.

Palabras clave: variedades de contacto, acciones tóricas.

Abstract

In this work, we will study contact manifolds obtained through the contact reduction method, initially demonstrated by Geiges and promoted by himself, E. Lerman among others. This result has its essence in the symplectic reduction theorem demonstrated by K. R. Meyer in 1973 and independently by J. Marsden and A. Weinstein in 1974. Both contributions to classical mechanics led to the search of generalization of these results to the contact case over the last few years. Therefore, a lot of attention is paid to the type of group of automorphisms that will act in the study manifold, with the aim of finding more information on the structure of the manifolds obtained after the reduction. The particularity in the examples that we will develop will be that the group acting in many cases will be a torus of a certain dimension, which will generate the so-called *contact toric manifolds*.

Keywords: contact manifolds, torus actions.

Dedicated to my parents

Acknowledgements

Agradezco a todos mis profesores de la PUCP y, en especial, aprecio enormemente las enseñanzas y valiosos consejos que recibí de mi asesor Jaime Cuadros y profesores Richard Gonzales y Roland Rabanal. Su influencia ha sido increíble, y sé que lo seguirá siendo.
Agradezco a mi padres por creer en mí y a muchas personas que conocí en el transcurso. Gracias a ellos estoy determinada a continuar investigando en este enigmático y fascinante paraíso de las matemáticas.

Contents

Introduction

The purpose of this thesis is to study the group of automorphisms of certain contact manifolds. To achieve this, we previously review the basic concepts of symplectic manifolds, and the contact manifolds, as well as their different incarnations in the Riemannian context, (Kähler and Sasaki structures, respectively). It will allow us to describe in detail the relationship between the contact manifolds and their symplectic cone.

Subsequently, we study the symplectic reduction, and we place emphasis on the reduction in the complex projective space. The latter will be very useful when we extend this technique to the case of interest in this work, *reduction on contact manifolds*. We give a detailed proof of the contact reduction theorem (originally given by Geiges in [8]) and its application via examples for the case of S^1-actions on certain manifolds. In the majority of examples that we exhibit, the $2n+1$- dimensional contact manifolds admit torus actions of dimension $(n+1)$ which preserves the contact form, manifolds with this quality are called *contact toric manifolds*. In the last years, the study of this type of contact manifolds allowed to find results of great importance in the area (see for example [4], [6], [7] and [13]).

Chapter 1

Almost complex structures and symplectic manifolds

In this chapter we will give a general picture of a symplectic manifold as a way to understand contact geometry, which can be viewed as the odd dimensional analog of symplectic geometry.

1.1 Symplectic manifolds

1.1.1 Symplectic vector spaces

Let V be a real vector space of dimension n. We will denote by V^* its dual space, and for $k \in \mathbb{N}$, let $\Lambda^k V^*$ be the space of antisymmetric (i.e., alternating) multilinear mappings from $\underbrace{V \times \cdots \times V}_{k \text{ times}}$ to $\mathbb{R}$. Certainly, for $k > n$, we have

$$\Lambda^k V^* = 0 \tag{1.1.1}$$

We get easily that

$$\Lambda^0 V^* = \mathbb{R}, \qquad \Lambda^1 V^* = V^*, \qquad \dim \Lambda^n V^* = 1. \tag{1.1.2}$$

A nonvanishing element of $\Lambda^n V^*$ defines an orientation of V. By taking antisymmetric multiplication, $\Lambda V^* = \oplus_{k=0}^n \Lambda^k V^*$ becomes an algebra with its $\mathbb{Z}$- grading induced by its degree.

We say that a bilinear form $\theta : V \times V \to \mathbb{R}$ is nondegenerate if , for $v \in V$, $\theta(v, \cdot\,) = 0$ implies that $v = 0$.

We say that a bilinear form $g : V \times V \to \mathbb{R}$ is a scalar product (or Euclidean metric) on V if g is symmetric and positive, i.e., for any $u, v \in V$,

$$\begin{aligned} \text{symmetric} \quad &: \quad g(u,v) = g(v,u), \\ \text{positive} \quad &: \quad g(u,u) > 0 \text{ if } u \neq 0. \end{aligned}$$

Definition 1.1. The vector space (V, ω) is called symplectic if V is a finite dimensional real vector space and $\omega : V \times V \to \mathbb{R}$ is a nondegenerate antisymmetric bilinear form. In this case, we call ω a symplectic form on V.

Definition 1.2. Let $(V_1, \omega_1), (V_2, \omega_2)$ be two symplectic vector spaces. A linear map $\phi : V_1 \to V_2$ is called symplectic if

$$\omega_1 = \phi^* \omega_2. \tag{1.1.3}$$

If the linear map $\phi : V_1 \to V_2$ is symplectic then, as ω_1 is nondegenerate, ϕ is injective. If ϕ is also an isomorphism, we call that ϕ is a symplectic isomorphism.

Proposition 1.3. *If (V, ω) is a symplectic vector space of dimension n, then n is even and $\omega^{n/2} \in \Lambda^n V^*$ is nonvanishing which defines an orientation of V. Moreover, the map*

$$v \in V \to \omega(v, \cdot\,) \in V^* \tag{1.1.4}$$

is an isomorphism.

Proof. Let $\langle \cdot\,,\cdot \rangle$ be a scalar product on V. Then there exists an antisymmetric invertible endomorphism $A \in \operatorname{End}(V)$ such that

$$\omega(\cdot\,,\cdot\,) = \langle \cdot\,, A\cdot \rangle\,. \tag{1.1.5}$$

Since

$$\det A = \det\left(-A^t\right) = (-1)^n \det A \tag{1.1.6}$$

thus n is even. If $\langle \cdot\,,\cdot \rangle'$ is another scalar product on V, and A' is the corresponding antisymmetric invertible endomorphism, then there is $P \in \mathrm{GL}(V)$ such that $PAP^t = A'$. Thus $\det A$ and $\det A'$ have the same signature. This means that V has a canonical orientation. In fact, this is equivalent to $\omega^{n/2} \in \Lambda^n V^*$ and $\omega^{n/2} \neq 0$.

As ω is nondegenerate, the map $v \in V \to \omega(v,\cdot\,) \in V^*$ is injective. As $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*$, (1.1.4) is an isomorphism. □

The basic example is the following.

Example 1.4. Let L be a vector space. Then $L \oplus L^*$ is a symplectic vector space with a symplectic form $\omega^{L\oplus L^*}$ defined by:

$$\omega^{L\oplus L^*}\left((l_1, l_1^*)\,, (l_2, l_2^*)\right) = \langle l_1, l_2^* \rangle - \langle l_2, l_1^* \rangle\,. \tag{1.1.7}$$

for every $(l_1, l_1^*), (l_2, l_2^*) \in L \oplus L^*$.

In particular, if we identify $\mathbb{R}^n$ with $\mathbb{R}^{n*}$ by the canonical scalar product of $\mathbb{R}^n$:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \tag{1.1.8}$$

for every $x = (x_1, \ldots, x_n)\,, y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we denote

$$\left(\mathbb{R}^{2n}, \omega_0\right) := \left(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega^{\mathbb{R}^n\oplus\mathbb{R}^{n*}}\right)$$

the standard symplectic space.

Furthermore, since $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, by replacing $z = x + iy$ in (1.1.7) and (1.1.8), we obtain

$$\omega_{st} = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \tag{1.1.9}$$

as the standard symplectic form for $\mathbb{C}^n$.

Let (V, ω) be a symplectic vector space. For $W \subset V$ a linear subspace, set

$$W^{\perp_\omega} = \{v \in V : \omega(v, w) = 0, \text{ for all } w \in W\}. \tag{1.1.10}$$

Definition 1.5. For W a subspace of a symplectic vector space (V, ω), we say

1. W is symplectic if $W \cap W^{\perp_\omega} = \{0\}$;
2. W is isotropic if $W \subset W^{\perp_\omega}$;
3. W is coisotropic if $W^{\perp_\omega} \subset W$;
4. W is Lagrangian if $W = W^{\perp_\omega}$.

Proposition 1.6. *For W a subspace of (V, ω), we have*

$$\dim W + \dim W^{\perp_\omega} = \dim V, \qquad \left(W^{\perp_\omega}\right)^{\perp_\omega} = W. \tag{1.1.11}$$

If W is symplectic, then $W^{\perp_\omega}$ is also symplectic and we have the direct decomposition of symplectic vector spaces

$$(V, \omega) = (W, \omega|_W) \oplus \left(W^{\perp_\omega}, \omega|_{W^{\perp_\omega}}\right). \tag{1.1.12}$$

Proof. Let $\langle \cdot, \cdot \rangle$ be a scalar product of V. Let $A \in \mathrm{End}(V)$ as in (1.1.5). Then $W^{\perp_\omega} = (AW)^{\perp}$.

Hence,

$$\dim W^{\perp_\omega} = \dim (AW)^{\perp} = \dim V - \dim (AW). \tag{1.1.13}$$

As A is invertible, by (1.1.13), we get the first equation of (1.1.11), in particular we have $\dim W = \dim \left(W^{\perp_\omega}\right)^{\perp_\omega}$. But by definition of $W^{\perp_\omega}$, we have $W \subset \left(W^{\perp_\omega}\right)^{\perp_\omega}$. This means the second equation of (1.1.11) holds.
If W is symplectic, then $W^{\perp_\omega} \cap \left(W^{\perp_\omega}\right)^{\perp_\omega} = W^{\perp_\omega} \cap W = \{0\}$, thus $W^{\perp_\omega}$ is symplectic. Now we get (1.1.12) by the first equation of (1.1.11) $\square$

1.1.2 Compatible complex structures

The following definition will expose the nature of the 2- form that will play a key role in defining a Kähler manifold in the next section.

Definition 1.7. Let V be a real vector space. If $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{Id}_V$, we call J a complex structure on V. Moreover, if ω is a symplectic form on V, such that

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot) \tag{1.1.14}$$

defines a scalar product on V, we call J a compatible complex structure on (V,ω). We denote by $\mathscr{J}(V,\omega)$ the space of compatible complex structures on (V,ω).

Let us recall that J is **antisymmetric with respect to** g if

$$g(X, J^tY) = -g(X, JY)$$

for every $X, Y \in V$.

Proposition 1.8. *If J is a compatible complex structure on a symplectic vector space (V,ω), then ω is J- invariant, i.e.,*

$$\omega\left(J\cdot, J\cdot\right) = \omega\left(\cdot,\cdot\right) \tag{1.1.15}$$

Proof. By (1.1.14), we have

$$\begin{aligned} \omega(\cdot,\cdot) &= g(\cdot,-J\cdot), \\ \omega(J\cdot,J\cdot) &= g(J\cdot,\cdot) = g\left(\cdot,J^t\cdot\right). \end{aligned} \tag{1.1.16}$$

As ω is antisymmetric, J is antisymmetric with respect to g. Then

$$g\left(\cdot,J^t\cdot\right) = -g(\cdot,J\cdot) = -\omega\left(\cdot,J^2\cdot\right) = \omega(\cdot,\cdot). \tag{1.1.17}$$

From (1.1.16), (1.1.17), we get (1.1.15). □

1.1.3 Symplectic vector bundles

Let M be a differentiable manifold, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.9. Let E be a manifold, $\pi : E \to M$ be a smooth surjective map. We say that E is a $\mathbb{K}$-vector bundle on M of rank m if π satisfy the following conditions.

1. For every $x \in M$, $E_x := \pi^{-1}(x)$ have the structure of a vector space over $\mathbb{K}$.

2. There exists an open covering $\{U_i\}$ of M and a family of diffeomorphisms

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^m \tag{1.1.18}$$

such that for every i,

(a) the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{K}^m \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

(b) for every $x \in U_i$, the induced mapping $\phi_{i,x} := pr_2 \circ \phi_i|_{E_x} : E_x \to \mathbb{K}^m$ is linear.

We denote by $m := rk(E)$. If $m = 1$, we say that E is a $\mathbb{K}$-line bundle. Let us recall that a section of E is an smooth mapping $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. We denote $C^\infty(M, E)$ as the space of sections of E over M.

If F is another $\mathbb{K}$-vector bundle on M, then we define the dual of E : $E^* = \bigcup\limits_{x \in M} \{E_x^*\}$, the direct sum of E and F: $E \oplus F = \bigcup\limits_{x \in M} \{E_x \oplus F_x\}$, the tensor product of E and F: $E \otimes F = \bigcup\limits_{x \in M} \{E_x \otimes F_x\}$. We also denote $\mathrm{Hom}\,(E, F) = E^* \otimes F$ and $C^\infty(M, E)$ as the space of sections of vector bun. Let us see some examples.

Example 1.10. For every $\mathbb{K}$-vector bundle V over M, $\mathrm{End}(V)$ and $\Lambda^2 E^*$ are $\mathbb{K}$-vector bundles over M.

A C^∞-map $\psi : E \to F$ is a morphism of $\mathbb{K}$-vector bundles over M if for any $x \in M$, ψ_x is a $\mathbb{K}$-linear map from E_x to F_x, i.e., $\psi \in C^\infty(M, \mathrm{Hom}\,(E, F))$. If for any $x \in M$, ψ_x is an isomorphism from E_x to F_x, then we say that ψ is an isomorphism of $\mathbb{K}$-vector bundles.

Definition 1.11. Let V be a real vector bundle on M, we say that (V, ω) is a symplectic vector bundle on M if $\omega \in C^\infty(M, \Lambda^2 V^*)$ and for any $x \in M$, (V_x, ω_x) is a symplectic vector space.

Definition 1.12. Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector bundles on M, $\psi \in C^\infty(M, \mathrm{Hom}\,(V_1, V_2))$. If for any $x \in M$, $\psi_x : (V_{1,x}, \omega_{1,x}) \to (V_{2,x}, \omega_{2,x})$ is a symplectic linear map, then we call ψ a symplectic morphism of symplectic vector bundles. If moreover ψ_x is an isomorphism for any $x \in M$, then we call ψ a symplectic isomorphism of symplectic vector bundles.

Definition 1.13. If $J \in C^{\infty}(M, \mathrm{End}(V))$ such that for any $x \in M$, $J_x^2 = -\mathrm{Id}_{V_x}$, we call J an **almost complex structure on** V. Moreover, if (V, ω) is a symplectic vector bundle on M, and for any $x \in M$, J_x is a compatible complex structure on (V_x, ω_x), we call J a compatible complex structure on (V, ω).

Remark 1.14. Let V be an n-dimensional vector space and $B \in \mathrm{End}(V)$. Then

$$\begin{aligned} \rho : \mathrm{End}(V) &\rightarrow V \otimes V^* \\ B &\rightarrow \rho(B) : V \times V^* \rightarrow \mathbb{R} \end{aligned}$$

is an isomorphism of algebras, where $\rho(B)(v, v^*) = \langle Bv, v^* \rangle$ for each $(v, v^*) \in V \times V^*$.

Definition 1.15. For a manifold M, if $J \in C^{\infty}(M, \mathrm{End}(TM))$ and for any $x \in M$, $J_x^2 = -\mathrm{Id}_{T_xM}$, we call that J is an **almost complex structure on** TM and (M, J) is an almost complex manifold.

Definition 1.16. A 2-form ω on a manifold M is called a symplectic form on M, if ω is real and closed, and if for any $x \in M$, $\omega_x \in \Lambda^2(T_x^*M)$ is nondegenerate. In this case, (M, ω) is called a **symplectic manifold**.

For a submanifold W of a symplectic manifold (M, ω), we call W a symplectic (resp. isotropic, coisotropic, Lagrangian) submanifold if for any $x \in M$, T_xW is a symplectic (resp. isotropic, coisotropic, Lagrangian) subspace of (T_xM, ω_x).

A diffeomorphism $\psi : M \rightarrow N$ is called a **symplectic diffeomorphism** or **symplectomorphism**) for two symplectic manifolds (M, ω), (N, ω_1) if $\psi^*\omega_1 = \omega$. And we can define $\mathrm{Sympl}(M, \omega)$ as the **group of symplectomorphisms** over M.

Let $J \in C^{\infty}(M, \mathrm{End}(TM))$ be an almost complex structure on a symplectic manifold (M, ω), then we say J is a compatible almost complex structure if $\omega(\cdot, J\cdot)$ defines a J-invariant Riemannian metric on TM.

