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Symmetry Breaking in Grand Unified Theories

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Abstract

In this work we review the symmetry breaking mechanism of gauge theories. On the first chapters of this thesis, we review the concept of symmetry as the action of a group that leaves an object invariant, in particular Lagrangians and actions, and then develop the corresponding globally and gauge symmetric theories and the relationship between them. It also reviewed the concept and general framework of the spontaneous breaking of a symmetry for renormalizable potentials. Correspondingly, two main results for global symmetries, Noether's theorem and Goldstone's Theorem, are reviewed in a general setting.

Chapter 3 is the most important part of this work. The Brout-Englert-Higgs mechanism is explained and used to retrieve the symmetry breaking patterns for the vector and all the second rank tensor irreducible representations of the $O(n)$ and $SU(n)$ groups. In general we will retrieve the vacuum expectation value (vev) for the particular representation and value of the parameters of the potential. Then, for this vev, we calculate the number of massive vector bosons of the theory. Following BEH mechanism and Goldstone's theorem, this number is equal to the number of broken generators delining thus the particular symmetry breaking pattern.

Chapter 4 is a review of the Standard Model with an aim towards Grand Unified Theories (GUTs). Lastly in Chapter 5 we review the group theory of the minimal model $SU(5)$ in a very exhaustive way and use the results of Chapter 3 to see the breaking patterns for this particular GUT.

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Chapter 1

Symmetries

In the following sections we will review classical field theory and introduce the importance of symmetries in Physics .

1.1 Symmetries in Physics

An object has a symmetry if it is invariant under the action of a group. Let X be an object such as a Hamiltonian or Lagrangian and G a group, we say that G is a symmetry of X if for any $g \in G$ the action of this element A_g leaves X invariant:

$$A_g(X) = X \qquad \forall g \in G \tag{1.1}$$

The way in which the group acts will have a deep connection to the way in which the group is represented. Usually the objects in which we are interested are Hamiltonians or Lagrangians but can also be other objects such as matrix elements.

We shall now overline a first example of symmetry[2] and see the consistency of the definition (1.1). Consider a system with a finite dimensional Hilbert space where pure states can be defined. From the standard axioms of Non relativistic Quantum Mechanics a symmetry transformation $g \in G$,with G the set of all transformation of the system, is defined as a transformation that is a one to one map between ray vectors [1] into ray vectors and conserve the transition probabilities. Thus, if ϕ, ψ are ray vectors; then:

$$|\langle\phi|\psi\rangle| = \left|\langle\phi_g|\psi_g\rangle\right| \tag{1.2}$$

where ϕ_g, ψ_g are the transformed ray vectors. Then, following Wigner's theorem, the only type of transformations that satisfy the last equation are linear unitary transformations:

$$\langle\phi_g|\psi_g\rangle = \langle\phi|\psi\rangle \tag{1.3}$$

[1] a pure state is a state where we have the maximum amount of quantum information available. A ray vector represents this state and is defined as the equivalence class $\{e^{i\alpha}\psi\}$ with ψ a ket in the Hilbert space

or antilinear antiunitary transformations [3]:

$$\langle \phi_g | \psi_g \rangle = \langle \phi | \psi \rangle^* \tag{1.4}$$

Thus we have:

$$\psi_g = U(g)\psi \tag{1.5}$$

where $U(g)$ is a continuous operator. The set of transformations G forms a group (group of transformations under compositions) but the $U(g)$ do not necessarily form the representation of the group but just a projective representation. For example, given $U(g_1), U(g_2)$ we have:

$$U(g_1)U(g_2) = W(g_1; g_2)U(g_1 g_2) \tag{1.6}$$

where $W(g_1; g_2)$ is a phase term that is not necessarily equal to one.

Since it is the Hamiltonian H that defines the dynamics of the system, a group $G' \subset G$ of transformation that conserves the dynamics of the system, or more succintly a dynamical symmetry of the system, is one such that:

$$\left| \langle \phi_g | H | \psi_g \rangle \right| = \left| \langle \phi | H | \psi \rangle \right| \qquad \forall g \in G' \tag{1.7}$$

then from Eq. (1.5) we have:

$$U(g) H U^\dagger(g) = H \qquad \forall g \in G' \tag{1.8}$$

with $U(g)$ linear unitary or antilinear antiunitary. But the term at the left is the transformation of H under g, so we have $H_g = H$. Then we see that a dynamical symmetry of the system is a symmetry of the Hamiltonian and it has the form Eq.(1.1) as we initially hypothesized.

It is well know that the symmetry group of H labels the states of the system. In fact from Eq.(1.2) if the group of transformations is a Lie Group, one can expand to first order to see that the generators of the Group also commute with H. Then, depending on the rank of the Algebra[2], one can construct an equal number of Casimir operators [4] that labels the states.

1.2 Field Theory and Continuous Global Symmetries

In this section we introduce a general framework for the field theories we will use in later chapters and some generalities about representations of the Lorentz and Poincaré group. We will denote with the symbol Φ a generic field and shall later differentiate the type of fields.

The importance of using a field theoretical point of view instead of the wave point of view can be seen mainly from the fact that in this framework the possibility of creating new particles is natural and paradoxes such as the Klein Paradox [5] are solved. In addition in this work we will focus only in Classical fields that is fields before quantization and not operators.

A first restriction in the possible fields theories is that since we will be dealing with fields in High Energy Physics (HEP), where gravitational effects are vanishingly small, the domain

[2]The rank of a Lie Algebra is the number of elements of the the maximal abelian subalgebra.

of the fields $\Phi(x)$ belonging to a certain theory will be the Minkowski Space $\mathcal{M}$. The metric we use is the usual one in particle physics:

$$\eta_{\mu\nu} = (1, -1, -1, -1). \tag{1.9}$$

As we have seen in Section (1.1), symmetries can be a guiding force in Physics. In particular in HEP they will be useful in model building to construct multiplets of particles and derive their dynamics (sec. 1.4). It can be shown that for each continuous symmetry of a system there is a corresponding Lie Group G that generates it. Suppose we have a set of fields: $\Phi_1, \cdots, \Phi_N$, then the most general transformation of this set of fields under a Lie group G of dimension d_G is as follows. The coordinate transformation is:

$$x^\mu \stackrel{g(\alpha)}{\rightarrow} x'^\mu = f^\mu(x; \alpha). \tag{1.10}$$

Here α is the set of d_G parameters that define an element of the Lie Group G [3]. This last equation induces a transformation on the fields:

$$\Phi_i(x) \stackrel{g(\alpha^{-1})}{\rightarrow} \Phi'_i(x') = G_i(\Phi(x), x; \alpha) \qquad i = 1, \cdots, N. \tag{1.11}$$

We are usually interested in infinitesimal transformations that act on fields and coordinates. From Eq.(1.10) and Eq.(1.11), the most general Lie transformation around the identity is:

$$x^\mu \rightarrow x'^\mu = x^\mu + \alpha_a \Gamma^\mu_a(x) \equiv x^\mu + \delta x^\mu \tag{1.12}$$

$$\Phi(x) \rightarrow \Phi'_i(x') = \Phi_i(x) + \alpha_a F_{i,a}(x, \Phi(x)) \equiv \Phi_i(x) + \bar{\delta}\Phi_i(x). \tag{1.13}$$

Here a runs on the parameters (coordinates of the Lie Group) of the transformation, i on the fields and α_a is now infinitesimal. From the last equations we implicitly have defined two different kinds of "variations".
The first one is the "form variation" and is defined for an infinitesimal transformation by:

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) \tag{1.14}$$

The second one is the "total variation" and is defined by:

$$\bar{\delta}\Phi(x) = \Phi'(x') - \Phi(x). \tag{1.15}$$

The relationship between them is:

$$\bar{\delta}\Phi(x) = \delta\Phi(x) + \delta x^\mu \partial_\mu \Phi(x). \tag{1.16}$$

[3]For example for $SO(3)$, the rotation group, we can assign α as the angles θ_1,θ_2,θ_3

Classically all the interesting physics can be derived from the action functional:

$$S[\Phi] := \int_V \mathcal{L}(\Phi_i, \partial_\mu\Phi_i, \partial_\mu\partial_\nu\Phi_i, \cdots, x)d^4x \qquad i = 1\cdots N. \tag{1.17}$$

For this particular Lagrangian we are not restricting the order of derivatives of the fields and V is a arbitrary volume of spacetime. Then, a Lie group transformation is a global symmetry of the Lagrangian $\mathcal{L}$ if:

$$\mathcal{L}(\Phi(x), \partial_\mu(x), \partial_\mu\partial_\nu\Phi_i, \cdots) = \mathcal{L}(\Phi'(x'), \partial_\mu\Phi'(x'), \partial_\mu\partial_\nu\Phi_i'(x'), \cdots) \tag{1.18}$$

Since the symmetry is continuous it will be enough to verify that the Lagrangian is invariant under G for infinitesimal transformations .

1.2.1 Restrictions on Field Theories

After having delineated basic concepts on symmetries and transformations we now proceed to state some restrictions on the classical Action Eq.(1.17) and Lagrangians that are needed to construct workable theories in HEP . As we will see, many of them are geometrically inspired.

(a) Following the principle of Special Relativity, the Action has to be invariant under the proper orthochronous Poincaré Group $\mathcal{P}_+^\uparrow$ i.e. : $S[\Phi'(x')] = S[\Phi(x)]$ for a Poincaré transformation $x' = \Lambda x + a$ where Λ denotes a proper ortochronous Lorentz transformation $\mathcal{L}_+^\uparrow$[4]. This implies that the Lagrangian, the integrand and the domain of integration all have to be $\mathcal{P}_+^\uparrow$ covariantly.