Remark 1.17. Let (M, ω) be a symplectic manifold. By Proposition 1.3, M has even dimension. Let $\dim M = 2n$. Then $\omega^n \neq 0 \in \Lambda^{2n}(T^*M)$ induces a canonical orientation on M.

Example 1.18. Let L be a manifold of dimension n, and $\pi : T^*L \to L$ be the natural projection. The Liouville form λ is a 1-form on T^*L which is defined as follows: for $q \in L$, $p \in T_q^*L$, $X \in T_{(q,p)}T^*L$,

$$\langle \lambda, X \rangle_{(q,p)} := \langle p, d\pi_{(q,p)}X \rangle_q . \tag{1.1.19}$$

Set

$$\omega^{T^*L} = -d\lambda. \tag{1.1.20}$$

Then ω^{T^*L} is a closed 2-form on T^*L. Let $\psi : U \subset L \to V \subset \mathbb{R}^n$, $q \to (q_1 = \psi_1(q), \cdots, q_n = \psi_n(q))$ be a local coordinate, then $\{\frac{\partial}{\partial q_j}\}$ is a local frame of TL, and $\{dq_j\}$ is a local frame of T^*L which gives the trivialization of T^*L on U. Thus

$$T^*L \to V \times \mathbb{R}^n, \qquad \left(q, \sum_i p_i \psi^*(dq_i)\right) \to (q_1, \cdots q_n, p_1, \cdots, p_n) \tag{1.1.21}$$

is the induced local coordinate of $T^*L|_U$, and $\{\frac{\partial}{\partial q_j}, \frac{\partial}{\partial p_j}\}$ is a local frame of $T(T^*L)$.

For $X = \sum_i X_i \frac{\partial}{\partial q_i} + P_i \frac{\partial}{\partial p_i}$, we have

$$\langle \lambda, X \rangle_{(q,p)} = \sum_{i=1}^{n} p_i X_i = \left\langle \sum_{i=1}^{n} p_i dq_i, X \right\rangle. \tag{1.1.22}$$

From (1.1.20) and (1.1.22), we get

$$\lambda = \sum_{i=1}^{n} p_i dq_i, \qquad \omega^{T^*L} = \sum_{i=1}^{n} dq_i \wedge dp_i. \tag{1.1.23}$$

Hence, ω^{T^*L} is nondegenerate, and $\left(T^*L, \omega^{T^*L}\right)$ is a symplectic manifold.

The next definition will be key to understand the geometry of the symplectic cone associated to a contact manifold.

Definition 1.19. Let (M, ω) be a symplectic manifold. **A Liouville vector field** is a vector field Ψ which satisfies that $\mathcal{L}_\Psi \omega = \omega$.

Remark 1.20. Notice that the flow φ_t corresponding to the Liouville vector field is such that ${\varphi_t}^*\omega = e^t\omega$, that is along the flow the symplectic form is rescaled exponentially. In fact, if we set $\lambda = \Psi \lrcorner\, \omega$ [1] a 1- form in M, by Cartan's formula we have that $d\lambda = \omega$ and $\mathcal{L}_\Psi \lambda = \lambda$.
For every $Y_p \in T_pM$ with $p \in M$,

$$\lambda_p(Y)_p = (\mathcal{L}_\Psi \lambda)_p(Y_p) = \lim_{t\to 0} \frac{(\varphi_t^*\lambda_p)(Y_p) - \lambda_p(Y_p)}{t}. \tag{1.1.24}$$

Thus, we can apply L'Hôpital's rule in (1.1.24) to obtain that

$$\lambda_p(Y_p) = \frac{d}{dt}|_{t=0} (\varphi_t^*\lambda_p)(Y_p). \tag{1.1.25}$$

Additionally,

$$\frac{d}{dt}|_{t=0} \left(e^t\lambda_p\right)(Y_p) = \frac{d}{dt}|_{t=0} (\varphi_t^*\lambda_p)(Y_p).$$

Finally, from the initial condition of the differential equation above, its unique solution must be

$$(\varphi_t^*\lambda_p)(Y_p) = \left(e^t\lambda_p\right)(Y_p).$$

Thus,

$$\varphi_t^*\lambda = e^t\lambda.$$

[1]The symbol $\lrcorner$ is the contraction of differential forms by a vector field.

Morevover,

$$\begin{aligned}\varphi_t^*\omega &= \varphi_t^*(d\lambda)\\ &= d\left(\varphi_t^*\lambda\right)\\ &= d(e^t\lambda)\\ &= e^t dt\wedge\lambda + e^t d\lambda = e^t\omega,\end{aligned}$$

since $\lambda = \Psi \lrcorner\, \omega$ implies that $dt\wedge\lambda = 0$ in M.

Let us see a useful example of a Liouville vector field in the Euclidean space.

Example 1.21. On $\mathbb{R}^{2n}$, the radial vector field $\partial_r = \sum\left(\frac{1}{2}x_j\partial_{x_j} + \frac{1}{2}y_j\partial_{y_j}\right)$ is a Liouville vector field.

1.2 Kähler manifolds

Let M be a complex manifold with an almost complex structure J. The almost complex structure J induces a splitting

$$TM^{\mathbb{C}} := TM\otimes_{\mathbb{R}}\mathbb{C} = T^{(1,0)}M\oplus T^{(0,1)}M, \tag{1.2.1}$$

where $T^{(1,0)}M = \{X - iJX \mid X\in TM\}$ and $T^{(0,1)}M = \{X + iJX \mid X\in TM\}$ are known as the eigenbundles of J corresponding to the eigenvalues i and $-i$, respectively. Let $T^{*(1,0)}M$ and $T^{*(0,1)}M$ be the corresponding dual bundles. Let

$$\Omega^{r,q}(M) := C^\infty\left(M, \Lambda^r\left(T^{*(1,0)}M\right)\otimes\Lambda^q\left(T^{*(0,1)}M\right)\right) \tag{1.2.2}$$

be the spaces of smooth (r,q)-forms on M.

On local holomorphic coordinates $(z_1,\cdots,z_n)$ with $z_j = x_j + iy_j$, we denote

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \tag{1.2.3}$$

$$dz_j = dx_j + i dy_j,\quad d\overline{z}_j = dx_j - i dy_j.$$

Then, on holomorphic coordinates $(z_1, \cdots, z_n)$ the $\partial, \overline{\partial}$-operators on functions are defined by

$$\partial f = \sum_j dz_j \frac{\partial}{\partial z_j} f, \quad \overline{\partial} f = \sum_j d\overline{z}_j \frac{\partial}{\partial \overline{z}_j} f \text{ for } f \in C^\infty(M). \tag{1.2.4}$$

They extend naturally to

$$\partial : \Omega^{\bullet,\bullet}(M) \to \Omega^{\bullet+1,\bullet}(M), \quad \overline{\partial} : \Omega^{\bullet,\bullet}(M) \to \Omega^{\bullet,\bullet+1}(M), \tag{1.2.5}$$

which verify the Leibniz rule for ∂ and $\overline{\partial}$. Besides, we have the decomposition

$$d = \partial + \overline{\partial}, \quad \partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{1.2.6}$$

The operator $\overline{\partial}$ is called the Dolbeault operator.

Definition 1.22. A **Kähler structure** on a Riemannian manifold (M^n, g) is given by a 2-form Ω and a field of endomorphisms of the tangent bundle J satisfying the following conditions:

- J is an almost complex structure.
- g is an Hermitian metric (also known as J-invariant metric), that is, $g(X,Y) = g(JX, JY)$, for every $X, Y \in TM$.
- $\Omega(X,Y) = g(JX, Y)$.
- Ω is a closed 2-form.
- J is integrable, that is J is a complex structure.

Certainly, any Kähler manifold is a symplectic manifold. Kähler manifolds represent an important class of symplectic manifolds. Let us exhibit one example that will be of great importance in this work.

Example 1.23. *(Projective space)* For $x, y \in \mathbb{C}^{n+1} \setminus \{0\}$, we say $x \sim y$ if there is $\lambda \in \mathbb{C}^*$ such that $x = \lambda y$. Then the complex projective space $\mathbb{CP}^n$ is defined as the quotient space $(\mathbb{C}^{n+1} \setminus \{0\}) / \sim$. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the standard projection map. For every $z \in \mathbb{C}^{n+1}$, we denote $[z] = [z_0 : z_1 : \ldots : z_n] = \pi(z)$ which is known as the homogeneous coordinate on $\mathbb{CP}^n$. Let $U_i = \{[z] \in \mathbb{CP}^n : z_i \neq 0\}$, then

$$\begin{aligned} \varphi_i : U_i &\to \mathbb{C}^n \\ [z] &\to \left(\frac{z_0}{z_i}, \ldots, \frac{\hat{z}_i}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right) \end{aligned}$$

defines an holomorphic local coordinate of $\mathbb{CP}^n$, where, as usual, the symbol "$\hat{}$" refers to omitting the i-th coordinate.

Let $\widetilde{\omega}_{FS,z}$ a real 2-form in $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$\widetilde{\omega}_{FS,z} = \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \left(\|z\|^2 \right).$$

(The notation FS is due to the fact that from $\widetilde{\omega}_{FS}$ we will exhibit the local expresion for the Fubini-Study form in $\mathbb{CP}^n$, which is going to be exposed as follows).

Equivalently,

$$\begin{aligned}
\widetilde{\omega}_{FS,z} &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(|z_0|^2+\cdots+|z_n|^2\right)\\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(z_0\overline{z}_0+\cdots+z_n\overline{z}_n\right)\\
&= \frac{\sqrt{-1}}{2\pi}\partial\Big[\sum_{j=0}^{n} d\overline{z}_j\frac{\partial}{\partial\overline{z}_j}\log\left(z_0\overline{z}_0+\cdots+z_n\overline{z}_n\right)\Big]\\
&= \frac{\sqrt{-1}}{2\pi}\partial\Big[d\overline{z}_0\frac{\partial}{\partial\overline{z}_0}\log\left(z_0\overline{z}_0+\cdots+z_n\overline{z}_n\right)+\cdots+d\overline{z}_n\frac{\partial}{\partial\overline{z}_n}\log\left(z_0\overline{z}_0+\cdots+z_n\overline{z}_n\right)\Big]\\
&= \frac{\sqrt{-1}}{2\pi}\partial\Big[\frac{z_0}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}d\overline{z}_0+\cdots+\frac{z_n}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}d\overline{z}_n\Big]\\
&= \frac{\sqrt{-1}}{2\pi}\left[\sum_{k=0}^{n} dz_k\frac{\partial}{\partial z_k}\left(\frac{z_0}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}d\overline{z}_0+\cdots+\frac{z_n}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}d\overline{z}_n\right)\right]\\
&= \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum_{k=0}^{n} dz_k\wedge d\overline{z}_k}{\|z\|^2}-\frac{\sum_{k=0}^{n}\overline{z}_k dz_k\wedge\sum_{j=0}^{n} z_j d\overline{z}_j}{\|z\|^4}\right]. \qquad (1.2.7)
\end{aligned}$$

Let U be an open set in $\mathbb{CP}^n$ and $\varphi : U \to \mathbb{C}^{n+1}\setminus\{0\}$ an holomorphic section, that is, φ is an holomorphic map with $\pi\circ\varphi = id_U$.

Claim 1.23.1. $\varphi^*\widetilde{\omega}_{FS}$ does not depend on the section φ.

Proof. Let $\varphi_1 : U \to \mathbb{C}^{n+1}\setminus\{0\}$ be another holomorphic section , then for every $[z]\in U$, there exists an holomorphic function

$$f : U \to \mathbb{C}^* \text{ with } \varphi_1([z]) = f([z])\varphi([z]).$$

Thus,

$$\begin{aligned}
\varphi_1^*\widetilde{\omega}_{FS,[z]} &= \widetilde{\omega}_{FS,\varphi_1([z])}\\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\|\varphi_1([z])\|^2\right)\\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\|f([z])\varphi([z])\|^2\right)\\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(|f([z])|^2\right)+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\|\varphi([z])\|^2\right)\\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(|f([z])|^2\right)+\varphi^*\widetilde{\omega}_{FS,[z]}. \qquad (1.2.8)
\end{aligned}$$

Besides,

$$\begin{aligned}
\partial\overline{\partial}\log\left(|f|^2\right) &= \partial\left(d\overline{z}\frac{\partial}{\partial\overline{z}}\log\left(f\overline{f}\right)\right)\\
&= \partial\left(d\overline{z}\frac{1}{f\overline{f}}\frac{\partial}{\partial\overline{z}}\left(f\overline{f}\right)\right)\\
&= \partial\left(d\overline{z}\frac{1}{f\overline{f}}\left[\frac{\partial f}{\partial\overline{z}}.\overline{f}+f.\frac{\partial\overline{f}}{\partial\overline{z}}\right]\right)\\
&= \partial\left(d\overline{z}\frac{1}{f\overline{f}}f.\frac{\partial\overline{f}}{\partial\overline{z}}\right)\\
&= \partial\left(d\overline{z}\frac{1}{\overline{f}}.\frac{\partial\overline{f}}{\partial\overline{z}}\right)\\
&= \frac{\partial}{\partial z}\left(\frac{1}{\overline{f}}.\frac{\partial\overline{f}}{\partial\overline{z}}\right)dz\wedge d\overline{z}\\
&= \left(\frac{\partial}{\partial z}\left(\frac{1}{\overline{f}}\right).\frac{\partial\overline{f}}{\partial\overline{z}}+\frac{1}{\overline{f}}.\frac{\partial}{\partial z}\left(\frac{\partial\overline{f}}{\partial\overline{z}}\right)\right)dz\wedge d\overline{z}\\
&= 0, \qquad (1.2.9)
\end{aligned}$$

since f is an holomorphic function and it is nonzero for every $[z]$ in U. Consequently, by replacing (1.2.9) in (1.2.8), we obtain that the claim is proved. $\square$

Therefore, by denoting $\omega_{FS} := \varphi^*\widetilde{\omega}_{FS}$, the previous claim implies that ω_{FS} independient of the election of the section φ, and since these sections exist locally, ω_{FS} is a global differential form in $\mathbb{CP}^n$.

Let us choose the following coordinate map

$$\begin{aligned}
\psi_0 : U_0 &\to \mathbb{C}^n\\
[z] &\to (w_1,\ldots,w_n) := w
\end{aligned}$$

with $w_i = \frac{z_i}{z_0}$. Thus, for the section

$$\begin{aligned}
\varphi : U_0 &\to \mathbb{C}^{n+1}\setminus\{0\}\\
[z] &\to (1,w)
\end{aligned}$$

we obtain the following expression for ω_{FS} respect to the coordinate ψ_0:

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(1+\|w\|^2\right). \tag{1.2.10}$$

Additionally, ω_{FS} is a real closed (1,1) form in $\mathbb{CP}^n$, indeed, since (1.2.10), the properties of ∂ and $\overline{\partial}$ stated in (1.2.6) and equation (1.2.7):

$$\begin{aligned}\omega_{FS} &= -\frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial\log\left(1+\|w\|^2\right)\\ &= -\frac{\sqrt{-1}}{2\pi}\overline{\partial\overline{\partial}}\log\left(1+\|w\|^2\right)\\ &= \overline{\omega}_{FS},\end{aligned} \tag{1.2.11}$$

and $d\omega_{FS} = d\left(\varphi^*\widetilde{\omega}_{FS}\right) = \varphi^* d\widetilde{\omega}_{FS} = 0$.

Finally, we only need to show that ω_{FS} is nondegenerate, (actually, positive definite) to see that $(\mathbb{CP}^n, \omega_{FS})$ is a symplectic manifold, where ω_{FS} is known as the **Fubini-Study form**. In fact, let

$$\begin{aligned}\Phi : \mathrm{U}_{n+1}\times\mathbb{C}^{n+1}\setminus 0 &\to \mathbb{C}^{n+1}\setminus 0, &\qquad \Phi' : \mathrm{U}_{n+1}\times\mathbb{CP}^n &\to \mathbb{CP}^n\\ (A,z) &\to Az & (A,[z]) &\to [Az]\end{aligned}$$

be U_{n+1}-actions on $\mathbb{C}^{n+1}\setminus 0$ and $\mathbb{CP}^n$, respectively, where U_{n+1} is the group of unitary matrices. It is easy to see that

$$\pi\circ\Phi_A = \Phi'_A\circ\pi. \tag{1.2.12}$$

First of all, we observe that U_{n+1} acts transitively on $\mathbb{CP}^n$ since $\mathbb{CP}^n \cong S^{2n+1}/S^1$ and U_{n+1} acts transitively on S^{2n+1}, which follows from the fact that every unit vector can be extended to an orthonormal basis in $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ and consequently, given two orthonormal bases in $\mathbb{C}^{n+1}$, the linear transformation which carries one basis to another corresponds to a unitary matrix.