For the Lagrangian this means that the Lagrangian itself must have the scalar representation

$$\mathcal{L} \to \mathcal{L}'(\Phi'(x'), \cdots, x') = \mathcal{L}(\Phi(x), \cdots, x), \tag{1.19}$$

so all the objects that compose it have to be Lorentz covariant i.e. transform under representations of $\mathcal{P}_+^\uparrow$ combined such that the result is a Lorentz scalar. Since we have $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the Lie Algebra of $\mathcal{L}_+^\uparrow$ can have two types of finite dimensional representation. The single valued representation that generates the Lie group $\mathcal{L}_+^\uparrow$, tensorial representations, and the double valued representations generated by $SL(2,\mathbb{C})$, spinorial representations. We can label all the finite irreps,[5] tensorial or spinorial, using the labels (j_L, j_R). For example for the left and right Weyl spinors we have $(1/2, 0)$ and $(0, 1/2)$ respectively. For the vectors $(1/2, 1/2)$ and for the Dirac spinor $(1/2, 0) \oplus (0, 1/2)$.

The integrand after a Poincaré transformation is:

$$d^4x' = |\det\Lambda| d^4x = d^4x \tag{1.20}$$

[4] $\mathcal{L}_+^\uparrow := \{\Lambda \in O(1,3) | \det\Lambda = 1, \Lambda_0^0 \geq 1\}$ with Λ is the fundamental representation of $O(1,3)$. $\mathcal{P}_+^\uparrow$ is the semidirect product of $\mathcal{L}_+^\uparrow$ and translations in four dimensions.

[5] irreps: irreducible representations

so it is clearly Poincaré invariant.

The covariance of the domain of integration can be solved imposing the use of volumes with spacelike boundaries [6].

(b) Fields vanish at the boundaries of spacetime: $\lim_{-|\vec{x}|\to\infty} \Phi(ct,\vec{x}) = 0$

(c) The action is a real functional of the fields and their derivatives. It can be shown that if this is not the case the S matrix unitary thus, after quantization, total probability is not a conserved quantity [6].

(d) Lagrangians are restricted such that they produce at most second order derivatives. This implies that at most, the Lagrangian is composed of second order derivatives for scalar fields and first order derivatives for spinor fields. A particular hindrance with order more than three in the derivatives is that usually these equations have non causal solutions, for example for a Lagrangian that generates the Abraham-Lorentz force these implies solutions where signals from the future affects the present [6].

(e) The interactions of the fields have to be local i.e. depend only on one set of coordinates. For example terms like $A_\mu(x)j^\mu(x)$ are accepted but $g_N \int d^4y\Phi(x)[(x-y)^2]^N\psi(y)$ are not. This will ensure that the dynamics of a field will be influenced by the other fields only in one point and not depend on all spacetime.

(f) The Lagrangian has to be renormalizable [7]. This means that the combination of fields and operators have to have mass dimension of order of max. 4 since only terms with coefficients of positive or zero order mass are renormalizable.

Following these restrictions the action is:

$$S[\Phi(x)] = \int_{\Sigma_1}^{\Sigma_2} \mathcal{L}(\Phi_i, \partial_\mu\Phi_i, x)d^4x \qquad i = 1,\cdots,N, \tag{1.21}$$

The Σ_i hyperspaces are space-like [7].

Using the conditions as in Eq.(1.21) we have that the fields $\Phi_i(x)$ with $\delta\Phi_i(x) = 0$ for all $x \in \Sigma_1, \Sigma_2$ satisfy Hamilton's principle i.e. $\delta S[\Phi] = 0$ if and only if they satisfy the N Euler-Lagrange equations:

$$\left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} - \frac{\partial\mathcal{L}}{\partial\phi_i}\right) = 0 \tag{1.22}$$

1.3 Noether's Theorem

Noether [8] derived two theorems and their inverses. The first one is the most commonly used. We, at first, define the symmetry of the system as continuous transformations that leave

[6] a spacelike manifold is one such that it's normal vector n^μ on each point satisfies $n^\mu n_\mu > 0$

[7] in practice what this means is that there exists a Poincaré transformation that transform into a surface with constant time

$\bar{\delta}S = 0.$

$$0 = \bar{\delta}S \equiv \int_{R'} \mathcal{L}(\Phi'(x'), \partial_\mu \Phi'(x'), x') dx'^4 - \int_R \mathcal{L}(\Phi(x), \partial_\mu \Phi(x), x) d^4x \tag{1.23}$$

Doing a change of variables in the integral:

$$\int_{R'} (\cdots) d^4x' \to \int_R (\cdots) \left(1 + \partial_\mu(\delta x^\mu)\right) d^4x \tag{1.24}$$

we obtain:

$$\bar{\delta}S = \int_R \mathcal{L}(\Phi'(x'), \partial_\mu \Phi'(x'), x') \left(1 + \partial_\mu(\delta x^\mu)\right) d^4x - \int_R L(\Phi(x), \partial_\mu \Phi(x), x) d^4x \tag{1.25}$$

Using the definition of $\bar{\delta}L$, from Eq.(1.15):

$$\bar{\delta}S = \int_R \left(\mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \bar{\delta}\mathcal{L}\right) \left(1 + \partial_\mu(\delta x^\mu)\right) d^4x - \int_R \mathcal{L} d^4x \tag{1.26}$$

Expanding to first order we have:

$$\bar{\delta}S = \int_R d^4x \left(\bar{\delta}\mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \mathcal{L}\partial_\mu \delta x^\mu + O(\delta x^v)^2\right) = 0 \tag{1.27}$$

From Eq.(1.16), since $\delta\mathcal{L} = \bar{\delta}\mathcal{L} - \delta x^\mu \partial_\mu \mathcal{L}$ we arrive at:

$$\begin{aligned} \bar{\delta}S &= \int_R d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta(\partial_\mu \Phi_i) + \partial_\mu \mathcal{L}\delta x^\mu + \mathcal{L}\partial_\mu \delta x^\mu\right) \\ &= \int_R d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\right)\delta\Phi_i + \int_R d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta\phi_i + \mathcal{L}\delta x^\mu\right) \end{aligned} \tag{1.28}$$

we arrive at Noether´s relation:

$$\sum_{i=1}^{N} \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} - \frac{\partial \mathcal{L}}{\partial \Phi_i}\right) \delta\Phi_i = \sum_{i=1}^{N} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\delta\Phi_i + \mathcal{L}\delta x^\mu\right) \tag{1.29}$$

To state Noether's first theorem we restrict to transformations of coordinates, fields and Lagrangians that additionally satisfy the following points:

- A general group transformation changes also the form of the Lagrangian that is:

$$\mathcal{L}(\Phi'(x'), \partial_\mu \Phi'(x'), x') \to \tilde{\mathcal{L}}(\Phi'(x'), \partial_\mu \Phi'(x'), x') \tag{1.30}$$

  Thus the most variation is then of the form:

$$\Delta\mathcal{L} := \tilde{\mathcal{L}}(\Phi'(x'), \partial_\mu \Phi'(x'), x') - \mathcal{L}(\Phi(x), \partial_\mu \Phi(x), x) \tag{1.31}$$

Instead, we restrict to Lagrangians such that :

$$\tilde{\mathcal{L}}(\Phi'(x'), \partial_\mu \Phi'(x'), x') = \mathcal{L}(\Phi'(x'), \partial_\mu \Phi'(x'), x') \tag{1.32}$$

So $\bar{\delta}\mathcal{L}$ is the most general variation.

- Similarly, we restrict to group transformations that leave : $\tilde{S}(\Phi'(x'), x') = S(\Phi'(x'), x')$ Later we will weaken this assumption. Note that all common global group transformations (Classical Lie groups and Poincaré) satisfy the last two conditions.
- Just to restate, fields vanish at the boundaries of spacetime: $\lim_{-|\vec{x}| \to \infty} \Phi(ct, \vec{x}) = 0$

From Eq.(1.12) and Eq.(1.13), the infinitesimal transformations for a Lie Group G of dimension d_G, with α the infinitesimal parameter independent of coordinates, satisfyng the above conditions are:

$$\delta x^\mu = \Gamma^\mu_a(x)\alpha_a \tag{1.33}$$

$$\bar{\delta}\Phi_i(x) = F_{ia}(x, \Phi)\alpha_a \tag{1.34}$$

Then, from Eq.(1.29), doing a variation around α and moving $\delta\alpha$ out of the divergence, we have:

$$\sum_{a=1}^{d_G}\sum_{i=1}^{N}\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} - \frac{\partial \mathcal{L}}{\partial \Phi_i}\right)\frac{\partial \delta \Phi_i}{\partial \alpha_a}\delta\alpha_a = \sum_{a=1}^{d_G}\sum_{i=1}^{N}\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial_\mu \Phi_i}\left[\frac{\partial(\delta\Phi_i)}{\partial \alpha_a}\right] + \mathcal{L}\frac{\partial(\delta x^\mu)}{\partial \alpha_a}\right)\delta\alpha_a \tag{1.35}$$

Defining:

$$j^\mu_a := -\sum_{i=1}^{N}\left(\frac{\partial \mathcal{L}}{\partial_\mu \Phi_i}\left[\frac{\partial(\delta\Phi_i)}{\partial \alpha_a}\right] + \mathcal{L}\frac{\partial(\delta x^\mu)}{\partial \alpha_a}\right) \tag{1.36}$$

and

$$\frac{\delta \mathcal{L}}{\delta \Phi_i} := \frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \tag{1.37}$$

Since the variations of α are not null and linearly independent, Eq.(1.35) implies the following d_G equations:

$$\sum_{i=1}^{N}\frac{\delta \mathcal{L}}{\delta \Phi_i}\frac{\partial \delta \Phi_i}{\partial \alpha_a} = \partial_\mu j^\mu_a \qquad a = 1, \cdots, d_G. \tag{1.38}$$

Now assuming the N L.E. equations are valid (N on shell conditions) we retrieve d_G conserved currents:

$$\partial_\mu j^\mu_a = 0 \qquad a = 1, \cdots, d_G. \tag{1.39}$$

and this is Noether's first theorem. Summarizing, let $\mathcal{L}$ be a Lagrangian such that $\bar{\delta}S = 0$ for a Lie transformation generated by a Lie Group G of dimension d_G. Suppose also that the fields satisfy the N Lagrange Euler equations. Then we obtain d_G conserved currents as shown in Eq.(1.39).

A more compact way to express Eq. (1.39) is using the total variation definition Eq.(1.34), Eq.(1.33). We have:

$$j_a^\mu = \Theta_\nu^\mu \Gamma^\nu_a(x) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} F_{ia}(x,\Phi) \tag{1.40}$$

with $\Theta_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \partial_\nu \Phi_i - \delta_\nu^\mu \mathcal{L}$

In the case of internal symmetries generated by a compact Lie Group we have:

$$\delta x^\mu = \Gamma_a^\mu(x)\alpha_a = 0 \rightarrow \Gamma_a^\nu(x) = 0 \tag{1.41}$$

$$\bar{\delta}\Phi_i(x) = \delta\Phi_i(x) = F_{ia}(\Phi)\alpha_a \tag{1.42}$$

A particular case is very important. Let $U(\alpha) = e^{iT_a\alpha_a}$ with $a = 1, \cdots, d_G$, a unitary N dimensional representation of a compact Lie group G in a vector space generated by N fields $\Phi_1(x), \cdots, \Phi_N(x)$. T_a are hermitian representations of the Lie Algebra. It is generally reducible. We have the transformation:

$$\Phi_i(x) \rightarrow \Phi_i'(x') = \Phi_i'(x) = [e^{iT_a\alpha_a}]_{ij}\Phi_j(x) \tag{1.43}$$

Infinitesimally:

$$\delta\Phi_i = i[T_a]_{ij}\Phi_j\alpha_a \tag{1.44}$$

So the currents using Eq.(1.40) are:

$$j_a^\mu = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}[T_a]_{ij}\Phi_j \tag{1.45}$$

The global symmetry of the Lagrangian implies that each field has to live in a particular irreducible representation. In particle physics this is translated as saying that fields are multiplets of the symmetry group [8]. In this case $[T_a]_{ij}$ is reducible in the N-dimensional representation but will be an irreducible representation for some subset of fields. An example of the use of Eq. (1.45) will be done in the Standard Model section.

Noether's theorem can be expanded such that the condition of a symmetry is one that transforms the action as:

$$\bar{\delta}S[\Phi] = \int dx^4 \partial_\mu K^\mu \tag{1.46}$$

Following the steps leading to Eq.(1.35) we have:

$$\sum_{i=1}^N \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} - \frac{\partial \mathcal{L}}{\partial \Phi_i} \right) \delta\Phi_i = \sum_{i=1}^N \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta\Phi_i + \mathcal{L}\delta x^\mu + K^\mu \right) \tag{1.47}$$

thus we have a more general current:

$$J_a^\mu = j_a^\mu + \frac{\partial K^\mu}{\partial \alpha_a} \tag{1.48}$$

[8]Though not always under global symmetry. In fact the dynamics comes using gauge symmetries

1.4 Abelian Gauge Transformations

The main result of last section is Noether's First theorem: a Lagrangian with a global continuous symmetry implies d_G continuity laws, Eq.(1.48), one for each dimension of the symmetry group. A natural question then is to ask what happens if instead of global transformations we use transformations that depend on spacetime.

An important consequence is that the initial Lagrangian, globally symmetric under a Classical Lie Group G, will be not invariant under this new local transformation. Thus in order to have a symmetric Lagrangian we need to modify the initial Lagrangian introducing a new vector field, the Yang Mills field, and a new kind of derivative, the covariant derivative.

Consider the Dirac Lagrangian with a spinor field ψ with mass m.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1.49}$$

A variation for ψ gives:

$$(-i\partial\!\!\!/ - m)\bar{\psi} = 0 \tag{1.50}$$

and for ψ

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{1.51}$$

So we can treat them as two independent fields ψ and $\bar{\psi}$. Eq.(1.49) is global invariant under $U(1)$, as we will see below.
Defining:

$$\psi \rightarrow \psi' = e^{i\alpha}\psi \equiv U(\alpha)\psi \tag{1.52}$$

as the transformation for ψ we see, using the induced representation on $\bar{\psi}$, that Eq.(1.49) is $U(1)$ global invariant.
The infinitesimal transformation for ψ is :

$$\delta\psi = i\alpha\psi \tag{1.53}$$

and the induced transformation in $\bar{\psi}$ is:

$$\delta\bar{\psi} = -i\alpha\bar{\psi} \tag{1.54}$$

Now assume the parameter depends on spacetime $\alpha(x)$, what will happen? In this case the transformation of $\partial_\mu\psi$ is:

$$\psi \rightarrow \partial_\mu(e^{i\alpha(x)}\psi) = e^{i\alpha(x)}\partial_\mu\psi + iq\partial_\mu\alpha(x)e^{i\alpha(x)}\psi \tag{1.55}$$

thus the Lagrangian becomes:

$$\mathcal{L} \rightarrow \mathcal{L}' = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \bar{\psi}\partial_\mu\alpha(x)\psi \tag{1.56}$$

So $\mathcal{L}$ is clearly not invariant under this transformation. In order to cancel this extra term we need to modify the derivative

$$\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu - igA_\mu \tag{1.57}$$

such that the Lagrangian (1.49) with the replacement (1.57) is invariant. A_μ is a vector field that has to transform such that:

$$\mathcal{D}_\mu\psi \to \mathcal{D}'_\mu\psi' = e^{i\alpha(x)}\mathcal{D}_\mu\psi \tag{1.58}$$

in order to cancel with the transformation of $\bar{\psi}$. Doing both simultaneous transformation on ψ and A_μ is defined as doing a gauge transformation. Solving last equation, A_μ has to transform as :

$$A'_\mu = U(\alpha)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^{-1}(\alpha) = A_\mu + \frac{1}{g}\partial_\mu\alpha(x) \tag{1.59}$$

The modified Lagrangian is then:

$$\mathcal{L}_M = \bar{\psi}(i\mathcal{D}_\mu - m)\psi = \bar{\psi}(i\partial_\mu - m)\psi + g\bar{\psi}\gamma^\mu\psi A_\mu \tag{1.60}$$

For the complete Lagrangian we have to include the kinetic term of the gauge vector. From Classical Electrodynamics it has to be a function $f(F_{\mu\nu})$ of the Electromagnetic tensor:

$$F_{\mu\nu} = \frac{1}{g}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{1.61}$$

This object is invariant under the transformation (1.59):

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = U(\alpha)F_{\mu\nu}U^{-1}(\alpha) = U(\alpha)U^{-1}(\alpha)F_{\mu\nu} = F_{\mu\nu}. \tag{1.62}$$

Where we have used the Abelian nature of the symmetry to commute the elements.

The kinetic term has to be Lorentz invariant so all indices have to be contracted. For an abelian symmetry there is only one form for $f(F_{\mu\nu})$, thus the final Lagrangian is:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\mathcal{D}_\mu\gamma^\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{1.63}$$

This is the classical QED Lagrangian[9] which is the most accurate theory in all physics. It leads to calculations that agree with experiment to more that 10 significant digits.

Summarizing, in this section we started with a globally invariant Lagrangian $\mathcal{L}$ under $U(1)$, Eq.(1.49), and then, following the requirement of invariance under transformations with α dependent on spacetime, we modified $\mathcal{L}$ replacing the normal derivative with a covariant derivative thus adding A_μ as a physical field and the requirement that both ψ and A_μ transforms simultaneously. The final Lagrangian (1.63) is invariant under both global transformations and gauge transformations. One consequence was the acquisition of a physically meaningful current-gauge interaction term included in $\mathcal{L}_{\text{QED}}$.

From another point of view, we started with a free Lagrangian for both ψ and A_μ:

$$\mathcal{L}_{\text{free}} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{1.64}$$

[9]More correctly, is the classical$\mathcal{L}_{\text{QED}}$ only ψ is in the representation $q = -1$ that is when we are dealing with electrons

and then after the replacement $\partial_\mu \to \mathcal{D}_\mu$ ("gauging") we obtained $\mathcal{L}_{\text{QED}}$ that is the theory of electromagnic interactions. This procedure of obtaining a Lagrangian with the correct interaction term after "gauging" a free Lagrangian is called the gauge principle [9]. Salam, in fact, hypotized that the interaction terms of weak and strong interactions could be generated after using the gauge principle on a free Lagrangian with an appropiate symmetry and representation. History shows he was correct.

1.5 Non Abelian Gauge Transformations

In this section we will introduce the main framework of Non-Abelian gauge theories and derive, as with QED, the gauge-matter interaction after applying the gauge principle to a globally non Abelian symmetric Lagrangian.