Claim 1.23.2. $\Phi_A^* \widetilde{\omega}_{FS} = \widetilde{\omega}_{FS}$.

Proof.

$$\begin{aligned}
\Phi_A^* \widetilde{\omega}_{FS,z} &= \widetilde{\omega}_{FS,\Phi_A(z)} \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \left(\|\Phi_A(z)\|^2 \right) \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \left(\|Az\|^2 \right) \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \left(\|z\|^2 \right) \\
&= \widetilde{\omega}_{FS,z},
\end{aligned}$$

where the fourth equality follows from the fact that A is a unitary matrix. □

Claim 1.23.3. $\pi^* \omega_{FS} = \widetilde{\omega}_{FS}$.

Proof. Since ω_{FS} does not depend on the election of the section, we can use the section φ_0 related to the chart ψ_0 defined in the coordinate open set U_0 as we did in (1.2.10) to obtain:

$$\begin{aligned}
\pi^* \omega_{FS,z} &= \omega_{FS,\pi(z)} \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \left(1 + \| \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0} \right) \|^2 \right) \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \left(\frac{\|z\|^2}{|z_0|^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \|z\|^2 - \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |z_0|^2 \\
&= \widetilde{\omega}_{FS,z},
\end{aligned}$$

where the last equality follows from (1.2.9), since z_0 is an holomorphic function defined in U_0 and take values in $\mathbb{C}^*$. □

Then ω_{FS} is positive definite in every element of $\mathbb{CP}^n$ if it is positive definite in just one point.

Thus, by working on the coordinate patch U_0, it follows from (1.2.7) at $z' = [1 : 0 : \ldots : 0]$ that

$$\omega_{FS,z'} = \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \tag{1.2.13}$$

which is positive definite, since $\omega_{FS} = \frac{1}{\pi}\omega_{st}$, where ω_{st} is the standard symplectic form of $\mathbb{C}^n$ (cf. (1.1.9)).

Claim 1.23.4. $\Phi_A'^*\omega_{FS} = \omega_{FS}$.

Proof. We have that

$$\begin{aligned}
\Phi_A'^*\omega_{FS,[z]} = \Phi_A'^*\left(\varphi_0^*\widetilde{\omega}_{FS}\right)_{,[z]} &= \widetilde{\omega}_{FS,\left(\varphi_0\circ\Phi'_A\right)([z])} \\
&= \widetilde{\omega}_{FS,\left(\varphi_0\circ\Phi'_A\circ\pi\right)(z)}.
\end{aligned}$$

From (1.2.12),

$$\begin{aligned}
\Phi_A'^*\omega_{FS,[z]} = \widetilde{\omega}_{FS,(\varphi_0\circ\pi\circ\Phi_A)(z)} &= \varphi_0^*\widetilde{\omega}_{FS,(\pi\circ\Phi_A)(z)} \\
&= \omega_{FS,(\pi\circ\Phi_A)(z)} \\
&= \pi^*\omega_{FS,\Phi_A(z)} \\
&= \widetilde{\omega}_{FS,\Phi_A(z)} \\
&= \Phi_A^*\widetilde{\omega}_{FS,z}.
\end{aligned}$$

Thus, from claims 1.23.2 and 1.23.3,

$$\begin{aligned}
\Phi_A'^*\omega_{FS,[z]} &= \widetilde{\omega}_{FS,z} \\
&= \pi^*\omega_{FS,z} \\
&= \omega_{FS,[z]}.
\end{aligned}$$

□

Chapter 2

Contact structures and Symplectic cones

In [1], Lerman defines the notion of a symplectic cone and its relationship with a given contact structure as a base space. Some results that will be exposed but not proved can be found in [3] and [5]. It is stated as follows.

2.1 Symplectic cones

Definition 2.1. A symplectic manifold (M, ω) is a **symplectic cone** if

- M is a principal $\mathbb{R}$ - bundle over some manifold B which is called the base of the cone, and
- the action of the real line $\mathbb{R}$ expands the symplectic form exponentially. That is, $\rho_\lambda^* \omega = e^\lambda \omega$, where ρ_λ denotes the diffeomorphism defined by $\lambda \in \mathbb{R}$.

Definition 2.2. An action of a Lie group G on a manifold M is **proper** if the map

$$G \times M \to M \times M$$
$$(g, m) \mapsto (g \cdot m, m)$$

is proper.

It follows that if a symplectic manifold (M, ω) has a complete vector field X, (that is, the flow of X is globally defined on $M \times \mathbb{R}$), with the following two properties:

1. the action of $\mathbb{R}$ induced by the flow of X is proper, and

2. the Lie derivative of the symplectic form ω with respect to the vector field X is again ω: $\mathcal{L}_X \omega = \omega$,

then (M, ω) is a symplectic cone relative to the induced action of $\mathbb{R}$.
In fact, if the action of $\mathbb{R}$ induced by the flow of X is proper, we obtain that M is a principal $\mathbb{R}$ - bundle over some manifold $B \cong M/\mathbb{R}$ because we additionally have that this action is free as it is globally defined on $M \times \mathbb{R}$ and generated by the flow of X. The second asumption comes from Remark 1.20.

Thus, we obtain an equivalent definition of a symplectic cone.

Definition 2.3. A **symplectic cone** is a triple (M, ω, X) where M is a manifold, ω is a symplectic form on M, X is a vector field on M generating a proper action of $\mathbb{R}$ such that $\mathcal{L}_X \omega = \omega$.

Remark 2.4. From Definition 1.19, we note that X is a Liouville vector field for the symplectic cone (M, ω, X).

Example 2.5. Let (V, ω_V) be a symplectic vector space. The manifold $M = V \setminus \{0\}$ is a symplectic cone with the action of $\mathbb{R}$ given by $\rho_\lambda(v) = e^\lambda v$. Clearly $\rho_\lambda^* \omega_V = e^\lambda \omega_V$. The base is a sphere.

Example 2.6. Let Q be a manifold. Denote the cotangent bundle of Q with the zero section deleted by $T^*Q \setminus 0$. There is a natural free action of $\mathbb{R}$ on the manifold $M := T^*Q \setminus 0$ given by dilations $\rho_\lambda(q,p) = (q, e^\lambda p)$. It expands the standard symplectic form on the cotangent bundle exponentially. Thus $T^*Q \setminus 0$ is naturally a symplectic cone. The base is the co-sphere bundle S^*Q.

2.2 Contact manifolds and contact transformations

The following definition is the basic one we need in order to introduce the notion of a contact manifold. As follows, we will develop some tools that will be key to understand the intrinsic structure of a contact manifold.

Definition 2.7. A 1-form η on a manifold B is a **contact form** if the following two conditions hold:

1. $\eta_b \neq 0$ for all points $b \in B$, where $\eta_b \in T_b^*B$. Hence $D := \ker\eta = \{(b,v) \in TB \mid \eta_b(v) = 0\}$ is a vector subbundle of the tangent bundle TB.

2. $d\eta|_D$ is a symplectic structure on the vector bundle $D \to B$(i.e. $d\eta|_D$ is nondegenerate).

Remark 2.8. If $D \to B$ is a symplectic vector bundle, then the dimension of its fibers is necessarily even. Hence if a manifold B has a contact form then B is odd-dimensional.

Remark 2.9. A 1-form η on $2n+1$ dimensional manifold B is contact if and only if the form $\eta \wedge (d\eta)^n$ is never zero, i.e. it is a volume form. This follows from the fact that $d\eta|_D$ is a symplectic structure on the vector bundle $D \to B$, which from Remark 1.17, gives us a nowhere vanishing $(2n+1)$-form, and conversely.

Remark 2.10.

$$\bigsqcup_{p \in M} D_p = D = \ker\eta$$

is not integrable. Indeed, the Frobenius integrability condition states that if $X, Y \in D$ then $[X, Y] \in D$. Besides, we have

$$d\eta(X, Y) = \eta(X)Y - \eta(Y)X - \eta[X, Y].$$

Thus $d\eta(X, Y) = -\eta[X, Y]$.
However, we have that $\eta \wedge (d\eta)^n \neq 0$ which implies that $\eta \wedge d\eta \neq 0$. We conclude that $\eta[X, Y]$ can not be zero, i,e. $D = \ker\eta$ is not integrable.

The previous remark allows us to notice that a contact form gives us the non integrable maximum condition for the distribution D. Now, let us observe some examples of contact forms.

Example 2.11. The 1-form $\eta = dz + xdy$ on $\mathbb{R}^3$ is a contact form: $\eta \wedge d\eta = dz \wedge dx \wedge dy$.

Example 2.12. Let $B = \mathbb{R} \times \mathbb{T}^2$. Denote the coordinates by t, θ_1 and θ_2 respectively. The 1-form $\eta = \cos t \, d\theta_1 + \sin t \, d\theta_2$ is contact.

We, indeed, can obtain a family of contact forms by a very easy but also useful observation.

Lemma 2.13. *Suppose η is a contact form on a manifold B. Then for any positive function f on B the 1-form $f\eta$ is also contact.*

Proof. Note first that since f is positive then in particular nowhere zero, $\ker f\eta = \ker\eta$. Thus to show that $f\eta$ is contact, it is enough to check that

$d(f\eta)|_D$ is nondegenerate, where $D = \ker\eta = \ker f\eta$. Now $d(f\eta) = df\wedge\eta + fd\eta$ and $\eta|_D = 0$. Therefore $d(f\eta)|_D = fd\eta|_D$. But f is nowhere zero and $d\eta|_D$ is nondegenerate since η is a contact form by assumption. Thus $d(f\eta)|_D$ is nondegenerate. □

Definition 2.14. We define the **conformal class** of a 1-form η on a manifold B to be the set $[\eta] = \{e^h\eta \,|\, h \in C^\infty(B)\}$, that is, the set of all 1-forms obtained from η by multiplying it by a positive function.

Thus if a 1-form η on a manifold B is a contact form, then its conformal class consists of contact forms all defining the same subbundle D of the tangent bundle of B.

Definition 2.15. M is **coorientable** if D is an orientable bundle.

Now, we have the enough machinery to define a contact structure.

Definition 2.16. A (co-orientable) **contact structure** D on a manifold B is a subbundle of the tangent bundle TB of the form $D = \ker\eta$ for some contact form η. The pair (B, D) is called a **contact manifold**.
A **co-orientation** of a contact structure D is a choice of a conformal class of contact forms defining the contact structure.

Remark 2.17. More generally a contact structure on a manifold B is a subbundle D of the tangent bundle TB such that for every point $x \in B$ there is a contact 1-form η defined in a neighborhood of x with $\ker\eta = D$. There exist contact structures which are not co-orientable. For such structures D a 1-form η with $\ker\eta$ exists only locally.

Hopefully, we can always have a contact form for a contact manifold, which resembles the contact form in $\mathbb{R}^{2n+1}$ in local coordinates. This is the purpose of the following theorem.

Theorem 2.18. ***(Darboux)*** *About each point of a contact manifold* (B^{2n+1}, η), *there exist local coordinates* $(x^1, ..., x^n, y^1, ..., y^n, z)$ *with respect to which*

$$\eta = dz - \sum_{i=1}^{n} y^i dx^i$$

Proof. See [3], page 26. □

Under the same conditions of the previous theorem, there is a 1-form $\eta_0 = dz - \sum_{i=1}^{n} y^i dx^i$ which is the standard contact form in $\mathbb{R}^{2n+1}$ where $\varphi_U : U \subset B \mapsto (\mathbb{R}^{2n+1}, \eta_0)$ is the local chart of B in the coordinate open U of B.

Thus

$$\eta_U = \varphi_U^* \eta_0 \tag{2.2.1}$$

where η_U is the 1-form defined in U. Then $\eta(X) = \eta_0((\varphi_U)_* X)$, where X is a vector field in U and $(\varphi_U)_* : TU \mapsto T\mathbb{R}^{2n+1}$.

Remark 2.19. By (2.2.1), it follows that $\eta_U \neq 0$ in U.

Remark 2.20. A 1-form η_U is a contact form in U, in fact,

$$\begin{aligned} \eta_U \wedge (d\eta_U)^n &= \varphi_U^* \eta_0 \wedge (d(\varphi_U^* \eta_0))^n \\ &= \varphi_U^* \eta_0 \wedge (\varphi_U^* d\eta_0)^n \\ &= \varphi_U^* \eta_0 \wedge \varphi_U^* (d\eta_0)^n \\ &\neq 0. \end{aligned}$$

Definition 2.21. A diffeomorphism ϕ of a $2n+1$-dimensional smooth manifold B, with the contact structure of the Darboux form of theorem 2.18, is called a **contact transformation** if there is a nowhere vanishing smooth function f such that

$$\phi^* \eta_0 = f \eta_0$$

If $f \equiv 1$ on U, then ϕ is called a **strict contact transformation**.

Let us recall that a **pseudrogroup** Γ on a topological space A is a collection of homeomorphisms between open subsets of A that is defined by a set of closure conditions (identity map in Γ, existence of inverse element in Γ and restriction of a map in Γ) by the composition operation and two special properties such as:

- (Restriction condition) If we have U an open set in A such that it is the union of open sets U_i and f is an homeomorphism from U to an open subset of A , and the restrictions of f to U_i is in Γ for all i then f is in Γ.

- (Gluing condition) If $f : U \to V$ and $f' : U' \to V'$ are in Γ, and the intersection $V \cap U'$ is not empty, then the following composition is in Γ:

$$f' \circ f : f^{-1}(V \cap U') \to f'(V \cap U').$$

Afterwards, the collection $\Gamma^{\mathfrak{Con}}$ of all such contact transformations forms a pseudogroup, called the **contact pseudogroup**. Besides, the subset of strict contact transformations forms a subpseudogroup denoted by $\Gamma^{\mathfrak{sCon}}$. Therefore, we can expose a more general definition of a contact manifold in terms of contact transformations. .

Definition 2.22. A $2n+1$ dimensional manifold B with a $\Gamma^{\mathfrak{Con}}$-structure is called a **contact manifold**. If B has a $\Gamma^{\mathfrak{sCon}}$-structure, then it is called a **strict contact manifold**. This structure is usually called the **contact structure in the wider sense**.

Definition 2.23. An **infinitesimal contact transformation** is a local vector field X defined on an open set $U \subset \mathbb{R}^{2n+1}$ that satisfies

$$\mathcal{L}_X \eta_0 = f \eta_0$$

where f is a smooth function on U.

If f vanishes on U, then X is called an **infinitesimal strict contact transformation**. Let $\mathfrak{scon}(U)$ and $\mathfrak{con}(U)$ denote the subsets of all vector fields on U consisting of infinitesimal strict contact transformations and infinitesimal contact transformations, respectively.

Definition 2.24. Let $(B_1, D_1 = \ker\eta_1)$ and $(B_2, D_2 = \ker\eta_2)$ be two co-orientable contact manifolds. A diffeomorphism $\varphi : B_1 \to B_2$ is a **contactomorphism** if the differential $d\varphi$ maps D_1 to D_2 preserving the co-orientations. That is, $\varphi^*\eta_2 = f\eta_1$ for some positive function f.

Definition 2.25. An action of a Lie group G on a manifold B **preserves a contact structure** D and its co-orientation if for every element $a \in G$ the corresponding diffeomorphism $a_B : B \to B$ is a contactomorphism. We will also say that the action of G on (B, D) is a **contact action**.

Definition 2.26. A vector field X on a contact manifold $(B, \xi = \ker\alpha)$ is called a **contact vector field** if its flow φ_t consits of contactomorphisms.