Let G be a $d_G > 1$ dimensional compact semi-simple Lie group[10], N fermionic fields ψ and M scalar bosonic fields ϕ such that each of them live in irreducible representations of G (multiplets). Then the most general free Lagrangian with global symmetry G is:

$$\mathcal{L}_{\text{free}} = \sum_\psi \mathcal{L}_\psi + \sum_\phi \frac{1}{2}\|\partial_\mu \phi\|^2. \tag{1.65}$$

Where schematically, we have denoted the kinetic term for the bosonic representations as scalar products in the internal space and the kinetic term for each ψ as $\mathcal{L}_\psi$, that is constructed such that it is globally G invariant. For a ψ transforming in the fundamental (vectorial) representation we have:

$$\mathcal{L}_\psi = i\bar{\psi}\partial_\mu\psi = i\bar{\psi}_i\partial_\mu\psi_i, \qquad i = 1, \cdots, d_G. \tag{1.66}$$

With each ψ_i a component of the multiplet ψ. For ϕ transforming in the fundamental representation we have:

$$\|\partial_\mu\phi\|^2 = (\partial_\mu\phi)^\dagger\partial^\mu\phi = \partial_\mu\phi_i^*\partial^\mu\phi_i \qquad i = 1, \cdots, d_G \tag{1.67}$$

This will be generalized to k-rank irreducible tensors in Section (3.5). We shall now follow the procedure outlined on the previous section and demand invariance under local transformations. We focus on the kinetic term for a field that transforms in the fundamental representation of G, Φ, a bosonic or fermionic field. We need that after replacing the derivative $(\partial_\mu \to \mathcal{D}_\mu)$ the kinetic term remains invariant under local transformation of G, similar to Eq.(1.66) and Eq.(1.67) with the new covariant derivative. We impose that is has to transform such that:

$$\mathcal{D}_\mu\Phi \to \mathcal{D}'_\mu\Phi' = U(\alpha)\mathcal{D}_\mu\Phi \tag{1.68}$$

[10] A simple group is a group without invariant subalgebras ex. $SU(2)$. A semi-simple group is a group without an invariant abelian subalgebra example product groups $SU(2) \times SU(2)$.

where $U(\alpha)$ with the fundamental representation of G. Taking as ansatz:

$$\mathcal{D}_\mu = \partial_\mu - igA_\mu \tag{1.69}$$

we see that for A_μ to satisfy Eq.(1.68) it has to transform as:

$$A'_\mu = U(\alpha)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^{-1}(\alpha) \tag{1.70}$$

It is interesting to note this transformation has the same form for the Abelian group $U(1)$, Eq.(1.59). We will also see that the components of the field A_μ tranform in a representation independent way. In this case A_μ will have more elements since it is constructed using the Lie Algebra of G that has more than one element. Thus it has an internal space indiced by a in addition to the spacetime indices. These fields transform under the adjoint representation of G. All in all, to construct the covariant derivative in analogy to QED in order to leave the leave the kinetic of the Lagrangian invariant we need the addition of a field of this type, the Yang-Mills field. The first exposition of this groundbreaking idea was done in the seminal paper [10].

To show what is stated above we do as follows. For group transformations near the identity we have:

$$U(\alpha) = 1 + i\alpha^a T^a \tag{1.71}$$

$$U(\alpha)^{-1} = 1 - i\alpha^a T^a \tag{1.72}$$

where T^a is a hermitian in the fundamental representation of the Lie algebra. Then we have using Eq.(1.70) to first order on the parameters:

$$A'_\mu = A_\mu + i[\alpha^a T^a, A_\mu] + \frac{1}{g}\partial_\mu \alpha^a T^a \tag{1.73}$$

For this equation to make sense, seeing the last term, A_μ has to live in the space generated by the T^a's since it is written as a linear combination of T_a's with arbitrary parameters. So:

$$A_\mu = A^a_\mu T^a \tag{1.74}$$

This is the Yang-Mills field expressed as combinations of generator of the Lie Algebra of G. Since gauge invariance implies global invariance the equation is valid also when each α_a changes in the same way over all the space, thus effectively having the last term null in Eq.(1.73). We see after expanding the second term that A_μ transforms as:

$$A'^a_\mu T^a = A^a_\mu T^a + i[\alpha^a T^a, A^b_\mu T^b] \tag{1.75}$$

Then using complexified structure equation of the Lie Algebra of G, that is representation independent, to calculate the Lie bracket:

$$[T^a, T^b] = i f^c_{ab} T^c \tag{1.76}$$

we have:

$$A'^c_\mu T^c = A^c_\mu T^c - f^c_{ab}\alpha^a A^b_\mu T^c \tag{1.77}$$

Then in direction T^c we have:

$$A^{c\,\prime}_\mu = A^c_\mu - f^c_{ab}\alpha^a A^b_\mu \tag{1.78}$$

But remembering that in the adjoint representation $D(t)$ the matrices $d_G \times d_G$ representing an element t^a of the Lie Algebra have the following form:

$$D(t_a)^c_b = f^c_{ab} \tag{1.79}$$

Thus we see that, for each internal coordinate c of the Yang-Mills field:

$$A^{c\,\prime}_\mu = A^c_\mu - D(t_a)^c_b \alpha^a A^b_\mu \tag{1.80}$$

And this is an infinitesimal transformation in the adjoint representation. Thus A_μ is a field with values in the Lie Algebra of G (spanned by t^a[11]) whose "internal coordinates" transforms in the adjoint representation of G . The explicit form of A_μ is modified according to the representation of the field to which it couples (see Eq.(3.382)).

For an arbitrary gauge transformation we then have, from Eq.(1.73)

$$A^{c\,\prime}_\mu = A^c_\mu - f^c_{ab}\alpha^a A^b_\mu + \frac{1}{g}\partial_\mu \alpha^c \tag{1.81}$$

We have shown the covariant derivative for a field in the fundamental representation. For an arbitrary representation is:

$$\mathcal{D}_\mu \Phi = \partial_\mu \Phi + igA_{\mu a}\Gamma(t^a)\Phi = \partial_\mu \Phi + ig\Gamma(A_\mu)\Phi \tag{1.82}$$

where $\Gamma(t^a)$ will give the Law of transformation in the specific representation of Φ.

To write a complete Lagrangian we need the kinetic term for this new field A_μ. The analogous of the electromagnetic field tensor, the field strenght of A_μ is defined as a vector in the internal space of the Lie Algebra of G :

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{g}[D_\mu, D_\nu] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)t^a + igA^a_\mu A^b_\nu [t^a, t^b] \\ &= (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu - g f^c_{ab} A^a_\mu A^b_\nu)t^c \end{aligned} \tag{1.83}$$

To construct the generalization of the kinetic energy for the Yang Mill's field we need to take its "norm squared". Since we have the scalar product in the abstract Lie algebra [12]:

$$\mathrm{Tr}[t^a, t^b] = \langle t^a, t^b \rangle = \delta^{ab} \quad a, b = 1, \cdots d_G \tag{1.84}$$

[11] the t^a are vectors of the Lie Algebra of G in the abstract space of the Lie algebra i.e. not representation dependent

[12] can change the normalization.

we see that the kinetic term for the Yang Mills field is:

$$\mathcal{L} = -\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle = -\frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} \tag{1.85}$$

Equivalently the kinetic term can be defined under any representation of G. Using $A_\mu = A^a_\mu \Gamma(t^a)$ and the fact that:

$$\mathrm{Tr}[\Gamma(t^a), \Gamma(t^b)] = \delta_{ab} I(\Gamma) \tag{1.86}$$

Where $I(\Gamma)$ is a constant that depends on the representation.[13] We have:

$$-\frac{1}{4}\frac{\mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}]}{I(\Gamma)} = -\frac{1}{4}\frac{F^{a\mu\nu}F^{b}_{\mu\nu}}{I(\Gamma)}\,\mathrm{Tr}[\Gamma(t^a)\Gamma(t^b)] = -\frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} \tag{1.87}$$

For example in the fundamental representation of $SU(N)$ we have $I(\Gamma) = 2$ and this gives the usual factor $-\frac{1}{8}$.

In addition of using $F^{\mu\nu a}$ we could use it's dual

$$F'^{\ a}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma} \tag{1.88}$$

to construct a kinetic term. Then, it is easy to see that $F'^{\ a}_{\mu\nu}F'^{\mu\nu a}$ is proportional to to the kinetic term defined above. On the other hand, imposing only gauge invariance on the Lagrangian (i.e. not imposing C,P, CP symmetries) we can add the following term to the Lagrangian:

$$\mathcal{L}_\Theta = \frac{1}{2}\Theta F'^{\ a}_{\mu\nu}F^{\mu\nu a} \tag{1.89}$$

This term is actually a total derivative so it does not affect the equations of motion neither Feynman rules, although it is important for non-perturbative effects only if the gauge group is non Abelian.

[13] Dynkin index

Chapter 2

Spontaneous Breaking of Global Symmetries

Section (2.1) is based on the Nobel Lectures and papers as shown in [11]. The rest of the Chapter is based mainly on [12] and [13].

2.1 The need for symmetry breaking

In Part (1.4) we already have seen an example of a gauge theory with the QED Lagrangian, where ψ are charged leptons and A_μ is the Electromagnetic field. At current date this is not the only interaction in Nature since there are two other experimentally verified interactions: Weak and Strong interactions. The next step in order to synthesize theories is to construct Lagrangians for other interactions that also include the electromagnetic one.
Using the field theoretical framework then, a first step into unification is extending a low energy Lagrangian into another one that includes this initial Lagrangian as a low energy limit. This is seen with the QED Lagrangian and the well known electroweak sector of the Standard Model.

In order to schematize this procedure, there are some properties we need to impose in the new Lagrangian that we can extract from the QED Lagrangian $\mathcal{L}_{QED}$. Recalling:

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\mathcal{D}_\mu\gamma^\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{2.1}$$

First, we can see a gauge symmetry given by the Lie Group $U(1)$ and an interaction term between the fermionic and bosonic fields (or gauge), dependent only on one coupling g, derived from the covariant derivative, that is equivalent to the one retrieved using the gauge principle on the free Lagrangian for ψ and A_μ. To not spoil $U(1)$ gauge symmetry the vector field A_μ has no mass term $\frac{m^2}{2}A_\mu A^\mu$. Experimentally a massless gauge vector is consistent with the the vector mediator of EM interactions, the photon, since these are long ranged. It is also renormalizable.

We want a gauge theory for weak interactions that is gauge invariant under a group G such that after applying the gauge principle to free global version we retrieve correct interaction terms, is renormalizable and gives adequate mass terms to the weak interaction mediators.