Proposition 2.27. *Let B be a $2n+1$-dimensional contact manifold with $D = \ker\eta$ as its contact bundle . Then*

1. *If n is odd, then B is orientable.*
2. *If n is even, then B is co-orientable. Thus, in this case B has a strict contact structure if and only if B is orientable.*

Proof. Let U_i, η_i, with U_i open sets in B and η_i their 1- forms defined in each U_i. Thus $\eta_i = f_{ij}\eta_j$ in $U_i \cap U_j$ and

$$\begin{aligned} d\eta_i &= d(f_{ij}\eta_j) \\ &= df_{ij} \wedge \eta_j + f_{ij} d\eta_j. \end{aligned}$$

Therefore

$$
\begin{aligned}
(d\eta_i)^n|_D &= (df_{ij} \wedge \eta_j + f_{ij} d\eta_j)^n|_D \qquad (2.2.2)\\
&= (f_{ij})^n (d\eta_j)^n|_D.
\end{aligned}
$$

On the other hand, we have

$$\eta_i \wedge (d\eta_i)^n = f_{ij}^{n+1}(\eta_j \wedge (d\eta_j)^n). \qquad (2.2.3)$$

Consequently, if n is odd , we obtain from (2.2.3) that the sign of the volume form depends only on D but not on the choice of η, so the contact structure D induces a natural orientation on B.
In case n is even, from (2.2.2) we will have that D is orientable, which means that B is co-orientable. In this case, B has a strict contact structure if and only if we can choose the f_{ij} all positive, that is, B is orientable. □

The importance of the following lemma relies on one of its main consequences (cf. Remark 2.29): the way we can characterize the tangent bundle of a contact manifold from the existence of a certain vector field.

Lemma 2.28. *Let (B^{2n+1}, η) be a strict contact manifold. Then, there is a unique vector field ξ, called the **Reeb vector field**, satisfying the following conditions*

1. $\eta(\xi) = 1$
2. $\xi \lrcorner\, d\eta = 0.$

Proof. As we have that $\eta \wedge (d\eta)^n \neq 0$, then $\eta \wedge (d\eta)^n$ is a volume form. And this gives the following isomorphism of $C^\infty(B)-$ modules

$$
\begin{aligned}
\eta \wedge (d\eta)^n : \mathfrak{X}^\infty(B) &\to \Omega^{2n}(B)\\
X &\to X \lrcorner\, (\eta \wedge (d\eta)^n).
\end{aligned}
$$

Therefore, by choosing $(d\eta)^n \in \Omega^{2n}(B)$, we have that there is a unique vector field $\widehat{\xi}$ defined in B such that

$$\widehat{\xi} \lrcorner (\eta \wedge (d\eta)^n) = (d\eta)^n. \tag{2.2.4}$$

Consequently, $\widehat{\xi} \lrcorner \widehat{\xi} \lrcorner (\eta \wedge (d\eta)^n) = \widehat{\xi} \lrcorner (d\eta)^n$ and we obtain that

$$\widehat{\xi} \lrcorner (d\eta)^n = 0. \tag{2.2.5}$$

By 2.2.4 and 2.2.5,

$$\begin{aligned} (\widehat{\xi} \lrcorner \eta) \wedge (d\eta)^n - \eta \wedge (\widehat{\xi} \lrcorner (d\eta)^n) &= (d\eta)^n \\ \eta(\widehat{\xi})(d\eta)^n &= (d\eta)^n. \end{aligned}$$

Thus $\eta(\widehat{\xi}) = 1$.

On the other hand,

$$\begin{aligned} \widehat{\xi} \lrcorner (d\eta)^n &= \widehat{\xi} \lrcorner (d\eta \wedge (d\eta)^{n-1}) \\ &= (\widehat{\xi} \lrcorner d\eta) \wedge (d\eta)^{n-1} + d\eta \wedge (\widehat{\xi} \lrcorner (d\eta)^{n-1}) \\ &= n(\widehat{\xi} \lrcorner d\eta) \wedge (d\eta)^{n-1} \end{aligned} \tag{2.2.6}$$

where the last equation is obtained by iterating $n-1$ times in the parentheses of the second term of the second line like we have done it in the first line, and $d\eta$ is a 2-form.

Finally, by (2.2.5), (2.2.6) and the fact that n is the rank of the 2-form $d\eta$, it follows that $\widehat{\xi} \lrcorner d\eta = 0$. □

Remark 2.29. The Reeb vector field ξ uniquely determines a 1-dimensional foliation F_ξ on (B, η) called the characteristic foliation. Let L_ξ be the trivial line bundle consisting of tangent vectors that are tangent to the leaves of F_ξ, then

$$TB = D \oplus L_\xi.$$

2.2.1 Examples of contact manifolds

As follows we will briefly expose an example of a noncoorientable contact manifold and coorientable ones that will be of big significance in the core of this work.

Example 2.30. ($\mathbb{R}^{n+1} \times \mathbb{RP}^n$)
Let us consider $B = \mathbb{R}^{n+1} \times \mathbb{RP}^n$, $\mathbb{R}^{n+1}$ with coordinates $(x^0, \ldots, x^n)$ and the real projective space $\mathbb{RP}^n$ with homogeneous coordinates, $(t_0, \ldots, t_n)$. If we set $U_i \subset \mathbb{R}^{n+1} \times \mathbb{RP}^n$ as the affine neighbourhood defined by $t_i \neq 0$. We have that $\{U_i\}_{i=0}^n$ cover B. We define the contact structure by a sequence of 1-forms η_i defined in U_i by

$$\begin{aligned}\eta_i &= dx_i + \sum_{j=0 \neq j} \frac{t_j}{t_i} dx_j \\ &= \sum_j \frac{t_j}{t_i} dx_j.\end{aligned}$$

In $U_i \cap U_j$, we have that

$$\eta_j = \frac{t_i}{t_j} \eta_i,$$

and this defines the contact line bundle $\mathcal{L}$ which is non-trivial since it is induced by the tautological line bundle on $\mathbb{RP}^n$. Hence, there is no globally defined contact 1-form on M which defines the contact structure, that is, the contact structure is not strict. We can obtain that

$$\eta_j \wedge (d\eta_j)^n = \left(\frac{t_i}{t_j}\right)^{n+1} \eta_i \wedge (d\eta_i)^n,$$

so M is orientable if and only if n is odd, and in this case M is not co-orientable.

The following lemma can be useful in obtaining new contact manifolds, roughly speaking, submanifolds of a certain symplectic manifold.

Lemma 2.31. *Let Ψ be a Liouville vector field on a symplectic manifold (M, ω) of dimension $2n{+}2$. Suppose that B is a codimension one submanifold of M transverse to Ψ. Then $\alpha = \Psi \lrcorner\, \omega$ is a contact form on B.*

Proof. In fact,

$$\begin{aligned}\alpha \wedge (d\alpha)^n &= (\Psi \lrcorner\, \omega) \wedge (d\,(\Psi \lrcorner\, \omega))^n \\ &= (\Psi \lrcorner\, \omega) \wedge \omega^n \\ &= \frac{1}{n+1} \Psi \lrcorner \left(\omega^{n+1}\right),\end{aligned}$$

where the second equality is a consequence of applying the Cartan's formula and using the definition of a Liouville vector field. The third equality can be proved inductively. Consequently, since Ψ is transversal to B, for each point $p \in B$ and every $v_1, \cdots, v_{2n+1}$ in T_pB, Ψ_p is linear independent with respect to each v_i. Thus, $\Psi \lrcorner\, (\omega^{n+1}) \neq 0$, and α is a contact form on B. □

Example 2.32. The radial vector field $\partial_r = \sum \left(\frac{1}{2} x_j \partial_{x_j} + \frac{1}{2} y_j \partial_{y_j}\right)$ (cf. Example 1.21) is transversal to the unit sphere S^{2n+1}. Thus, from Lemma 2.31, $\partial_r \lrcorner\, \omega_{st} = \frac{1}{2} \sum d\theta_j$ is a contact form on S^{2n+1}, where ω_{st} is the standard symplectic form of $\mathbb{C}^{n+1} \cong R^{2n+2}$ stated in (1.1.9), this time exposed in polar coordinates.

Subsequently, let us study the group of transformations that will provide the necessary structure for strict contact manifolds in order to study the moment maps in the contact case.

Definition 2.33. Let B be a strict contact manifold, and let $Con(B, D)$ denote the group of global contact transformations, that is, the subgroup of the group $\mathrm{Diff}\,(B)$ of diffeomorphisms of B that leaves the contact distribution D invariant. Alternatively fixing a contact form η such that $D = \ker\eta$, then

$Con(B,D)$ can be characterized as

$$Con\,(B,D) = \{\phi \in \mathrm{Diff}\,(B)\,|\phi^*\eta = f\eta \text{ for } f \in C^\infty(B) \text{ nowhere vanishing}\}$$

With the 1-form η fixed we are also interested in the subgroup $Con(B,\eta)$ of global strict contact transformations defined by the condition $\phi^*\eta = \eta$.

The Lie algebras of $Con(B,D)$ and $Con(B,\eta)$ denoted by $\mathfrak{con}(B,D)$ and $\mathfrak{con}(B,\eta)$, respectively, can be characterized as follows:

$$\mathfrak{con}(B,D) = \{X \in \mathfrak{X}^\infty(B)|\mathcal{L}_X\eta = g\eta \text{ for some } g \in C^\infty(B)\}$$

$$\mathfrak{con}(B,\eta) = \{X \in \mathfrak{X}^\infty(B)|\mathcal{L}_X\eta = 0\},$$

where $\mathfrak{X}^\infty(B)$ denotes the Lie algebra of smooth vector fields on B.

Those Lie algebras are associated with the corresponding pseudogroups $\Gamma^{\mathfrak{Con}}$, not groups of global transformations.

Moreover, according to Definition 2.26, we can characterize $\mathfrak{con}(B,D)$ as the Lie algebra which consits of contact vector fields in B.

Lemma 2.34. *If $X \in \mathfrak{con}(B,\eta)$ then $\mathcal{L}_X\xi = [X,\xi] = 0$.*

Proof. We have that

$$\begin{aligned} \mathcal{L}_X\left(\eta(\xi)\right) &= \left(\mathcal{L}_X\eta\right)(\xi) + \eta\left([X,\xi]\right) \\ 0 &= 0 + \eta\left([X,\xi]\right). \end{aligned} \tag{2.2.7}$$

Then

$$\begin{aligned} \mathcal{L}_{[X,\xi]}\eta\left([X,\xi]\right) &= \mathcal{L}_{[X,\xi]}\eta + \eta\left([[X,\xi],[X,\xi]]\right) \\ 0 &= \mathcal{L}_{[X,\xi]}\eta + 0. \end{aligned} \tag{2.2.8}$$

Thus, since (2.2.7), (2.2.8) and Cartan's formula:

$$\begin{aligned} \mathcal{L}_{[X,\xi]}\eta &= d\left([X,\xi] \lrcorner\, \eta\right) + [X,\xi] \lrcorner\, d\eta \\ 0 &= 0 + [X,\xi] \lrcorner\, d\eta. \end{aligned} \tag{2.2.9}$$

Then, by using the nondegeneracy of $d\eta$ in $\ker\eta$, (2.2.7) and (2.2.9) imply that:

$$[X, \xi] = 0 \text{ for every } X \in \mathfrak{con}(B, \eta).$$

□

The following proposition will be useful as it will allow us to notice the well definition of a contact moment map.

Proposition 2.35. *Let $(B, D = \ker\eta)$ be a contact manifold. The linear map from contact vector fields to smooth functions given by $X \to f^X := \eta(X)$ is one-to-one and onto.*

Proof. Let us observe that, by taking the Reeb vector field ξ:

$$\eta(\xi) = 1.$$

Thus, $\eta\left(X - \eta(X)\xi\right) = 0$, which means that $X - \eta(X)\xi \in D$ for every vector field X in B.
As $d\eta|_D$ is nondegenerate, $X - \eta(X)\xi$ is uniquely determined by

$$(X - \eta(X)\xi) \lrcorner\, d\eta|_D \tag{2.2.10}$$

For every section v of $D \to B$ and every contact vector field X in B,

$$\begin{aligned} \left(\mathcal{L}_X\eta\right)(v) &= 0 \\ \left(X \lrcorner\, d\eta + d\left(X \lrcorner\, \eta\right)\right)(v) &= 0 \\ d\eta(X, v) + d\left(\eta(X)\right)(v) &= 0. \end{aligned}$$

Let us define the linear map from contact vector fields to smooth functions by

$$X \to f^X := \eta(X).$$

Thus, for every section v of D and for every contact vector field X in B,

$$\begin{aligned} d\eta(X,v) &= -d\left(\eta(X)\right)(v) \\ &= -df^X(v). \end{aligned} \tag{2.2.11}$$

Moreover,

$$\begin{aligned} \left(X - \left(X - f^X\xi\right)\right) \lrcorner\, d\eta &= f^X\xi \lrcorner\, d\eta \\ &= 0, \end{aligned} \tag{2.2.12}$$

as ξ is the corresponding Reeb vector field.

Thus, in particular, from (2.2.11) and (2.2.12):

$$\begin{aligned} (X - \eta(X)\xi) \lrcorner\, d\eta|_D &= X \lrcorner\, d\eta|_D \\ &= -df^X|_D, \end{aligned} \tag{2.2.13}$$

for every contact vector field X in B. Consequently, if we assume that $f^X = f^Y$ for every contact vector fields X, Y in B, it follows from what we observed in (2.2.10) and (2.2.13) that $X = Y$ in D, and, as $TB = D \oplus \mathbb{R}\xi$ where $X = X - \eta(X)\xi + \eta(X)\xi$ with $\eta(X)\xi \in \mathbb{R}\xi$, we obtain that the linear map is 1-1.

We are going to see that the linear map is onto. Indeed, for every $f \in C^\infty(B)$ and from (2.2.13), there exists a unique section X'_f of D, such that:

$$X'_f \lrcorner\, d\eta|_D = -df|_D. \tag{2.2.14}$$

Let us define the following vector field in B,

$$X_f := X'_f + f\xi;$$

we observe that

$$\begin{aligned} \eta(X_f) = \eta(X'_f + f\xi) &= \eta(X'_f) + f\eta(\xi) \\ &= \eta(X'_f) + f = f. \end{aligned} \tag{2.2.15}$$

It is only left to prove that X_f is a contact vector field.

In fact, for every $w = w_1 + w_2 \in TB = D \oplus \mathbb{R}\xi$:

$$\begin{aligned}\mathcal{L}_{X_f}\eta(w) &= (d(X_f \lrcorner\, \eta) + X_f \lrcorner\, d\eta)\,(w) \\ &= d\,(\eta(X_f))\,(w_1) + d\,(\eta(X_f))\,(w_2) + (X_f \lrcorner\, d\eta)(w_1) + (X_f \lrcorner\, d\eta)(w_2).\end{aligned}$$

From (2.2.14), (2.2.15) and the definition of the Reeb vector field ξ:

$$\begin{aligned}\left(\mathcal{L}_{X_f}\eta\right)(w) &= d(f)(w_1) + d(f)(w_2) - d(f)(w_1) + (X_f \lrcorner\, d\eta)(w_2) \\ &= df(w_2) + (X_f \lrcorner\, d\eta)(w_2) \\ &= d(\eta(X_f))(w_2) + (X_f \lrcorner\, d\eta)(w_2) \\ &= \left(\mathcal{L}_{X_f}\eta\right)(w_2). \qquad (2.2.16)\end{aligned}$$

Now, as $w_2 \in \mathbb{R}\xi$, we can write $w_2 = t\xi$ for some $t \in \mathbb{R}$, then:

$$\begin{aligned}\left(\mathcal{L}_{X_f}\eta\right)(w) = \left(\mathcal{L}_{X_f}\eta\right)(w_2) &= \mathcal{L}_{X_f}(\eta(w_2)) - \eta\left([X_f, w_2]\right) \\ &= -\eta\left([X_f, w_2]\right). \qquad (2.2.17)\end{aligned}$$

If t was zero, we would obtain immediately that X_f is a contact vector field. Otherwise, we can write:

$$\left(\mathcal{L}_{X_f}\eta\right)(w) = g.\eta(w) \qquad (2.2.18)$$

where $g = -\frac{\eta([X_f, w_2])}{t} \in C^\infty(B)$ and $\eta(w) = t$, which implies that X_f is a contact vector field in B. $\square$

2.3 The link between Symplectic cones and Contact manifolds

The following propositions in this section are going to show how a contact manifold B and its symplectic cone M are intimately related.

Proposition 2.36. *Any principal $\mathbb{R}$-bundle $\mathbb{R} \to M \stackrel{\varpi}{\to} B$ is trivial.*

Proof. Note first that if $s : B \to M$ is a (local) section of $M \stackrel{\varpi}{\to} B$ and $f \in C^\infty(B)$ is a function, then $s - f$ is again a (local) section of $M \stackrel{\varpi}{\to} B$. To prove that a principal bundle is trivial it is enough to construct a global section. To this end choose an open cover $\{U_\alpha\}$ of B such that for each U_α there is a section $s_\alpha : U_\alpha \to M$. Choose a partition of unity τ_α subordinate to the cover $\{U_\alpha\}$. Two sections of a principal $\mathbb{R}$-bundle differ by real-valued function. Thus by abuse of notation on an intersection $U_\alpha \cap U_\beta$, $s_\alpha - s_\beta$ is a real-valued function. Now define for each index α

$$s'_\beta = s_\beta - \sum_{\alpha \neq \beta} \tau_\alpha (s_\beta - s_\alpha)$$

Then on an intersection $U_\alpha \cap U_\beta$

$$\begin{aligned} s'_\beta - s'_\gamma &= \left(s_\beta - \sum_{\alpha \neq \beta} \tau_\alpha (s_\beta - s_\alpha) \right) - \left(s_\gamma - \sum_{\alpha \neq \gamma} \tau_\alpha (s_\gamma - s_\alpha) \right) \\ &= s_\beta - s_\gamma - \left(\sum_{\alpha \neq \beta, \gamma} \tau_\alpha (s_\beta - s_\gamma) \right) + \tau_\beta (s_\gamma - s_\beta) - \tau_\gamma (s_\beta - s_\gamma) \\ &= s_\beta - s_\gamma - (\sum_\alpha \tau_\alpha)(s_\beta - s_\gamma) \\ &= 0. \end{aligned}$$

Therefore, the collection of local sections $\{s'_\alpha\}$ defines a global section of $\varpi : M \to B$. Consequently the bundle is trivial. $\square$

Thus any symplectic cone is of the form $B \times \mathbb{R}$ where $B = M/\mathbb{R}$ is an odd-dimensional manifold.

Proposition 2.37. *Let (M, ω, X) be a symplectic cone, let B be its base and let $\varpi : M \to B$ denote the projection. Pick a trivialization $\varphi : B \times \mathbb{R} \to M$.*

Then $\varphi^*\omega = d(e^t\eta)$ where t is a coordinate on $\mathbb{R}$ and η is a contact form on B. Conversely, if η is a contact form on B then $(B \times \mathbb{R}, d(e^t\eta), \frac{\partial}{\partial t})$ is a symplectic cone.

Proof. By Proposition 2.36, the principal $\mathbb{R}$-bundle $\overline{\omega} : M \to B$ is trivial. Let us choose a trivialization

$$\begin{aligned} \varphi : B \times \mathbb{R} &\to M \\ (p,t) &\to \varphi(p,t) := \rho_t(s(p)) \end{aligned} \tag{2.3.1}$$

where ρ_t is the flow generated by the Liouville vector field X according to Definition 2.3 and $s : B \to M$ is a global section of $\overline{\omega} : M \to B$.
Under this identification the vector field X becomes $\frac{\partial}{\partial t}$.
As $d\omega = 0$ and $\mathcal{L}_X\omega = \omega$, then

$$\omega = \mathcal{L}_X\omega = d(X \lrcorner\, \omega). \tag{2.3.2}$$

Let us call $\beta := X \lrcorner\, \omega$ in M. Then $X \lrcorner\, \beta = X \lrcorner\, (X \lrcorner\, \beta) = 0$ and

$$\begin{aligned} \mathcal{L}_X\beta = d(X \lrcorner\, \beta) + X \lrcorner\, d\beta &= X \lrcorner\, d\beta \\ &= \beta. \end{aligned} \tag{2.3.3}$$

As $\beta(X) = 0$ and $X = \frac{\partial}{\partial t}$, we obtain that $\varphi^*\beta = \rho_t^*\beta$ (1-forms in $B \times \mathbb{R}$) does not depend on dt. So we can set

$$(\rho_t^*\beta)_{(p,0)} := \eta_{(p)} \tag{2.3.4}$$

as a 1-form in B, for every $p \in B$.
Let $Y \in TM$, then, in local coordinates:

$$Y = (a^1_{1,2,\ldots,2n+2})\frac{\partial}{\partial x_1} + \ldots + (a^{2n+1}_{1,2,\ldots,2n+2})\frac{\partial}{\partial x_{2n+1}} + (a^{2n+2}_{1,2,\ldots,2n+2})\frac{\partial}{\partial t}, \tag{2.3.5}$$

where $a^{1}_{1,2,\ldots,2n+2},\cdots,a^{2n+2}_{1,2,\ldots,2n+2}$ are C^∞ functions in M.

From (2.3.3), we obtain that:

$$\begin{aligned}(\beta(Y))_{s(p)} &= \lim_{\lambda\to 0}\frac{(\rho_\lambda^*\beta(Y))_{s(p)}-(\beta(Y))_{s(p)}}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{(\beta\left(\rho_{\lambda *}(Y)\right))_{(s(p))}-(\beta(Y))_{s(p)}}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{\beta\left(d\rho_{\lambda_{\rho^{-1}_{\lambda(s(p))}}}\left(Y_{\rho_\lambda^{-1}(s(p))}\right)\right)-(\beta(Y))_{s(p)}}{\lambda}.\end{aligned} \tag{2.3.6}$$

Thus, from (2.3.5) and the fact that we are identifying X with $\frac{\partial}{\partial t}$, we can apply L'Hôpital's rule in (2.3.6) to obtain that, due to a solution of a differential equation,

$$(\varphi^*\beta)_{(p,t)} = (\rho_t^*\beta)_{s(p)} = e^t(\rho_t^*\beta)_{(p,0)}. \tag{2.3.7}$$

Consequently, by taking the exterior derivative in (2.3.7) and in view of the identification made above of η in B:

$$(\varphi^*\omega)_{(p,t)} = d(e^t\eta)_p, \tag{2.3.8}$$

for every $p\in B$.

Let us prove that η is a contact form in B. By setting $n=\frac{1}{2}\dim M-1$, we know that $\omega^{n+1}\neq 0$ in M, and since (2.3.8) and the fact that φ is a trivialization, we obtain :

$$\left(d(e^t\eta)\right)^{n+1}\neq 0. \tag{2.3.9}$$

On the other hand,

$$\begin{aligned}\left(d(e^t\eta)\right)^{n+1} &= e^{t(n+1)}\left(dt\wedge\eta+d\eta\right)^{n+1}\\ &= e^{t(n+1)}\left((n+1)dt\wedge\eta\wedge(d\eta)^n+(d\eta)^{n+1}\right)\\ &= (n+1)e^{t(n+1)}\left(dt\wedge\eta\wedge(d\eta)^n\right)\neq 0\end{aligned}$$

But in B, we know that there is no depency on the variable t, so $\eta\wedge(d\eta)^n \neq 0$.

Conversely, let us suppose that η is a contact 1-form on B. Let $\omega = d(e^t\eta)$ and let $X = \frac{\partial}{\partial t}$. Then

$$\mathcal{L}_X\omega = d(\frac{\partial}{\partial t} \lrcorner\, d(e^t\eta)) = d(\frac{\partial}{\partial t} \lrcorner\, (e^t dt \wedge \eta + e^t d\eta)) = d(e^t\eta + 0) = \omega,$$

that is, X is a Liouville vector field on M.
It remains to check that ω is nondegenerate. For any $(b,t) \in B \times \mathbb{R}$, the tangent space $T_{(b,t)}(B \times \mathbb{R})$ decomposes as $T_{(b,t)}(B \times \mathbb{R}) = \ker \eta_b \oplus \mathbb{R}\xi(b) \oplus \mathbb{R}$ where ξ is the Reeb vector field of η (cf. Remark 2.29).
Since η is contact, then $d\eta_b|_{\ker \eta_b}$ is nondegenerate. The restriction $dt \wedge \eta_b$ to $\mathbb{R}\xi(b)\oplus\mathbb{R}$ is nondegenerate as well. Hence $\omega = e^t(dt\wedge\eta+d\eta)$ is nondegenerate. This proves that $(B \times \mathbb{R}, d(e^t\eta), \frac{\partial}{\partial t})$ is a symplectic cone. $\square$

Let (B,η) be a strict manifold of dimension $2n+1$ with Reeb vector field ξ, and M its symplectic cone. On M we define $\mathfrak{S}(M,\omega)$ as the group of symplectomorphisms of (M,ω), and $\mathfrak{S}_0(M,\omega)$ the subgroup of $\mathfrak{S}(M,\omega)$ that commutes with homotheties, which are the ones that satisfy $\rho_\lambda^*\omega = e^\lambda\omega$ with $\rho_\lambda \in \mathrm{Diff}(M)$ and $\lambda \in \mathbb{R}$.

Their correspondings Lie algebras are denoted by $\mathfrak{s}(M,\omega)$ and $\mathfrak{s}_0(M,\omega)$, which can be characterized respectively as:

$$\mathfrak{s}(M,\omega) = \{X \in \mathfrak{X}^\infty(M) | \mathcal{L}_X\omega = 0\} \tag{2.3.10}$$

$$\mathfrak{s}_0(M,\omega) = \{X \in \mathfrak{X}^\infty(M) | [X,\psi] = 0\}, \tag{2.3.11}$$

where ψ is the Liouville vector field which generates the flow of the homotheties.
According to the definitions given in Definition 2.33,

Proposition 2.38. *There exists an isomorphism $\mathfrak{S}_0(M,\omega) \simeq Con(B,\eta)$ of topological groups, which is induced by the natural inclusion $B \to M \simeq B \times \mathbb{R}$.*

Proof. cf. [12], page 314. □

The following proposition characterizes the group of automorphisms of our interest with the symplectic cone scenario.

Proposition 2.39. *Infinitesimally, there are Lie algebra isomorphisms*

$$\mathfrak{s}_0(M,\omega) \simeq \mathfrak{con}(B,\eta) \simeq C^\infty(B)^\xi,$$

where

$$C^\infty(B)^\xi = \{f \in C^\infty(B) | \varphi_t^* f = f\},$$

where φ_t is the flow generated by the Reeb vector field ξ. Moreover, ξ is in the center of $\mathfrak{con}(B,\eta)$.

Proof.
- $\underline{\mathfrak{con}(B,\eta) \simeq C^\infty(B)^\xi}$: In fact, we observe that we can use the same linear map $X \to \eta(X)$, in this case, for every $X \in \mathfrak{con}(B,\eta)$, cf. Proposition 2.35 . Consequently, we have that the linear map is 1-1.
  From Proposition 2.35, we obtain that $\mathfrak{con}(B,D) \simeq C^\infty(B)$.
  Then, there exists $X \in \mathfrak{con}(B,D)$, that is, $\mathcal{L}_X\eta = h\eta$ for some $h \in C^\infty(B)$, such that $f = \eta(X)$. Let us prove that our map is onto.
  Let $f \in C^\infty(B)^\xi$, that is,

$$\begin{aligned} \mathcal{L}_\xi(\eta(X)) &= 0 \\ (\mathcal{L}_\xi\eta)(X) + \eta([\xi,X]) &= 0 \\ 0 + \eta([\xi,X]) &= 0 \end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathcal{L}_X \eta)(\xi) &= (h\eta)(\xi) \\
\mathcal{L}_X(\eta(\xi)) - \eta([X,\xi]) &= h \\
0 &= h,
\end{aligned}$$

which implies that $X \in \mathfrak{con}(B,\eta)$.

- $\underline{\mathfrak{s}_0(M,\omega) \simeq \mathfrak{con}(B,\eta)}$: Let us define a map

$$\begin{aligned}
\mathfrak{s}_0(M,\omega) &\to \mathfrak{con}(B,\eta) \\
X &\to X_B
\end{aligned}$$

Since $X \in \mathfrak{s}_0(M,\omega)$, $[X,\psi] = 0$ where ψ is the Liouville vector field in M. This allow us to choose X_B as X. Indeed, let X be in local coordinates as $(a^1_{1,\ldots,2n+2})\frac{\partial}{\partial x^1} + \cdots + (a^{2n+2}_{1,\ldots,2n+2})\frac{\partial}{\partial t}$ and $\psi := \frac{\partial}{\partial t}$. Since $[X,\psi] = 0$, in local coordinates this means:

$$\left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} = 0 \tag{2.3.12}$$

where the V^i's and W^j's are the coefficients for X and ψ, respectively. By calculating we obtain that:

$$\begin{aligned}
\left(-\frac{\partial a^1_{1,\ldots,2n+2}}{\partial t}\right)\frac{\partial}{\partial x^1} - \left(\frac{\partial a^2_{1,\ldots,2n+2}}{\partial t}\right)\frac{\partial}{\partial x^2} \cdots - \left(\frac{\partial a^{2n+2}_{1,\ldots,2n+2}}{\partial t}\right)\frac{\partial}{\partial t} &= 0 \\
\frac{\partial a^1_{1,\ldots,2n+2}}{\partial t} = 0 = \cdots = \frac{\partial a^{2n+2}_{1,\ldots,2n+2}}{\partial t}&, \qquad (2.3.13)
\end{aligned}$$

which means that our vector field X has no any coefficient in the t coordinate. So we can set $X_B := X$.
It only remains to prove that our map is well defined. In fact, let γ_t be the flow generated by X, φ the trivialization taken in Proposition 2.37

and ρ_t the flow generated by the Liouville vector field ψ.

Since $\varphi_t^*\omega = \omega$,

$$\begin{aligned}
\varphi^*(\gamma_t^*\omega) &= \varphi^*\omega \\
\rho_\lambda^*\left(\gamma_t^*\omega\right) &= d(e^\lambda \eta) \\
\gamma_t^*(\rho_\lambda^*\omega) &= d(e^\lambda \eta) \\
\gamma_t^* d(e^\lambda \eta) &= d(e^\lambda \eta) \\
d\left(\gamma_t^* e^\lambda \eta\right) &= d(e^\lambda \eta). \qquad (2.3.14)
\end{aligned}$$

This implies that:

$$d\left(\gamma_t^* e^\lambda \eta - e^\lambda \eta\right) = 0, \qquad (2.3.15)$$

and if we we had that there exists a smooth function f in B such that

$$\gamma_t^* e^\lambda \eta - e^\lambda \eta = df \qquad (2.3.16)$$

for every λ and t in $\mathbb{R}$, it would imply that $df = 0$, so we conclude in particular that $\gamma_t^*\eta = \eta$, and $\mathcal{L}_X \eta = 0$, obtaining the well definition of our map and the isomorphism follows immediately.

- ξ is in the center of $\mathfrak{con}(B, \eta)$: This is exactly what we proved in Lemma 2.34.

□

Chapter 3

Contact reduction and Contact Toric Manifolds

The notion of contact reduction arises from the natural interplay between the symplectic cones and contact manifolds. As expected, the notion of symplectic reduction plays a key role in understanding the concept of reduction at the level of contact structures. We begin this chapter explaining symplectic reduction performed on the complex projective space $\mathbb{CP}^n$. This example will prove to be very useful, at the level of contact reduction, especially for contact toric manifolds. First some preliminaries.

3.1 Symplectic moment maps

Now, let (M, ω) be a symplectic manifold, G a Lie group acting in M, $\mathfrak{g}$ its Lie algebra and $\phi : G \to \mathrm{Sympl}(M, \omega)$ be a symplectic action, that is, $\phi^*\omega = \omega$.

Definition 3.1. ϕ is a **Hamiltonian action** if there is a map $\mu : M \to \mathfrak{g}^*$ such that

1. - For every $X \in \mathfrak{g}$, let

$$\begin{aligned} \mu^X : M &\to \mathbb{R} \\ \mu^X(p) &:= \langle \mu(p), X \rangle \end{aligned}$$

   be the component of μ along X.

   - Let $X^\#$ be the vector field in M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subset G$

   Then

$$d\mu^X = X^\# \lrcorner\, \omega.$$

   That is, μ^X is a Hamiltonian function for the vector field $X^\#$.

2. μ is equivariant with respect to the action ϕ of G in M and the coadjoint action Ad^* of G in $\mathfrak{g}^*$, that is,

$$\mathrm{Ad}_g^* \circ \mu = \mu \circ \phi_g.$$

(M, ω, G, μ) is called a G- **Hamiltonian space** and μ is called the **moment map**.