The first model for weak interactions was developed in analogy to QED. It was the phenomenologically four fermion interaction (Fermi's theory). It satisfied universality since all interactions depended on G_F (Fermi's constant), was chiral, but did not have any symmetry nor was renormalizable. Experimentally, the theory was shown to be inconsistent, for example, with scattering experiments of muon neutrinos with electrons since the scattering section diverged for high energies:

$$\sigma_{\text{tot}} = \frac{G_F^2}{\pi} \frac{(s - m_\mu^2)^2}{s} \tag{2.2}$$

The next model, was a theory with an intermediate massive vector boson (IMVB). The interaction terms of the Lagrangian were modified from a current-current one into a vector-current one. Problems continued, since it was not renormalizable, (order 6 of mass in interaction terms) and phenomenologically had divergences for other high energies processes such as $\nu\bar{\nu} \to W^+W^-$.

At the beginning of the 60s, the individually universal and vectorial nature of weak and electromagnetic interactions suggested a possible unification via a gauge theory. At this point, it was already known the existence of $W^\pm$ charged vectors bosons and the usual A electromagnetic gauge boson and it was suggested, from the decay of strange particles, that at least one other neutral vector was needed, thus restricting the possible gauge group to one of dimension at least four.

A, possibly first, model using the gauge theory framework [14] was done using $SU(2)_L$ as the chiral gauge group for weak interactions and leptons in the fundamental representation. Glashow proposed another model [15] where it was found the correct symmetry group $SU(2)_L \times U(1)$. In this model, leptons were expressed in the adjoint representation of $SU(2)_L$. Since the representation is real it only can be constructed as Majorana spinor thus the chiral spinor were hidden in it. Contrary to the SM, though, the masses for $W^\pm$ and Z were added explicitly, breaking gauge symmetry thus explaining the qualification of "partially symmetric" since only the Lagrangian minus these last terms was gauge invariant. The main consequence of this fact is that a mass inserted in this way gave a non renormalizable Lagrangian, as was later found. So, we see that the problems of mass, gauge symmetry and renormalizability are all connected.

Finally in the late 60's Salam [16] and Weinberg [17] independently built the original $SU(2)_L \times U(1)_Y$ electroweak sector of the Standard Model using leptons in the fundamental representation. Besides being compatible with old observables as charged currents and providing some new, as neutral currents, the Weinberg angle $\sin\theta_W$ and the GIM mechanism, a main feature was the use of a particular mechanism to give mass to the $W^\pm$ and Z bosons without explicitly breaking gauge symmetry. This is the Brout-Englert-Higgs (BEH) mechanism that is based on the spontaneous symmetry breaking (SSB) of a gauge theory. Coloquially speaking, the SSB phenomena happens when the ground state of a Lagrangian has a lower symmetry than the complete Lagrangian. Additionally, the model, was shown to be renormalizable some years later; T'Hooft showed the renormalizability of both massless [18] and massive[1] [19] of general Yang-Mills field theories of which the Salam and Weinberg model is a particular casel.

[1] with masses obtained after SSB

Also, the low energy limit Lagrangian included the QED Lagrangian and the IMVB models, thus delineating in some sense an electroweak unification.

In this chapter we discuss the preliminaries for the BEH mechanism: the conceptual foundations of spontaneous symmetry breaking and some applications in theories with global symmetries.

2.2 Simple mechanical model exhibiting a spontaneous symmetry breaking

As seen in Section (1.1) symmetries are everywhere in Nature. There is much information one can recover using symmetry principles but there is a also a lot to gain when one is not manifested. A particular example of this is the spontaneous breaking of a symmetry.

Since this is a framework independent concept, we can discuss it conceptually using the following simple system [20]. Consider a ring with center at O and radius r that rotates around an axis parallel to the plane of the ring with constant angular velocity. In addition, there is a material point located in a non fixed point M and mass m that can move along the ring without any friction. The angle θ is defined as the angle between the segment OM and the axis of rotation, as shown in the Fig.(2.1). Having fixed the axis of rotation, the azimuth angle is ϕ and the angular velocity is:

$$\frac{d\phi}{dt} = \Omega. \tag{2.3}$$

Fixing a frame that rotates with the ring the equilibrium position along the tangent to the ring

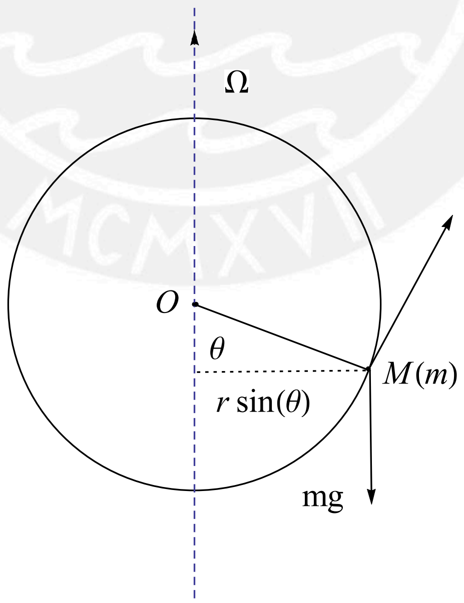


Figure 2.1 Simple system

is:

$$F(\theta) = -mg\sin\theta + mr\Omega^2\sin\theta\cos\theta = 0 \tag{2.4}$$

where $a\sin\theta$ is the distance from M to the rotation axis. The first term is the projection along the tangent of the ring of the weight mg and the second is the centrifugal force projected also along the tangent. Since $dV = -Fdx = Frd\theta$ where dx is an infinitesimal displacement along the circumference of the ring, integrating we have:

$$\begin{aligned} V(\theta) =& mgr\cos\theta + m\Omega^2\frac{r^2}{4}\cos 2\theta + k = mgr\cos\theta + m\Omega^2\frac{r^2}{4}(1-2\sin^2\theta) + k \\ =& -mgr\left(\frac{\Omega^2 r}{2g}\sin^2\theta + \cos\theta\right) + m\Omega^2\frac{r^2}{4} + k \end{aligned} \tag{2.5}$$

where k is the integrating constant. In the frame of reference fixed above we choose $V(\theta = 0) = 0$ so:

$$V(\theta) = mgr\left[1 - \cos\theta - \frac{1}{2}\frac{\Omega^2}{\Omega_c^2}\sin^2\theta\right] \tag{2.6}$$

where we have defined

$$\Omega_c^2 = \frac{g}{r}. \tag{2.7}$$

In this potential we see the parity ($\theta \to -\theta$) symmetry since all functions of θ are even.

Using $\frac{\partial V}{\partial\theta} = 0$ in order to minimize we have:

$$\left.\frac{\partial V}{\partial\theta}\right|_{\theta=\langle\theta\rangle} = mgr\sin\langle\theta\rangle\left(1 - \frac{\Omega^2}{\Omega_c^2}\cos\langle\theta\rangle\right) \tag{2.8}$$

From Eq.(2.8) we have the extrema $\langle\theta\rangle = 0, \pi$ and if $\Omega > \Omega_c$ also the set $\langle\theta\rangle = \{\pm\theta_0\}$, with $\theta_0 = \arccos\left((\frac{\Omega_c}{\Omega})^2\right)$. Using:

$$\frac{\partial^2 V}{\partial\theta^2} = mrg\left(\cos\theta - \frac{\Omega^2}{\Omega_c^2}\cos 2\theta\right) \tag{2.9}$$

we have that $\langle\theta\rangle = 0$ is a minimum for $\Omega < \Omega_c$ otherwise a local maximum, $\theta = \pi$ is always a maximum and that the set $\{\pm\theta_0\}$ are minima for $\Omega > \Omega_c$. This also can be also be see from the plots (2.2) and (2.3).

The value of the parameter Ω, the angular velocity, modifies the form of the potential and the number of minima of the potential. This fact can be interpreted as a phase transition, taking Ω as the control parameter and defining the order parameter as $\langle\varphi\rangle = \frac{\langle\theta\rangle}{\theta_0}$. We see that when $\Omega < \Omega_c$, we have a single minimum so $\langle\theta\rangle = 0$, and when $\Omega > \Omega_c$ we have two minima $\langle\theta\rangle = \pm\theta_0$, thus $\langle\varphi\rangle \neq 0$. In both case the potential has parity symmetry. Hidden, there is a particular reduction in the symmetry of the system when $\Omega > \Omega_c$. To see this, one can select $\langle\theta\rangle = +\theta_0$ as the chosen ground state after the transition. Then, one can express the variable θ around the ground state θ_0:

$$\theta \to \theta_0 + \chi \tag{2.10}$$

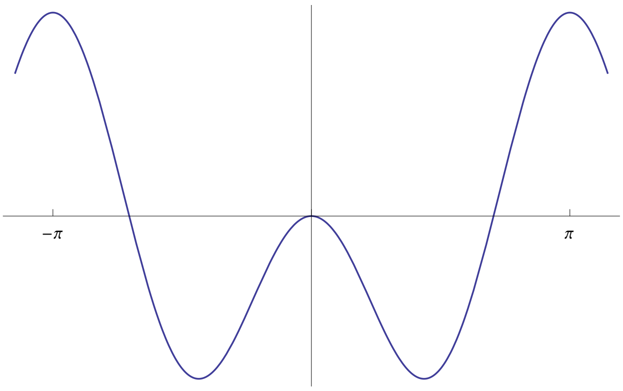


Figure 2.2 Potential $V(\theta)$ for $\Omega > \Omega_c$

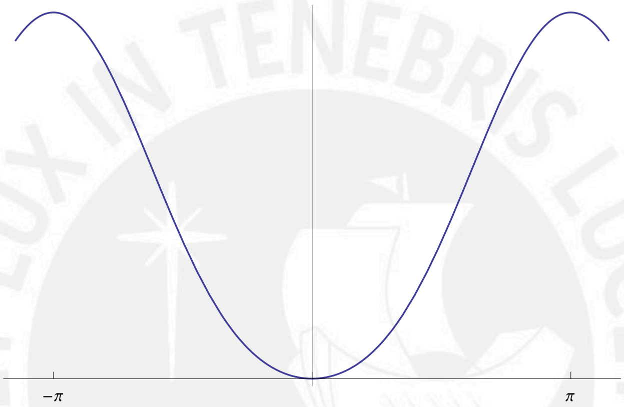


Figure 2.3 Potential $V(\theta)$ for $\Omega < \Omega_c$

we have that the new potential V_χ is not parity invariant under $\chi \to -\chi$. In fact :

$$\begin{aligned} V_\chi(\chi) =& mgr\left[1-\cos(\theta_0+\chi)-\frac{1}{2}\frac{\Omega^2}{\Omega_c^2}\sin^2(\theta_0+\chi)\right] \\ =& mgr\Bigg[1-\cos\theta_0\cos\chi-\frac{1}{2}\frac{\Omega^2}{\Omega_c^2}(\sin^2\theta_0\cos^2\chi+\cos^2\theta_0\sin^2\chi) \\ &+\left(\sin\theta_0-\frac{1}{2}\frac{\Omega^2}{\Omega_c^2}\sin 2\theta_0\cos\chi\right)\sin\chi\Bigg] \end{aligned} \tag{2.11}$$

This potential is odd so parity symmetry is broken for a perturbation around the ground state. Thus, after the transition we have a system that is symmetric except at the ground state. This is an example of a system whose symmetry has been spontaneously broken.

In general a phenomenon with spontaneous symmetry breaking will consist of the following elements: a parameter F called "control parameter" whose influence depends on a critical value F_{crit}, a symmetry of the system given by a group G that breaks i.e. the symmetry group is reduced, and an order parameter φ, an observable of the system whose value will indicate the existence or not of the spontaneous breaking phenomena. In the example above we had

$\Omega = F$, $F_{\text{crit}} = \Omega_c, \varphi = \varphi$ and the symmety group was $\mathbb{Z}_2$.

Initially consider the system with $F < F_{\text{crit}}$. We have that the configuration of the system, represented by the action in Field theories, is invariant under the action of the group G. Also we have that the order parameter has a null value: $\langle\varphi\rangle = 0$ (one minima in the example above). Then after $F > F_{\text{crit}}$ the order parameter is not zero $\langle\varphi\rangle \neq 0$, and the symmetry group of the system is broken, usually leading to a smaller symmetry group, having thus a phase transition.

2.3 Spontaneous symmetry breaking in an Abelian model

In High Energy Physics we are interested mostly in continuous symmetries. In this case the results of spontaneously broken symmetries will vary if the symmetry is global or local. Following the discussion of Section (2.2), the translation of the main elements of spontaneously broken symmetries into the Lagrangian formalism of classical field theory is done as follows: the control parameter F will be the "mass squared" of the field [2], usually a self interacting scalar field ϕ , the symmetry will be a symmetry of the Lagrangian given by a Lie Group G and the order parameter will be the vacuum expectation value (vev) equivalently minimum field configuration of the field in question: $\langle\phi\rangle$, F_{crit} will be model dependent but usually $F_{\text{crit}} = 0$.

In practice what will happen is that after the control parameter passes some threshold, the classical vev of the theory will not be the null field, $\langle\phi\rangle \neq 0$ and we will have a degenerate set of minima given by an algebraic equation. After this, we can arbitrarily select one of the equivalent possible ground states. Then a perturbation around a particular chosen ground state will get us into a perturbed Lagrangian that has a smaller symmetry H, having so a spontaneous breaking of the symmetry G. An equivalent condition to see the spontaneous breaking of continuous symmetries is having a non null set of generators $\{T_a\}$ of the large group G, such that:

$$T_a \langle\phi\rangle \neq 0 \tag{2.12}$$

These will be called the broken generators. The generators that do not belong to this class wi generate the Little group (stability group, residual grioup) of $\langle\phi\rangle$. The equivalence of both SSB conditions will be seen at Section (2.4).

Following the discussion of Section (2.1), the SSB of the gauge symmetry $SU(2)_L \times U(1)_Y$ provided the needed masses to $W^\pm$ and Z. In order to understand why this happens, first we need to understand the SSB of a global symmetry and the consequences of it. The first important result in this context was the discovery in the 60's that the spontaneous breaking of a global continuous symmetry ($G \to H$) implied the existence of at least $d_G - d_H$ real scalar particles with zero mass. This result is known as Goldstone's Theorem [21], [22] and will be explained in detail in Section (2.4).

[2]this is at least true for all the models treated in this thesis but in general it is not, instead we have a combination of parameters of the potential that act as control parameters whose combination gives the SSB condition see chapter 12.2 of [13]

In the rest of the section we give a first example of the Goldstone's theorem with the following Lagrangian with global symmetry $U(1)$ and with complex a scalar field ϕ. The mirror example with gauge symmetry is done in Section (3.1).
The Lagrangian is:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left\|\partial_\mu \phi\right\|^2 - \frac{\lambda}{4}\left(\|\phi\|^2 + \frac{m^2}{\lambda}\right)^2 \tag{2.13}$$

with potential:

$$V(\phi) = \frac{\lambda}{4}\left(\|\phi\|^2 + \frac{m^2}{\lambda}\right)^2 \tag{2.14}$$

The complex scalar field ϕ can be represented as a 2 dimensional real vector i.e. as $\phi = \phi_1 + i\phi_2$ or in a more vectorial form $\phi = (\phi_1, \phi_2)$. Seen as two real fields, it is a vector in the vector space of the fundamental representation of $SO(2)$. Remembering, the Energy functional for a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is:

$$E(\phi_i, \partial_\mu \phi_i) = \int dx^3 \mathcal{H} = \int dx^3 \left(\pi_i^0 \partial_0 \phi_i - \mathcal{L}\right) \tag{2.15}$$

where $\pi_i^0 \equiv \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i}$ is the conjugate momentum for ϕ_i. We minimize the Energy functional to calculate the vev : $\langle \phi \rangle$. For Eq.(2.13) we have:

$$E(\varphi, \partial_\mu \varphi) = \int d^3x \left[\sum_{i=1}^{2}\left(\frac{1}{2}(\partial_0 \phi_i)^2 + \frac{1}{2}(\partial_j \phi_i)^2\right) + \frac{\lambda}{4}\left(\|\phi\|^2 + \frac{m^2}{\lambda}\right)^2\right]. \tag{2.16}$$

Since the vacuum has to be invariant under translations we need that $(\partial_\mu \langle \phi_i \rangle)^2 = 0$ in each field and the so possible minima are constants. For the rest of the section, this will be a universal condition for all matter fields when trying to evaluate the extrema of the Energy functional. Since the energy has to be bounded from below we need to have $\lambda > 0$. The last term is the potential and the minimization is done using:

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i = \langle \phi_i \rangle} = 0 \qquad i = 1, 2. \tag{2.17}$$

We obtain:

$$\left(\frac{m^2}{\lambda} + \|\langle \phi \rangle\|^2\right)\langle \phi_i \rangle = 0 \qquad i = 1, 2. \tag{2.18}$$

Having retrieved the equation that defines the possible vev, we will see now how m^2, the control parameter will change the order parameter, $\langle \phi \rangle$. From Eq.(2.18) we see that the critical point is $m^2_{\text{crit}} = 0$. So we will have a symmetric phase for $m^2 \geq 0$ and a spontaneously broken phase $m^2 \leq 0$. In each phase we will count the number of massless bosons in two distinct forms, first directly using a perturbation around the ground state of the phase and also seeing how the generators of the symmetry group act on the selected ground state as in Eq.(2.12) and the using Goldstone's Theorem.

If $m^2 \geq 0$ the ground state is unique and we have $\langle \phi \rangle = \phi_0 = (0, 0)$ and the symmetry is not broken. A perturbation around this ground state shows that the fields (ϕ_1, ϕ_2) have equal mass m. This is called a Wigner-Weyl realization of symmetry. Analogously we now see how the generators act on this ground state. T, the only generator of the group $SO(2)$, in the fundamental representation T is:

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.19}$$

Since $\phi_0 = (0,0)$ we have $T\phi_0 = (0,0)$ thus $T\phi_0 = \phi_0$ so we don't have any breaking. The little group of ϕ_0 is all $SO(2)$.
When $m^2 = -\mu^2 < 0$, the Hessian of V at the configuration $(0,0)$ is negative definite and so is a maximum implying that at least one component of the vev is different from zero thus $\langle\phi\rangle \neq 0$ and so we are at the SSB phase. From Eq.(2.18) components of the vev have to satisfy:

$$\sum_{i=1}^{2} \langle\phi_i\rangle^2 = \frac{\mu^2}{\lambda} \equiv v^2 \tag{2.20}$$

For convenience we select the ground state as $\langle\phi\rangle = \phi_0 = (v, 0)$. The only generator is broken since are at the SSB phase. Explicitly, from the form of the generators applied to the vev, we have that:

$$T\phi_0 = (-v, 0) \neq (0,0) \tag{2.21}$$

Then Goldstone's theorem implies that there will be one massless real scalar boson. Correspondingly, the little group of ϕ_0 is null as there is not even one generator that is not broken.