Theorem 3.2. *Let ϕ be a symplectic action of G in (M, ω) with moment map μ. Suppose $H : M \to \mathbb{R}$ is invariant under the action ϕ. ($H(x) = H(\phi_g(x))$ for every $x \in M, g \in G$), then μ is an integral for X_H (that is, if F_t is the flow of X_H then $\mu(F_t(p)) = \mu(p)$ for every p, t where F_t is defined).*

Proof. From the nondegeneracy of ω, it follows that for every 1-form α, there is a unique vector field Ω_α such that

$$\Omega_\alpha \lrcorner\, \omega = \alpha.$$

As H is invariant,

$$H\left(\phi_{\exp(tX)}(p)\right) = H(p)$$

for every $X \in \mathfrak{g}$. Besides, as μ is a moment map for ϕ, it follows that $\Omega_{d\mu^X} = X^{\#}$, so by differentiating over $t = 0$ we have

$$\begin{aligned} 0 &= dH(p) \cdot X^{\#}(p) \\ &= \mathcal{L}_{X^{\#}} H \\ &= \{H, \mu^X\} \\ &= \omega(\Omega_{dH}, X^{\#}) \\ &= -\omega(X^{\#}, \Omega_{dH}) \\ &= -d\mu^X(p) \cdot \Omega_{dH}(p) \end{aligned}$$

where $X_H = \Omega_{dH}$ and the last equality follows from the fact that μ is a moment map of ϕ. This implies that $\mu(F_t(p)) = \mu(p)$ for every $p \in M$ and F_t flow of Ω_{dH}. □

Theorem 3.3. *Let ϕ be a symplectic action of a Lie group G in a symplectic manifold (M, ω). Suppose that $\omega = -d\theta$ and the action leaves invariant θ, that is*

$$\phi_g^* \theta = \theta$$

for every $g \in G$. Then $\mu : M \to \mathfrak{g}^*$, defined by

$$\langle \mu(p), \xi \rangle = \left(\xi^{\#} \lrcorner\, \theta \right)(p),$$

is an Ad^* *equivariant map for ϕ.*

Proof. By invariance of θ, we have

$$0 = \frac{d}{dt} |_{t=0}\, \phi^*_{\exp(t\xi)}(\theta) = \mathcal{L}_{\xi^{\#}} \theta.$$

Thus, by Cartan's formula,

$$\begin{aligned} 0 &= \xi^{\#} \lrcorner\, d\theta + d(\xi^{\#} \lrcorner\, \theta) \\ &= \xi^{\#} \lrcorner -\omega + d(\xi^{\#} \lrcorner\, \theta). \end{aligned}$$

That is,

$$d(\xi^{\#} \lrcorner\, \theta) = \xi^{\#} \lrcorner\, \omega$$

which proves that $\mu^{\xi} = \xi^{\#} \lrcorner\, \theta$ is a moment map for ϕ.

μ is Ad^*- equivariant:

We want to prove that

$$\langle \mu(\phi_g(p)), \xi \rangle = \left\langle \mathrm{Ad}_g^*(\mu(p), \xi) \right\rangle$$

which is equivalent to prove

$$\begin{aligned} & \mu^{\xi}\left(\phi_g(p)\right) = \langle \mu(p)), \mathrm{Ad}_{g^{-1}}\xi \rangle \\ \Leftrightarrow \quad & \mu^{\xi}\left(\phi_g(p)\right) = \mu^{\mathrm{Ad}_{g^{-1}}\xi}(p) \\ \Leftrightarrow \quad & \left(\xi^{\#} \lrcorner\, \theta\right)\left(\phi_g(p)\right) = \left((\mathrm{Ad}_{g^{-1}}\xi)^{\#} \lrcorner\, \theta\right)(p) \end{aligned}$$

Additionally, we have the following property:

$$(\mathrm{Ad}_{g^{-1}}\xi)^{\#} = \phi_g^*(\xi^{\#})$$

which proof is as follows:

$$\begin{aligned} (\mathrm{Ad}_g\xi)_p^{\#} &= \frac{d}{dt}\phi_{\exp t\mathrm{Ad}_g\xi}(p)|_{t=0} \\ &= \frac{d}{dt}\phi_{g(\exp t\xi)g^{-1}}(p)|_{t=0} \\ &= \frac{d}{dt}\left(\phi_g \circ \phi_{\exp t\xi}\right)\left(g^{-1}p\right) \\ &= \left(\phi_g\right)_{*g^{-1}p}\left(\xi^{\#}\right)_{g^{-1}p} \\ &= \left(\phi_{g^{-1}}^*\left(\xi^{\#}\right)\right)_p . \end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}\left((\mathrm{Ad}_{g^{-1}}\xi)^{\#} \lrcorner\, \theta\right)(p) &= \left(\phi_g^*(\xi^{\#}) \lrcorner\, \theta\right)(p) \\ &= \left(\phi_g^*(\xi^{\#}) \lrcorner\, \phi_g^*\theta\right)(p) \\ &= \left(\phi_g^*\left(\xi^{\#} \lrcorner\, \theta\right)\right)(p) \\ &= \left(\xi^{\#} \lrcorner\, \theta\right)(\phi_g(p))\end{aligned}$$

for every $p \in M$, and $g \in G$. $\square$

Let us exhibit some examples which will be very useful in defining an analogue of a moment map in the contact case scenario. In fact, we will see that this analogy is not as easily seen as we would imagine.

Example 3.4. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ where every element can be represented by e^{it} for $t \in \mathbb{R}$. Thus, its Lie algebra is $\mathfrak{g} = \{it : t \in \mathbb{R}\} \simeq \mathbb{R}$. Let us consider the action

$$\begin{aligned}\varphi : S^1 \times \mathbb{C} &\to \mathbb{C} \\ \left(e^{it}, z\right) &\mapsto e^{it}z.\end{aligned}$$

By setting ω as the standard symplectic form in $\mathbb{C}$, we want to find the corresponding moment map $\mu : \mathbb{C} \to \mathbb{R}^*$ which must satisfy, by definition, that $d\mu^X = X^{\#} \lrcorner\, \omega$, for every X in $\mathbb{R}^*$.
Indeed,

$$\begin{aligned}X^{\#}(z) &= \frac{d}{dt}\Big|_{t=0} \exp(tX).z \\ &= \frac{d}{dt}\Big|_{t=0}\varphi(e^{it}, z) \\ &= \frac{d}{dt}\Big|_{t=0}e^{it}z \\ &= iz.\end{aligned} \tag{3.1.1}$$

At the same time, $z = x + iy = r\cos\theta + ir\sin\theta$, then:

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= -r\sin\theta + ir\cos\theta \\ &= iz. \end{aligned} \tag{3.1.2}$$

Thus, from (3.1.1) and (3.1.2):

$$X^{\#} = \frac{\partial}{\partial \theta}.$$

If we want $X^{\#}$ to be expressed in z and $\overline{z}$ coordinates,

$$\begin{aligned} X^{\#} &= \frac{\partial}{\partial z}.\frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \overline{z}}.\frac{\partial \overline{z}}{\partial \theta} \\ &= (iz).\frac{\partial}{\partial z} + (-i\overline{z})\frac{\partial}{\partial \overline{z}} \\ &= i\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right). \end{aligned} \tag{3.1.3}$$

Additionally, from (1.2.3), (3.1.3) becomes, in real coordinates

$$\begin{aligned} X^{\#} &= i\left((x+iy)\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - (x-iy)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right) \\ &= i\left(-i\left(x\frac{\partial}{\partial y}\right) + i\left(y\frac{\partial}{\partial x}\right)\right) \\ &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \end{aligned} \tag{3.1.4}$$

Since (1.1.9), $\omega = \frac{i}{2}dz \wedge d\overline{z}$ in $\mathbb{C}$, and from (3.1.3):

$$\begin{aligned} \left(X^{\#} \lrcorner\, \omega\right)_{(.)} &= \left(i\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right) \lrcorner \left(\frac{i}{2}dz \wedge d\overline{z}\right)\right)_{(.)} \\ &= \left(-\frac{1}{2}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right) \lrcorner\, (dz \wedge d\overline{z})\right)_{(.)} \\ &= -\frac{1}{2}\left(dz_{\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right)}.d\overline{z}_{(.)} - dz_{(.)}.d\overline{z}_{\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right)}\right) \\ &= -\frac{1}{2}\left(zd\overline{z}_{(.)} + \overline{z}dz_{(.)}\right) \\ &= \left(-\frac{1}{2}\left(zd\overline{z} + \overline{z}dz\right)\right)_{(.)} \\ &= d\left(-\frac{1}{2}\left(z.\overline{z}\right)\right)_{(.)}. \end{aligned} \tag{3.1.5}$$

Thus, by integrating in (3.1.5) over $\mathbb{C}$, we obtain the following moment map $\mu : \mathbb{C} \to \mathbb{R}^*$:

$$\mu(z) = -\frac{1}{2}|z|^2 + C$$

for every $z \in \mathbb{C}$ and $C \in \mathbb{R}$ a constant.

Example 3.5. Let G be a Lie group acting on a manifold M of dimension n with its corresponding action $\phi_g : M \to M$.
By setting

$$\begin{aligned} \tilde{\phi}_g : T^*M &\to T^*M \\ T_q^*M &\to T^*_{\phi_g(p)}M \\ (q,p) &\to \left(\phi_g(q), \phi^*_{g^{-1}}p\right), \end{aligned}$$

it is easily seen that this is a left action of G on T^*M, for every $q \in M$ and $p \in T_q^*M$.
We observed in Example 1.18 that if (q_i) is a local coordinate system on M, and (q_i, p_i) is the corresponding coordinate system on T^*M, it follows that

$$\lambda = \sum_{i=1}^{n} p_i dq_i. \tag{3.1.6}$$

By its canonical expression, λ is G-invariant. Therefore, there is a moment map for this G-action on T^*M given by

$$\langle \mu, X \rangle = -\left(X^{\#} \lrcorner \lambda\right), \tag{3.1.7}$$

for every $X \in \mathfrak{g}$.
Thus, if $X^{\#} = \sum_{i=1}^{n} X_i(q) \frac{\partial}{\partial q_i}$ in local coordinates, we have

$$\langle \mu(q,p), X \rangle = \sum_{i=1}^{n} p_i X_i(q) = \left\langle p, X_q^{\#} \right\rangle. \tag{3.1.8}$$

We notice that this map is, indeed, a moment map (that is, it satisfies the second condition of being a moment map) because it verifies the hypotheses of Theorem 3.3.

Next, we state one important result that determines how to produce from coisotropic submanifolds of a symplectic manifold (the zero set of a moment map coming from a Hamiltonian action) symplectic quotients.

3.2 The Marsden-Weinstein-Meyer Reduction Theorem

This remarkable theorem serves as a preamble of what we are going to expose about contact manifolds obtained by the reduction process. This result can be appreciated naturally in Example 3.7 that follows.

Theorem 3.6. *Let G be a Lie group and suppose we have a Hamiltonian action of G on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. If G acts freely and properly on $\mu^{-1}(0)$ (with zero as a regular value for μ and then $\mu^{-1}(0)$ is a manifold), then the orbit $M_G := \mu^{-1}(0)/G$ is a smooth manifold, the natural projection $\pi : \mu^{-1}(0) \to M_G$ is a principal G-bundle, and there exists a unique symplectic form ω_G on M_G satisfying $\pi^*\omega_G = \omega|_{\mu^{-1}(0)}$.*

Proof. See [14]. Page 175. □

Example 3.7. Let

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim \quad := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^* \tag{3.2.1}$$

be the complex projective space.

Let us consider the following diagram (cf. Example 1.23), where π is the standard projection map restricted to S^{2n+1} and i is the inclusion map:

$$\begin{tikzcd} S^{2n+1} \arrow[r, "i"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \\ (\omega_{FS}, \mathbb{CP}^n) & \end{tikzcd}$$

We found in Claim 1.23.3 that:

$$\pi^*\omega_{FS} = i^*\omega. \tag{3.2.2}$$

The diagram above represents a symplectic reduction of $\mathbb{C}^{n+1}$, where S^1 acts on $\mathbb{CP}^n$ with the following moment map:

$$\mu : \mathbb{C}^{n+1} \to \mathbb{R}$$
$$z \mapsto -\frac{\|z\|^2}{2} + \frac{1}{2}.$$

In fact, let

$$\begin{aligned} \omega &= \frac{\sqrt{-1}}{2}\sum_{i=1}^{n+1} dz_i \wedge d\overline{z}_i \\ &= \sum_{i=1}^{n+1} dx_i \wedge dy_i \\ &= \sum_{i=1}^{n+1} r_i dr_i \wedge d\theta_i \end{aligned}$$

be the standard symplectic form in $\mathbb{C}^{n+1}$.

By considering the action of S^1 on $(\mathbb{C}^{n+1}, \omega)$:

$$e^{it} \in S^1 \mapsto \Psi := \text{ multiplication by } e^{it},$$

we observe that it is the same action as :

$$j(S^1) \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$
$$\big(j(e^{it}), (z_1, \dots, z_{n+1})\big) \mapsto (e^{it}z_1, \dots, e^{it}z_n),$$

where

$$\begin{aligned} j : S^1 &\to \mathbb{T}^{n+1} = S^1 \times \ldots \times S^1 \\ e^{it} &\mapsto (e^{it}, \ldots, e^{it}) \end{aligned}$$

is the inclusion map.

Thus, by noting that the Lie algebra of $\mathbb{T}^{n+1}$ is isomorphic to $\mathbb{R} \oplus \ldots \oplus \mathbb{R} \simeq \mathbb{R}^{n+1}$, we can proceed as we did in Example 3.4 to obtain that

$$X^{\#}(z_1, \ldots, z_{n+1}) = i(z_1, \ldots, z_{n+1}),$$

which is equivalent to

$$\begin{aligned} X^{\#} &= \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} + \ldots + \frac{\partial}{\partial \theta_{n+1}} \\ &= i \sum_{k=1}^{n+1} \left(z_k \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial \overline{z}_k} \right). \end{aligned} \tag{3.2.3}$$

Now, let us consider

$$\begin{aligned} \mu : \mathbb{C}^{n+1} &\to \mathbb{R} \\ z &\mapsto -\frac{\|z\|^2}{2} + \text{ ct}, \end{aligned}$$

since

$$d\mu = -\frac{1}{2} d(\sum_{i=1}^{n+1} r_i^2)$$

we have that

$$\begin{aligned} \left(X^{\#} \lrcorner\, \omega\right)(v) &= \left(\sum_{i=1}^{n+1} r_i dr_i \wedge d\theta_i \right) (X^{\#}, v) \\ &= \left(- \sum_{i=1}^{n+1} r_i dr_i \right) (v) \\ &= \left(-\frac{1}{2} \sum d(r_i)^2 \right) (v) = d\mu(v). \end{aligned}$$

Then, the action Ψ is Hamiltonian with moment map μ.

Besides, by taking $\frac{1}{2}$ as the constant,

$$\begin{aligned} \mu^{-1}(0) &= \left\{ z \in \mathbb{C}^{n+1} / -\frac{\|z\|^2}{2} + \frac{1}{2} = 0 \right\} \\ &= S^{2n+1}, \end{aligned}$$

where

$$\begin{aligned} \mu : \mathbb{C}^{n+1} &\to \mathbb{R} \\ z &\mapsto -\frac{\|z\|^2}{2} + \frac{1}{2}. \end{aligned}$$

Consequently, $\mu^{-1}(0)/S^1 = S^{2n+1}/S^1 = \mathbb{CP}^n$, that is, $\mathbb{CP}^n$ is the symplectic reduction of $\mathbb{C}^{n+1}$.

3.3 Contact moment maps

An alternative way to define a cooriented contact structure for a manifold is stated in terms of the annihilator of a certain distribution of T^*B which is going to be useful in understanding the way G acts in T^*B.

Definition 3.8. A codimension-1 distribution ζ on a manifold B is **co-orientable** if its annihilator $\zeta^\circ \subset T^*B$ is an oriented line bundle, that is, has a nowhere vanishing global section. It is co-oriented if one component ζ°_+ of $\zeta^\circ \setminus 0$ is chosen.

Let $D \subset TB$ be a distribution of codimension 1. We define $D^\circ \subset T^*B$ as

$$D^\circ = \{\beta \text{ 1-form } \mid \beta(X) = 0 \text{ for every } X \in D\}$$

Then

$$D^\circ = \{0\} \cup \{f\eta \mid f > 0\} \cup \{f\eta \mid f < 0\}.$$

Definition 3.9. A **co-oriented contact structure** D on a manifold M is a co-oriented codimension-1 distribution such that $D^\circ \setminus 0$ is a symplectic submanifold of the cotangent bundle T^*B (the cotangent bundle is given the canonical sympletic form). We denote the chosen component of $D^\circ \setminus 0$ by D°_+ and refer to it as the symplectization of (B, D).