A second way to see the fact that we have one massless boson is doing perturbations around the vev. These will be shown using two different parameterizations of the field and do the transformation $\phi \to v + \chi$ (a "perturbation around the minimum"). The first one is a cartesian parametrization:

$$\phi_1 = v + \chi_1 \tag{2.22}$$

$$\phi_2 = \chi_2 \tag{2.23}$$

The perturbed potential is:

$$\begin{aligned} V(\|v+\chi\|) =& \frac{\lambda}{4}(\|v+\chi_1+i\chi_2\|^2 - v^2)^2 = \frac{\lambda}{4}\left((v+\chi_1)^2 + \chi_2^2 - v^2\right)^2 \\ =& \frac{\lambda}{4}(2\phi_o\chi_1 + \chi_1^2 + \chi_2^2)^2 = \mu^2\chi_1^2 + \sqrt{\mu^2\lambda}(\chi_1^3 + \chi_1\chi_2^2) + \frac{\lambda}{4}(\chi_1^2+\chi_2^2)^2 \end{aligned} \tag{2.24}$$

The new Lagrangian to second order is then :

$$\mathcal{L}_\chi = \frac{1}{2}(\partial_\mu\chi_1)^2 + \frac{1}{2}(\partial_\mu\chi_2)^2 - \mu^2\chi_1^2 \tag{2.25}$$

Note that the field χ_2 does not have mass and corresponds to the Goldstone boson. This resultant Lagrangian does not have $U(1)$ symmetry. This is seen explicitly using the real

representation of $U(1)$.
The other parametrization is using radial coordinates:

$$\phi = (v+\rho)e^{i\eta/v} \tag{2.26}$$

It is clear that the potential does not depend on η. In fact:

$$V(\|\phi\|) = \frac{\lambda}{4}(\rho^2+2v\rho)^2 = \mu^2\rho^2 + \sqrt{\lambda}\mu\rho^3 + \frac{\lambda}{4}\rho^4 \tag{2.27}$$

The perturbed Lagrangian to second order is then:

$$\mathcal{L}_\rho = \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}(1+\frac{\rho}{v})^2(\partial_\mu\eta)^2 - \mu^2\rho^2 \tag{2.28}$$

In this parametrization $\eta(x)$ corresponds to the Goldstone boson. An additional comment, before finishing this section is that using a certain parametrization we explicitely saw the full perturbed Lagrangian with the physical Goldstone bosons whereas when dealing with generators acting on the vev we just know that there exist a certain number of Goldstone bosons without getting the explicit form of them. Sometimes, when it is difficult to found a parametrization like the radial one it is better to see the SSB pattern using the generators.

2.4 Classical Goldstone's Theorem

For the enunciation of this version of the theorem, we restrict to theories with N complex scalar fields that live in a unitary representation (generally reducible) of real dimension $d_\Phi = 2N$, of a compact Lie group G of dimension d_G. We also restrict to Lagrangians $\mathcal{L}$ of the form:

$$\mathcal{L} = \frac{1}{2}\|\partial_\mu\phi\|^2 - V(\phi_i) \tag{2.29}$$

Here $\|\|^2$ is the N complex scalar product i.e. $\|\phi\|^2 = \sum_{i=1}^N \phi_i^*\phi_i$. It is easy to see that the kinetic term is invariant under global G transformations for an appropiate N. Let ϕ' the transformed field:

$$\left\|\partial_\mu\phi'\right\|^2 = \partial^\mu\phi_i^{*\prime}\partial_\mu\phi_i' = \partial^\mu U_{ij}^*\phi_j^*\partial_\mu U_{ij'}\phi_{j'} = \partial^\mu\phi_i^*\partial_\mu\phi_i = \left\|\partial_\mu\phi\right\| \tag{2.30}$$

where we have used the unitarity of U. $V(\phi_i)$ is the potential of the Lagrangian, that will be composed of invariant polynomials under global and local G symmetry of at most order four.

We also suppose that the potential has a degenerate set of ground states that is connected for some elements of the group. Selecting a particular ground state ϕ_0 the elements $h \in G$ with property:

$$U(h)\phi_0 = \phi_0 \tag{2.31}$$

form a subgroup H of G of dimension R_H. Note that it contains the identity element. Natu-

rally, H will have a Lie algebra $\mathfrak{h}$. This group is commonly called the "little group of ϕ_0"[3]. Note also that the elements of G that does not have this property does not form a subgroup of G but are elements of the coset group G/H.
As usual we have:

$$U(g) = e^{\alpha_a T_a}, \tag{2.32}$$

with T_a the antihermitian representation of an element of $\mathfrak{g}$ the Lie algebra of G. Restricting ourselves to elements of the group H and using the expansion Eq.(2.32) the equivalent definition given by Eq.(2.31) in terms of Lie Algebras is:

$$T_h\phi_0 = 0 \tag{2.33}$$

that is valid in each order k of α_h^k with h denoting the directions restricted to the group H by the d_H independent elements of $\mathfrak{h}$. For the elements (labeled by g) that do not belong to $\mathfrak{h}$ we have $T_g\phi_o \neq 0$ thus for a linear combination of them we have:

$$\alpha_g T_g \phi_0 \neq 0 \qquad \{\alpha_g\}_{\text{arbitrary}} \tag{2.34}$$

so they are closed as a vector space thus generate the coset group G/H.

In this way the generators of G can be divided in two disjoint sets : the generators of H, $\{T_h\}$, and the $R_G - R_H$ "broken generators" $\{T_g\}$ that generate the coset group G/H. From now on we will use h and g as indices over their respective subspaces. Fields near ϕ_0 can be divided in fields in the directions generated by the broken generators around ϕ_0, that is, proportional to $T_g\phi_0$ and fields $H(x)$ orthogonal to this space:

$$[T_g\phi_0]_i H(x)_i = 0, \qquad i = 1, \cdots, N \tag{2.35}$$

We shall call these spaces Goldstone and Higgs space, respectively. As perturbations in the $d_G - d_H$ directions of Goldstone space are linearly independent we have $d_G - d_H$ fields, $\alpha_g(x)\hat{T}_g\phi_0$, in this space. The dimension of the Higgs space is then $N_H = d_\phi - d_G + d_H$. Thus a general perturbation is $\phi_0 \to \phi_0 + \chi(x)$ with :

$$\chi(x) = \tau_g(x)\hat{T}_g\phi_0 + H(x) \tag{2.36}$$

Notice that $\alpha_g(x)$ are dimensionless. The perturbed Lagrangian is defined as:

$$\mathcal{L}_\chi(\chi) := \mathcal{L}(\phi_0 + \chi) = \frac{1}{2}\left\|\partial_\mu \chi\right\|^2 - V(\phi_0 + \chi). \tag{2.37}$$

Now we are ready to state Goldstone's Theorem using the scheme outlined above. Suppose we have a Lagrangian $\mathcal{L}$ as Eq.(2.29) with global and continuous symmetry for a group G in the SSB phase corresponding to the vev ϕ_0. Then the potential of the perturbed Lagrangian $\mathcal{L}_\chi$ to second order depends only on the Higgs field space, which implies that fields in the $d_G - d_H$ dimensional Goldstone subspace are massless.

[3] The little group H or stability group of a vector ϕ_0 are the $h \in G$ that leave ϕ_0 invariant: $U(h)\phi_0 = \phi_0$

To see this let's start from the invariance condition of the potential:

$$V\left(U(\alpha)(\phi_0 + \chi(x))\right) = V(\phi_0 + \chi(x)) \tag{2.38}$$

with α the coordinates of an arbitrary $g \in G$. This is also valid also for local transformations i.e. $\alpha(x)$. Then from Eq.(2.32) we have for an arbitrary direction:

$$U(\alpha(x)) = 1 + \alpha_a(x)T_a + \frac{1}{2}(\alpha_a T_a)^2 + \mathcal{O}(\alpha^3) \tag{2.39}$$

So, using Eq.(2.38) as Eq.(2.39) for perturbations only on the Higgs space $(\chi(x) = H(x))$ we have:

$$\begin{aligned} V(\phi_0 + H(x)) =& V\left(U(\alpha(x))(\phi_0 + H(x))\right) \\ =& V(\phi_0 + H(x) + \alpha_a(x)T_a\phi_0 + \alpha_a(x)T_aH(x) + A\alpha^2) \\ =& V(\phi(x) + \alpha_a(x)T_aH(x) + A\alpha^2) \\ \equiv& V(\bar{\phi}(x)) \end{aligned} \tag{2.40}$$

where the terms of order α^2 are written in symbolic form as $A\alpha^2$. Expanding around the perturbed field $\phi(x)$ we have to first order:

$$V(\phi_0 + H(x)) \approx V(\phi(x)) + \left[\frac{\partial V}{\partial \bar{\phi}}\right]_i \Big|_{\bar{\phi}=\phi} (\alpha_a(x)T_aH(x) + A\alpha^2)_i \tag{2.41}$$

We shall now do a second expansion around ϕ_0 for the second term

$$\left[\frac{\partial V}{\partial \bar{\phi}}\right]_i \Big|_{\bar{\phi}=\phi} \approx \left[\frac{\partial V}{\partial \bar{\phi}}\right]_i \Big|_{\bar{\phi}=\phi_0} + \left[\frac{\partial^2 V}{\partial \bar{\phi}^2}\right]_{ij} \Big|_{\bar{\phi}=\phi_0} (H(x) + \alpha_a(x)T_a\phi_0)_j \tag{2.42}$$

The first term vanishes. Combining (2.41) and (2.42), we see we do not have any term quadratic in $\alpha_a(x)$ and not involving $H(x)$. Thus to second order in $\alpha(x)$ and $H(x)$:

$$V(\phi_0 + H(x)) \approx V\left(\phi_0 + \alpha_a(x)T_a\phi_0 + H(x)\right) \tag{2.43}$$

Relabeling $\alpha_a(x) = \tau_a(x)$ and since we only have non null values in directions directions on the Goldstone space we have $a = g$, thus to second order:

$$V_2\left(\phi_0 + \chi(x)\right) = V_2\left(\phi_0 + \tau_a(x)T_a\phi_0 + H(x)\right) = V_2(\phi_0 + H(x)). \tag{2.44}$$

This means there are no terms of the type $T_g\Phi_0T_{g'}\phi_0\tau_g(x)\tau_{g'}(x)$, which implies that the fields in Goldstone space do not acquire a mass term. A diagonalization in this space gives the Goldstone bosons.