Definition 3.10. If a Lie group G acts on a manifold B preserving a 1-form η, the corresponding η-**moment map** $\Psi_\eta : B \to \mathfrak{g}^*$ determined by η is defined by

$$\langle \Psi_\eta(x), X \rangle = \eta_x \left(\underline{X}_x \right)$$

for all $x \in B$ and all vectors X in the Lie algebra $\mathfrak{g}$ of G, where, as above, $\underline{X}$ denotes the vector field induced by X : $\underline{X}_x = \frac{d}{dt}|_{t=0} \left(\exp tX \right) \cdot x$

If $d\eta$ is a symplectic form then, up to a sign convention, Ψ_η is a symplectic moment map. If η is a contact form then Ψ_η is a candidate for a contact moment map. Note however that if f is a G - invariant function, then $e^f\eta$ is also a contact form defining the same contact distribution, while clearly $\Psi_{e^f\eta} = e^f\Psi_\eta$. That is, this definition of the moment map depends on a particular choice of a contact form and not just on the contact structure.
Indeed, if η is a contact 1-form , with $\Psi_\eta : M \to \mathfrak{g}^*$ as its η - moment map and if $f \in C^\infty(B)$ is G - invariant, then $\ker(e^f\eta) = \ker\eta$.
Let us call $e^f\eta = \hat{\eta}$. Thus $\ker(\eta) = \ker(\hat{\eta}) = D$.
Then

$$\begin{aligned} \langle \Psi_\eta(x), X \rangle &= \eta \left(\underline{X}_x \right) \\ \langle \Psi_{\hat{\eta}}(x), X \rangle &= \hat{\eta} \left(\underline{X}_x \right). \end{aligned}$$

If we assume that $\omega = d\eta$, in D:

$$\begin{aligned} d\hat{\eta} = d(e^f\eta) &= d(e^f)\wedge\eta + e^f d\eta \\ &= e^f d\eta \end{aligned}$$

In particular, by the bilinearity of $\langle,\rangle$, we obtain that

$$\Psi_{e^f\eta} = e^f\Psi_\eta. \tag{3.3.1}$$

Remark 3.11. From this last equation it is clear that the moment map depends upon the 1-form η (to be more precise, it depends on the conformal class of the contact form) and not on the contact structure. In [11], Lerman proposes the definition of a "universal" moment map which depends on the contact structure and not only on the contact form. This generalisation of the contact map will be explained in the next subsection. Nevertheless, the restricted notion of a contact moment map given in Definition 3.10 will suffice to exhibit examples of contact reduction in Chapter 3.

3.3.1 Construction of a universal moment map

If we suppose again that a Lie group G acts on a manifold B preserving a co-oriented contact structure D, that is, we have the action $\phi_g : B \to B$ where $(\phi_g)_*|_D(D) = D$.
We have seen in Example 3.5 that there is an action of G on T^*B

$$(q,p) \mapsto \left(\phi_g(q), \phi^*_{g^{-1}}p\right),$$

for every $q \in B$ and $p \in T^*B$.
This action preserves D° and D°_+.
In fact, for every $v \in TM$,

$$p(v) = p(v^H + v^V) = p(v^H) + p(v^V) = p(v^V).$$

Thus

$$\begin{aligned}
\phi_g^*(p)(v) &= p\left(\phi_{g_*}v\right) \\
&= p\left(\phi_{g_*}\left(v^H+v^V\right)\right) \\
&= p\left(\phi_{g_*}(v^H)+\phi_{g_*}(v^V)\right) \\
&= p\left(\hat{v}^H\right)+p\phi_{g_*}(v^V) \\
&= p\left(\phi_{g_*}(v^V)\right)
\end{aligned}$$

Therefore, the action preserves D°. On the other hand, for every $p \in D_+^\circ$ we have that $\phi_g^*(p)(X) = p\left(\left(\phi_g\right)_*(X)\right) > 0$ because the G action preserves the cooriented structure and $p(X^v) > 0$ where X^v is the vertical vector of X.
The restriction $\Psi = \Phi|_{D_+^\circ}$ of the moment map Φ for the action of G on T^*B to D_+° depends only on the action of the group and on the contact structure. Moreover, since $\Phi : T^*B \to \mathfrak{g}^*$ is given by the formula (cf. (3.1.8)),

$$\langle \Phi(q,p), X\rangle = \left\langle p, \underline{X}_q\right\rangle$$

for all $q \in B$, $p \in T_q^*B$ and $X \in \mathfrak{g}$, we see that if η is any invariant contact form with ker$\eta = D$ and $\eta(B) \subset D_B^\circ$ then

$$\langle \eta^*\Psi\left(q\right), X\rangle = \langle \eta^*\Phi(q), X\rangle = \left\langle \eta_q, \underline{X}_q\right\rangle = \langle \Psi_\eta(q), X\rangle$$

where $\eta^*\Phi(q) := (\Phi\circ\eta)(q) = \Phi(q,\eta_q)$. Thus $\Psi\circ\eta = \Psi_\eta$, that is, $\Psi = \Phi|_{D_+^\circ}$ can be considered a universal moment map.

$$\begin{tikzcd}
B \arrow[r, "\eta"] \arrow[dr, "\Psi_\eta"'] & D_+^\circ \arrow[d, "\Psi"] \\
 & \mathfrak{g}^*
\end{tikzcd}$$

3.4 The contact reduction theorem

We follow Geiges in [8], and study how we can construct other manifolds if we choose a Lie group G acting in a contact manifold B, such that this group gives us some kind of symmetry, more explicitly, the group of automorphisms $Con(B, \eta)$.

First of all, let us take a look at the case where such group is S^1.

Proposition 3.12. *Let (B, η) be a contact manifold with a strict contact S^1 - action, generated by the flow of a vector field $\underline{X}$ in B. Then $\underline{X}$ is tangent to the level sets of the moment map Ψ_η. The value 0 is a regular value of Ψ_η if and only if $\underline{X}$ is nowhere zero on the level set $\Psi_\eta^{-1}(0)$. Hence, in this case the S^1 action on B restricts to a locally free action on $\Psi_\eta^{-1}(0)$. If this restricted action is free, η induces a contact form on the quotient manifold $\Psi_\eta^{-1}(0)/S^1$.*

Proof. We compute

$$d\Psi_\eta = d(\eta(\underline{X})) = \mathcal{L}_{\underline{X}}\eta - \underline{X} \lrcorner\, d\eta = -\underline{X} \lrcorner\, d\eta. \tag{3.4.1}$$

Thus, $d\Psi_\eta(\underline{X}) \equiv 0$, which proves the first statement.

We see that, by definition, $p \in \Psi_\eta^{-1}(0)$ if and only if $\underline{X}_p \in \ker\eta_p$. Hence, along the 0–level of Ψ_η, the fact that $\eta \wedge (d\eta)^n \neq 0$ and (3.4.1) gives us that 0 is a regular value of Ψ_η if and only if $\underline{X}$ is nowhere zero in the level set $\Psi_\eta^{-1}(0)$.

Now assume that 0 is indeed a regular value of Ψ_η. The conditions $\mathcal{L}_{\underline{X}}\eta \equiv 0$ and $\eta(X) \equiv 0$ along $\Psi_\eta^{-1}(0)$ imply that η descends to a well-defined 1–form on the quotient manifold $\Psi_\eta^{-1}(0)/S^1$.

The restriction of the 2–form $d\eta$ to $T_p(\Psi_\eta^{-1}(0)) \cap \ker\eta_p$ has 1–dimensional kernel, indeed, $\ker\eta_p$ is 1- dimensional for every $p \in \Psi_\eta^{-1}(0)$ because $\underline{X}_p \neq 0$ lies in this kernel, and if $\ker d\eta_p$ had dimension more than 1, it would imply that $\eta \wedge (d\eta)^n$ will be zero in some point in $\Psi_\eta^{-1}(0)$.

(For example, if $n=1$, then

$$\begin{aligned}(\eta_p \wedge d\eta_p)_{(\underline{X}_p, Y_p, Z_p)} &= \eta_p(\underline{X}_p) d\eta_p(Y_p, Z_p) - \eta_p(Y_p) d\eta(\underline{X}_p, Z_p) + \eta_p(Z_p) d\eta(\underline{X}_p, Y_p) \\ &= 0\end{aligned}$$

for every Y_p, Z_p in $T_p(\Psi_\eta^{-1}(0)) \cap \ker \eta_p$ with $d\eta(\underline{X}_p, Z_p)$ and $d\eta(\underline{X}_p, Y_p)$ both zeros if we assume that $\ker d\eta_p$ has dimension more than 1.)
When we pass to the quotient $\Psi_\eta^{-1}(0)/S^1$, the 1–form induced by η is given by restricting η to hyperplanes in $T_p(\Psi_\eta^{-1}(0))$ complementary to $\underline{X}_p$. Similarly, the differential of the induced 1–form is given by restricting $d\eta$ to such hyperplanes. It follows, as claimed, that η induces a contact form on $\Psi_\eta^{-1}(0)/S^1$. □

Lemma 3.13. *The moment map Ψ_η is equivariant with respect to the given G-action on B and the coadjoint action of G on $\mathfrak{g}^*$, that is,*

$$\Psi_\eta(g \cdot m) = g(\Psi_\eta(m)) \text{ for all } g \in G, m \in B$$

Proof.

$$\begin{aligned}\underline{X}_{g\cdot m} &= \frac{d}{dt}\left(\exp(tX)\, g \cdot m\right)|_{t=0} \\ &= \frac{d}{dt}\left(\left(gg^{-1}\exp(tX)\, g\right) \cdot m\right)|_{t=0} \\ &= g_{*m}\left(\frac{d}{dt}\left(g^{-1}\exp(tX)\, g\right) \cdot m|_{t=0}\right) \\ &= g_{*m}\left(\underline{\mathrm{Ad}_{g^{-1}}(X)}_m\right).\end{aligned}$$

Thus, for every $X \in \mathfrak{g}$,

$$\begin{aligned}
\langle \Psi_\eta(g \cdot m), X \rangle &= \eta_{g\cdot m}(\underline{X}_{g\cdot m}) \\
&= \eta_{g\cdot m}\left(g_{*_m}\left(\underline{\mathrm{Ad}_{g^{-1}}(X)}_m\right)\right) \\
&= (g^*\eta)_m\left(\underline{\mathrm{Ad}_{g^{-1}}(X)}_m\right) \\
&= \eta_m\left(\underline{\mathrm{Ad}_{g^{-1}}(X)}_m\right) \\
&= \langle \Psi_\eta(m), \mathrm{Ad}_{g^{-1}}(X) \rangle \\
&= \langle g\left(\Psi_\eta(m)\right), X \rangle .
\end{aligned}$$

□

Lemma 3.14. *(a) For all $p \in B$, $v \in T_pB$, and $X \in \mathfrak{g}$, we have*

$$\langle d_p\Psi_\eta(v), X \rangle = d\eta(v, \underline{X}_p);$$

here we identify $T_{\Psi_\eta(p)}\mathfrak{g}^*$ with $\mathfrak{g}^*$.

(b) The flow of the Reeb vector field ξ preserves the level sets of Ψ_η.

(c) If $\Psi_\eta(p) = 0$, then $T_p(G \cdot p)$, the tangent space to the orbit through p, is an isotropic subspace of the symplectic vector space $(\mathrm{ker}\eta_p, d\eta_p)$.

(d) If 0 is a regular value of Ψ_η, then the isotropic subspace in (c) is of the same dimension as G, and it equals the symplectic orthogonal complement of $\mathrm{ker}\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right)$.

Proof. (a) As we have that $\mathcal{L}_{\underline{X}}\eta \equiv 0$, the Cartan's formula yields to

$$d(\underline{X}_p \lrcorner\, \eta) + \underline{X}_p \lrcorner\, d\eta = 0 \tag{3.4.2}$$

Let $v \in T_pB$, thus

$$d\left(\eta\left(\underline{X}_p\right)\right)(v) + d\eta(\underline{X}_p, v) = 0. \tag{3.4.3}$$

Let us define $\langle d_p\Psi_\eta(v), X\rangle := d\langle \Psi_\eta, X\rangle(v)$ for every $v \in T_pB$. Since we identify $T_{\Psi_\eta(p)}\mathfrak{g}^*$ with $\mathfrak{g}^*$, this definition makes sense.
By definition of the moment map Ψ_η,

$$\langle d_p\Psi_\eta(v), X\rangle = d\langle \Psi_\eta, X\rangle(v) = d\big(\eta(\underline{X}_p)\big)(v).$$

Hence, by (3.4.3), we obtain

$$\langle d_p\Psi_\eta(v), X\rangle = d\eta(v, \underline{X}_p)$$

for every $p \in B, v \in T_pB$, and $X \in \mathfrak{g}$.

(b) From Lemma 2.28 and item (a),

$$\langle d_p\Psi_\eta(\xi), X\rangle = d\eta(\xi, \underline{X}_p) = 0 \text{ for all } X \in \mathfrak{g}, \tag{3.4.4}$$

Thus, $d_p\Psi_\eta(\xi) = 0$ which means that the flow of the Reeb vector field ξ preserves the level sets of Ψ_η.

(c) The tangent space $T_p(G \cdot p)$ is spanned by vectors of the form $\underline{X}_p$ with X lying in $\mathfrak{g}$ by the isomorphism between $\mathfrak{g}$ and T_eG . In particular, it is a subspace of kerη_p, since $\Psi_\eta(p) = 0$ then $\eta_p(\underline{X}_p) = \langle\Psi_\eta(p), X\rangle = 0$. If we take $v = \underline{Y}_p$ for some $Y \in \mathfrak{g}$ and $\underline{X}_p$ both in $T_p(G.p)$ we obtain from (3.4.3) that

$$\begin{aligned} d\eta_p(\underline{Y}_p, \underline{X}_p) &= d(\eta(\underline{Y}_p))(\underline{X}_p) \\ &= 0, \end{aligned}$$

which means that $T_p(G.p)$ is an isotropic subspace of (ker$\eta, d\eta_p$).

(d) In order to prove that $\dim T_p(G \cdot p) = \dim G$, we need to show that $\underline{X}_p \neq 0$ for any non-zero $X \in \mathfrak{g}$. Given such an X, the fact that 0 is a regular value of Ψ_η allows us to choose a tangent vector $v \in T_pB$ such that $\langle d_p\Psi_\eta(v), X\rangle \neq 0$ because $d_p\Psi_\eta$ is surjective for every $p \in \Psi_\eta^{-1}(0)$. Then $\underline{X}_p \neq 0$ follows from (a) for every $p \in \Psi_\eta^{-1}(0)$.

We have that 0 is a regular value of Ψ_η, and the intersection of the hyperplane $\ker\eta_p$ with $T_p\left(\Psi_\eta^{-1}(0)\right)$ is transverse in the sense that it is a manifold for points in $\Psi_\eta^{-1}(0)$ such that the Reeb vector field ξ belongs to $T_p\left(\Psi_\eta^{-1}(0)\right)$, as ξ is not in $\ker\eta_p$ and by (b) the flow of ξ preserves the level sets of Ψ_η so in particular it preserves the zero level set of Ψ_η. Consequently, by the item (c), we obtain that

$$T_p\left(G\cdot p\right) \text{ and } \ker\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right)$$

are of complementary dimension in $\ker\eta_p$.

From (a), for every $v \in \ker\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right)$, $p \in \Psi_\eta^{-1}(0)$ and $\underline{X}_p \in T_p(G\cdot p)$ (we can use a linear combination of fundamental vector fields but we will obtain the same result), it follows that

$$\begin{aligned} d\eta\left(v, \underline{X}_p\right) &= \langle d_p\Psi_\eta(v), X\rangle \\ &= 0. \end{aligned}$$

Thus,

$$\left(T_p\left(G\cdot p\right)\right)^\perp \supset \ker\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right).$$

Hence, as $\left(T_p\left(G\cdot p\right)\right)^\perp$ and $\ker\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right)$ have the same dimension by the the results obtained above, this inclusion must be an equality.

□

Theorem 3.15. *(Contact reduction) Let G be a compact Lie group acting by strict contact transformations on the contact manifold (B,η). If $0 \in \mathfrak{g}^*$ is a regular value of the moment map Ψ_η of this action, then G acts locally freely on the level set $\Psi_\eta^{-1}(0)$. If the action is free, η induces a contact form on the quotient manifold $\Psi_\eta^{-1}(0)/G$.*

Proof. First of all, we are going to show that G acts locally freely on the level set $\Psi_\eta^{-1}(0)$. Indeed, let us call $G_p = \{g \in G \mid g.p = p\}$ the isotropy group of p, $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid \underline{X}_p = 0\}$ its correspondent Lie algebra, and $\operatorname{Ann}\mathfrak{g}_p = \{T \in \mathfrak{g}^* \mid \langle T, X\rangle = 0, \forall X \in \mathfrak{g}_p\}$ the annihilator of $\mathfrak{g}_p$ for every $p \in B$.

Since $0 \in \mathfrak{g}^*$ is a regular value for Ψ_η, it follows that

$$\operatorname{Im} d_p\Psi_\eta = \mathfrak{g}^*.$$

for every $p \in \Psi_\eta^{-1}(0)$. On the other hand, $\operatorname{Im} d_p\Psi_\eta \subset \operatorname{Ann}\mathfrak{g}_p$ since for every $T \in \operatorname{Im} d_p\Psi_\eta$, that is, $T = d_p\Psi_\eta(v)$ for some $v \in T_pB$,

$$\begin{aligned}\langle T, X\rangle &= \langle d_p\Psi_\eta(v), X\rangle\\ &= d\eta(v, \underline{X}_p)\\ &= 0,\end{aligned}$$

for every $X \in \mathfrak{g}_p$ (the second equality is obtained from item a) of Lemma 3.14).

Thus $\mathfrak{g}^* = \operatorname{Ann}\mathfrak{g}_p$ and this implies that $\mathfrak{g}_p = 0$ and we obtain that $\dim G_p = 0$ for every $p \in \Psi_\eta^{-1}(0)$ which means that G acts locally freely on the level set $\Psi_\eta^{-1}(0)$, moreover those isotropy groups are finite since we are assuming that G is compact.

Let us show that η induces a contact action on the quotient if the action is free. In fact, if we assume that the action of G on B is free, it follows from the compactness of G that $\Psi_\eta^{-1}(0)/G$ is a manifold. Since we have that $g^*\eta = \eta$ for every $g \in G$ (where $g^*\eta$ represents the pullback of the action map of G on B over η), we have $\mathcal{L}_{\underline{X}}\eta \equiv 0$ and from item (c) of the Lemma 3.14, $\eta(\underline{X}) \equiv 0$ along $\Psi_\eta^{-1}(0)$, where $\underline{X}$ is generated by the flow of the action of G over B . Therefore η descends to a well-defined 1–form on the quotient manifold $\Psi_\eta^{-1}(0)/G$.

The restriction of the 2–form $d\eta_p$ to ${T_p(\Psi_\eta^{-1}(0)) \cap \ker\eta_p}/{T_pG \cdot p}$ allows us to obtain $\eta_p \wedge (d\eta_p)^n \neq 0$, since $T_pG \cdot p$ is an isotropic subspace of $\ker\eta_p$ and we found that $(T_p(G \cdot p))^\perp = \ker\eta_p \cap T_p\left(\Psi_\eta^{-1}(0)\right)$, so the only obstruction for $\eta_p \wedge (d\eta_p)^n$ to be nowhere zero is in $T_p(G \cdot p)$ and this is the reason why $d\eta_p$ is nonzero in $T_p(\Psi_\eta^{-1}(0)) \cap \ker\eta$. Therefore, the induced 1- form by η is a contact form for the quotient $\Psi_\eta^{-1}(0)/G$. □

In fact, the examples exhibiting contact reduction that will be presented in the next section, are *contact toric manifolds*, these are manifolds with a large group of automorphisms which allows the manifold to admit very symmetric groups acting on them in an appropriate fashion. We have the following definition.

Definition 3.16. An action of a torus G on a contact manifold (B, D) is **completely integrable** if it is effective, preserves the contact structure D and if $2\dim G = \dim B + 1$. A **contact toric G-manifold** is a co-oriented contact manifold (B, D) with a completely integrable action of a torus G.

Remark 3.17. Lemma 3.14 reveals an important difference between the contact and the symplectic case: in the proof of d), one notices that another possible regular value besides zero can not ensure that $T_p(G \cdot p)$ is a subspace of $\ker\eta_p$. So contact reduction, stated as Theorem 3.15, only works for zero as a regular value. There is a variation of this notion, given by Willet in [20] where it is possible to contactify quotients for certain non-zero regular values.

3.5 Examples of contact toric reduction

As follows, we will compute some examples of contact manifolds obtained by the reduction process, some of them proposed in [9] but not developed in detail, and this is the purpose of this section.

Example 3.18. Let

$$S^7 = \{z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4; |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\},$$

with $z_j = x_j + iy_j$, then the contact form on S^7 can be written as

$$\eta = \sum_{j=0}^{3} (x_j dy_j - y_j dx_j)\,,$$

and its Reeb vector field is

$$\xi = \sum_{j=0}^{3} (x_j \partial y_j - y_j \partial x_j)\,.$$

Let S^1 act on S^7 by

$$\begin{aligned} \phi : S^1 \times S^7 &\to S^7 \\ (e^{it}, (z_0, z_1, z_2, z_3)) &\mapsto \left(e^{-it} z_0, e^{-it} z_1, e^{it} z_2, e^{it} z_3\right). \end{aligned} \tag{3.5.1}$$

The associated fundamental vector field of this action is (in real coordinates),

$$\begin{aligned} X_0 \;=\; & -\left(x_0 \frac{\partial}{\partial y_0} - y_0 \frac{\partial}{\partial x_0}\right) - \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1}\right) + \left(x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}\right) \\ & + \left(x_3 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial x_3}\right). \end{aligned}$$

We can proceed by the same way as we did in Example 3.4 and we will obtain that the moment map $\mu : S^7 \to \mathbb{R}$ is then stated (up to a factor $-\frac{1}{2}$) as

$$\mu(z) = \eta_z(X_0) = -|z_0|^2 - |z_1|^2 + |z_2|^2 + |z_3|^2,$$

with zero level set

$$\mu^{-1}(0) = \{z \in S^7; |z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^2\}$$

Now, since $z \in S^7$, every element of $\mu^{-1}(0)$ satisfies that:

$$|z_0|^2 + |z_1|^2 + |z_0|^2 + |z_1|^2 = 1$$

and

$$|z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2 = 1.$$

Thus,

$$\mu^{-1}(0) = S^3\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right). \tag{3.5.2}$$

Clearly, 0 is a regular value for μ. Thus, the reduced space can be identified with $(S^3 \times S^3)/S^1$ which, by the contact reduction theorem 3.15, is a contact manifold.

Let us identify more explicitly the manifold $(S^3 \times S^3)/S^1$. In order to do this, let us consider the following diffeomorphism

$$\begin{aligned} F : S^3 \times S^3 &\to S^3 \times S^3 \\ (z_0, z_1, z_2, z_3) &\mapsto (z_0 z_3 + \overline{z_1 z_2}, z_0 z_2 - \overline{z_1 z_3}, z_2, z_3) \end{aligned}$$

and the following S^1 action

$$\begin{aligned} \psi : S^1 \times S^7 &\to S^7 \\ (e^{it}, (z_0, z_1, z_2, z_3)) &\mapsto (z_0, z_1, e^{it} z_2, e^{it} z_3). \end{aligned}$$

In one hand we have that, for every $(z_0, z_1, z_2, z_3) \in S^3 \times S^3$:

$$\begin{aligned} F \circ \phi(z_0, z_1, z_2, z_3) &= F\left(e^{-it} z_0, e^{-it} z_1, e^{it} z_2, e^{it} z_3\right) \\ &= \left(z_0 z_3 + \overline{z_1 z_2}, z_0 z_2 - \overline{z_1 z_3}, e^{it} z_2, e^{it} z_3\right) \end{aligned} \tag{3.5.3}$$

Besides,

$$\begin{aligned}\psi\circ F(z_0,z_1,z_2,z_3) &= \psi\left(z_0z_3+\overline{z_1z_2}, z_0z_2-\overline{z_1z_3}, z_2, z_3\right)\\ &= \left(z_0z_3+\overline{z_1z_2}, z_0z_2-\overline{z_1z_3}, e^{it}z_2, e^{it}z_3\right), \qquad (3.5.4)\end{aligned}$$

That is, F is an equivariant diffeomorphism under the S^1-actions ϕ and ψ. Thus

$$\left(S^3\times S^3\right)/S^1 \cong S^3\times\left(S^3/S^1\right)\cong S^3\times S^2.$$

If we set $G=\mathbb{T}^4$, $B=S^7$ and consider the action in (3.5.1), we observe that S^7 becomes a contact toric manifold.

Example 3.19. Let us consider the weighted action of S^1 on $S^{2n-1}\subset\mathbb{C}^n$ by

$$\left(e^{it},(z_0,\dots,z_{n-1})\right)\mapsto\left(e^{\lambda_0 it}z_0,\dots,e^{\lambda_{n-1}it}z_{n-1}\right), \qquad (3.5.5)$$

where $(\lambda_0,\dots,\lambda_{n-1})\in\mathbb{Z}^n$. Additionaly, let us recall that S^{2n-1} has a standard contact structure given by

$$\eta=\sum_{i=0}^{n-1}(x_idy_i-y_idx_i),\ \xi=\sum_{i=0}^{n-1}(x_i\partial_{y_i}-y_i\partial_{x_i}). \qquad (3.5.6)$$

The associated moment map,

$$\mu(z)=\lambda_0|z_0|^2+\cdots+\lambda_{n-1}|z_{n-1}|^2,$$

which has zero as a regular value for any $(\lambda_0,\dots,\lambda_{n-1})$ such that $\lambda_0\cdots\lambda_{n-1}\neq 0, \gcd(\lambda_0,\dots,\lambda_{n-1})=1$ and at least two λ's have different signs.
Now, let us take $\lambda_0=\cdots=\lambda_k=a$ and $\lambda_{k+1}=\cdots=\lambda_{n-1}=-b$, $a,b\in\mathbb{Z}^+$ relative prime. Then , by the same procedure made in the previous example,

$$\mu^{-1}(0)\cong S^{2k+1}\left(\sqrt{a/a+b}\right)\times S^{2(n-k)-1}\left(\sqrt{b/a+b}\right),$$

and the reduced space is

$$\mu^{-1}(0)/S^1 = S^{2k+1}\left(\sqrt{a/a+b}\right) \times S^{2(n-k)-1}\left(\sqrt{b/a+b}\right)/S^1, \qquad (3.5.7)$$

where the S^1-action is

$$(e^{it}, (x, y)) \mapsto (e^{iat}x, e^{-ibt}y) \qquad (3.5.8)$$

for every $x \in S^{2k+1}\left(\sqrt{a/a+b}\right)$ and $y \in S^{2(n-k)-1}\left(\sqrt{b/a+b}\right)$.
It is worth noting that S^{2n-1} is a contact toric manifold with a natural extension of the action settled in (3.5.5) to $\mathbb{T}^n$.
Moreover, the maximal torus $\mathbb{T}^n$ is generated by the vector fields $H_i = x_i\partial_{y_i} - y_i\partial_{x_i}$, for $i = 0, \ldots, n-1$ and we observe that the Reeb vector field ξ in (3.5.6) belongs to the subspace generated by the vectors H_i.

One would like to generalise the Example 3.18, at least if we consider some convenient weights in the associated S^1-action. This leads us to a remarkable result of M. Y. Wang and W. Ziller in [19], where they use topological arguments to obtain relevant properties of certain type of manifolds which apart from being contact manifolds, admit Riemannian metrics with quite interesting properties, for instance these manifolds admit Einstein metrics (a manifold has Einstein metric if its Ricci curvature is proportional to its metric, cf. [2]). These manifolds will be explained briefly in the following example.

Example 3.20. The Wang-Ziller manifold $M^{p;q}_{k,l}$ given in [19] is defined as the total space of the S^1-bundle over $\mathbb{CP}^p \times \mathbb{CP}^q$ whose Euler class is $k\alpha_1 + l\alpha_2$ where α_1 and α_2 are the positive generators of $H^2(\mathbb{CP}^p)$ and $H^2(\mathbb{CP}^q)$, respectively and k and l are integers.
In the 5-dimensional case, Wang and Ziller obtained that, for $p = q = 1$, the

manifolds $M_{k,l}^{1,1}$ are diffeomorphic to $S^3 \times S^2$. To achive this, they used arguments involving calculations of espectral sequences and a famous theorem of Smale on the clasification of 5-dimensional manifolds in [16] . They show that these manifolds are spin and simply connected and $H^2(M_{k,l}^{1,1}, \mathbb{Z}) = \mathbb{Z}$, hence applying Smale theorem, they concluded that all these manifolds are diffeomorphic to $S^3 \times S^2$. For a detailed argument cf. [19] or Appendix in [7]. We note that the in example 3.18 , the manifold we studied is a Wang-Ziller manifold with weights $k = -1$ and $l = 1$.

Example 3.21. Let us consider the following action on S^7

$$e^{it} \mapsto \left(e^{-kit} z_0, e^{it} z_1, e^{it} z_2, e^{it} z_3\right) \tag{3.5.9}$$

with k a positive integer. Thus, its corresponding moment map will be

$$\mu(z) = -k|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2,$$

and, by proceeding as we did to obtain 3.5.2, we will have that:

$$\mu^{-1}(0) = S^1(\sqrt{k/(k+1)}) \times S^5(\sqrt{1/(k+1)}).$$

Now, if we consider the following k-fold covering map p:

$$p : S^1 \times S^5 \to S^1 \times S^5$$
$$(z_0, z_1, z_2, z_3) \mapsto \left((z_0)^{-k}, z_1, z_2, z_3\right)$$

we will obtain that the following diagram commutes

$$\begin{array}{ccc} S^1 \times S^5 & \xrightarrow{\quad p \quad} & S^1 \times S^5 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ (S^1 \times S^5)/S^1 & \xrightarrow{\hat{p}} & \mu^{-1}(0)/S^1 \end{array}$$

where π_1 is the quotient map respect to the diagonal S^1-action on $S^1 \times S^5$ and π_2 corresponds to the action which has been defined in (3.5.9).
Besides, the following diffeomorphism

$$\begin{aligned} G : S^1 \times S^5 &\to S^1 \times S^5 \\ (z_0, z_1, z_2, z_3) &\mapsto (z_0, \overline{z_0} z_1, \overline{z_0} z_2, \overline{z_0} z_3) \end{aligned}$$

and the S^1-actions

$$\begin{aligned} &\phi_1 : (z_0, z_1, z_2, z_3) \mapsto (e^{it} z_0, e^{it} z_1, e^{it} z_2, e^{it} z_3) \text{ (the diagonal } S^1\text{-action)} , \\ &\phi_2 : (z_0, z_1, z_2, z_3) \mapsto (e^{it} z_0, z_1, z_2, z_3) \end{aligned}$$

satisfy that:

$$\begin{aligned} G \circ \phi_1(z_0, z_1, z_2, z_3) &= G(e^{it} z_0, e^{it} z_1, e^{it} z_2, e^{it} z_3) \\ &= (e^{it} z_0, \overline{z_0} z_1, \overline{z_0} z_2, \overline{z_0} z_3) \\ \phi_2 \circ G(z_0, z_1, z_2, z_3) &= \phi_2(z_0, \overline{z_0} z_1, \overline{z_0} z_2, \overline{z_0} z_3) \\ &= (e^{it} z_0, \overline{z_0} z_1, \overline{z_0} z_2, \overline{z_0} z_3). \end{aligned}$$

Therefore, G is an equivariant diffeomorphism respect to the actions ϕ_1 and ϕ_2, so we get that $(S^1 \times S^5)/S^1 \cong S^1/S^1 \times S^5 \cong S^5$. Consequently, since the diagram above commutes, we obtain that our reduced space is

$$\mu^{-1}(0)/S^1 \cong \left((S^1 \times S^5)/S^1\right)/\mathbb{Z}_k \cong S^5/\mathbb{Z}_k. \tag{3.5.10}$$

It is important to notice that what we have found is not a manifold, but an orbifold, roughly speaking, a topological space which is locally the Euclidean space quotiened by a finite group, (cf. [17]).

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