2.5 Spontaneous symmetry breaking for a Lagrangian with Non Abelian Global Symmetry

A simple and clear example of the application of Goldstone's theorem for a non Abelian group is using a Lagrangian with $SO(3)$ symmetry:

$$\mathcal{L} = \frac{1}{2}\left\|\partial_\mu \phi\right\|^2 - \frac{\lambda}{4}\left(\|\phi\|^2 + \frac{m^2}{\lambda}\right)^2 \tag{2.45}$$

with the field $\phi = (\phi_1, \phi_2, \phi_3)$ in the fundamental representation of $SO(3)$. The symmetry operators have the form $U(g) = e^{\alpha_a T_a}$, with T_a the fundamental representation of $\mathfrak{so}(3)$, the Lie Algebra of $SO(3)$. The explicit matrices are:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.46}$$

The minimum conditions are retrieved in the usual way:

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi=\langle\phi\rangle} = 0 \qquad i = 1, 2, 3. \tag{2.47}$$

We have:

$$\left(\frac{m^2}{\lambda} + \left\|\langle\phi\rangle\right\|^2\right)\langle\phi_i\rangle = 0 \qquad i = 1, 2, 3. \tag{2.48}$$

As before, when $m^2 > 0$ we have the symmetric case with vev $\langle\phi\rangle = (0, 0, 0)^T$. For $m^2 < 0$ we have the SSB case and explicitly we see that the set of ground states that minimize the potential have to satisfy the following equation:

$$\sum_{i=1}^{3} \langle\phi_i\rangle^2 = -\frac{m^2}{\lambda} = v^2 \tag{2.49}$$

As before, we select a certain configuration of fields that satisfy equation Eq.(2.49) , a particular vev, for example $\phi_0 = (0, 0, v)$. Since $SO(3)$ has 3 generators T_a we apply them into ϕ_0 and we have that the set of broken generators ($T_b\phi_0 \neq 0$) are T_1, T_2 and the unbroken one is T_3 with $T_3\phi_0 = 0$. Now applying Goldstone's theorem we see that we have two Goldstone bosons corresponding to the ϕ_1 and ϕ_2 fields and the perturbed Lagrangian has a smaller continuous symmetry generated only by T_3. Since there is only one subgroup of $SO(3)$ with one generator we see that $SO(2)$ is the new global symmetry. Thus the spontaneous symmetry breaking pattern is:

$$SO(3) \rightarrow SO(2) \tag{2.50}$$

Doing a perturbation around the vev $\phi = \phi_0 + \chi_i$ with i labeling the directions for the potential we have:

$$V(\chi) = \frac{\lambda}{4}((v+\chi_3)^2 + \chi_2 + \chi_1 - v^2)^2 \tag{2.51}$$

Then the perturbed Lagrangian (2.45) to second order is:

$$\mathcal{L} = \partial_\mu \chi_i \partial^\mu \chi_i - \lambda v^2 \chi_3^2 \tag{2.52}$$

showing explicitly the result of Goldstone's theorem with χ_1 and χ_2 as the massless Goldstone's bosons and the only Higgs χ_3 with mass $\sqrt{2\lambda}v$.

The residual symmetry group $SO(2)$ is seen from the full Lagrangian:

$$\begin{aligned}\mathcal{L} =& \partial_\mu \chi_i \partial^\mu \chi_i - \lambda v^2 \chi_3^2 + \frac{\lambda}{2}\chi_3^2 + \frac{\lambda}{4}\chi_3^4 \\ &+ \lambda v \chi_3 (1+\chi_3^2)(\chi_1^2+\chi_2^2) + \frac{\lambda}{4}(\chi_1^2+\chi_2^2)^2\end{aligned} \tag{2.53}$$

where χ_3 is a singlet under this symmetry and the two Goldstone bosons transform as the fundamental representation of $SO(2)$.

The result of the massless scalar bosons of the Goldstone was seen as a puzzle at the time of it's discovery since in those times people worked with global symmetries of phenomenological flavor models and the result implied long ranged interactions that were clearly not observed. Contemporaneously, there was also the puzzle of having an adequate propagator for a massive vector boson such that only the timelike polarization cancels. These two problems and their solutions are related, as we will see in the next Chapter.

Chapter 3

Generalized Brout-Englert-Higgs Mechanism

The discussion of Section (3.1) is based on [12]. For the rest of the chapter we mainly deal with the pedagogical exposition of the seminal paper [23]. The group theoretical methods were mainly acquired from [24],[25],[26], [27] and in particular the basis of the Classical Algebras are taken as defined in [28]. The form of presentation was inspired by [13].

3.1 Brout-Englert-Higgs Mechanism in an Abelian Model

Following Section (2.1) the model of Glashow (1962), although important since it found the adequate symmetry group of Electroweak interactions was not renormalizable nor solved the problem for the massive vector boson propagator. In 1964, however, it was found a circumvention for Goldstone's Theorem. Three different contemporaneous papers ([29], [30],[31]), each one with different methods, showed that after upgrading the global symmetry into a gauge symmetry, thus adding the respetive gauge field, the massless bosons dissapeared. Instead, they were "eaten" by the gauge field such that it acquired mass and, surprise, it also gave an adequate propagator for the gauge field.

We start with an example using a $U(1)$ gauge invariant Lagrangian:[1]

$$\mathcal{L}(\phi, \phi^*, A^\mu, \partial_\mu A^\nu) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left\| \mathcal{D}_\mu \phi \right\|^2 - V(\phi, \phi^*), \tag{3.1}$$

with:

$$\mathcal{D}_\mu \phi = (\partial_\mu - igA_\mu)\phi \tag{3.2}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{3.3}$$

[1] we set the label of $U(1)$ to $q = 1$

and potential:

$$V(\phi, \phi^*) = \frac{\lambda}{4}\left(\|\phi\|^2 - \frac{\mu^2}{\lambda}\right)^2. \tag{3.4}$$

The gauge transformations for $U(1)$ are:

$$A_\mu \to A_\mu + \frac{1}{g}\partial^\mu \alpha(x) \tag{3.5}$$

$$\phi \to e^{i\alpha(x)}\phi. \tag{3.6}$$

From the energy functional:

$$E(A_\mu, \phi) = \int dx^3 \left(\frac{1}{2}F^{0i}F_{0i} + \frac{1}{4}F_{ij}F^{ij} + \|\mathcal{D}_\mu \phi\|^2 + V(\|\phi\|)\right), \tag{3.7}$$

we calculate the fields that minimize it. To minimize the kinetic term of the boson field A_μ we do:

$$\left.\frac{\partial E}{\partial(\partial_\mu A_\nu)}\right|_{A_\mu = \langle A_\mu \rangle} = 0 \to F^{\mu\nu} = 0. \tag{3.8}$$

Then A_μ has to be a pure gauge i.e.

$$\langle A_\mu \rangle = \frac{1}{g}\partial_\mu \alpha(x), \tag{3.9}$$

The minimization of the potential is the same as in the globally symmetric ones. The interesting case is only when there is Spontaneous Symmetry Breaking (SSB): $\mu^2 \geq 0$.

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i = \langle \phi_i \rangle} \left(-\frac{\mu^2}{\lambda} + \|\langle \phi \rangle\|^2\right) \langle \phi \rangle_i = 0 \qquad i = 1, 2. \tag{3.10}$$

The set of all possible vevs of ϕ has to satisfy the equation:

$$\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2 = \frac{\mu^2}{\lambda} \equiv v^2. \tag{3.11}$$

Though, differing from the globally symmetric examples of Chapter (2), it is not sufficient to fix the complex vev (constant) $\langle \phi \rangle = \phi_0 = \langle \phi_1 \rangle + i \langle \phi_2 \rangle$ in order to get a physical vev; we still can have $U(1)$ gauge transformations on ϕ_0 whose results, new vevs, still satisfy Eq. (3.11). Thus we need to fix the gauge at the level of the vev to cancel the extra degrees of freedom. To do this we minimize the third term. We have using Eq.(3.9):

$$\mathcal{D}_\mu \phi|_{\phi = \langle \phi \rangle, A_\mu = \langle A_\mu \rangle} = \partial_\mu \langle \phi \rangle - i\left(\partial_\mu \alpha(x)\right)\langle \phi \rangle = 0. \tag{3.12}$$

From Eq.(3.9) and Eq.(3.11) this implies that:

$$\langle \phi(x) \rangle = e^{i\alpha(x)}\phi_0. \tag{3.13}$$

This is the set of solutions defining all the possible ground states, the same as the case with global symmetry. For convenience we choose $\alpha(x) = 0$ that implies:

$$\langle A_\mu \rangle = 0, \quad \langle \phi \rangle = \phi_0. \tag{3.14}$$

From last equation, the fact that we fix the gauge of the vev implies that we only need to look at the potential to evaluate the ground state. Using a real ϕ_0, ($\langle \phi_2 \rangle = 0$) we have doing an infinitesimal perturbation in Cartesian parametrization in the matter field:

$$\phi = \phi_0 + \chi_1 + i\chi_2, \tag{3.15}$$
$$A_\mu = \langle A_\mu \rangle + A_\mu = 0 + A_\mu. \tag{3.16}$$

The perturbed Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\|\partial_\mu\chi_1 + i\partial_\mu\chi_2 - ig\phi_0 A_\mu - igA_\mu\chi_1 + gA_\mu\chi_2\|^2 - f(\chi_1,\chi_2), \tag{3.17}$$

where using $\phi_0 = \sqrt{\frac{\mu^2}{\lambda}}$:

$$f(\chi_1,\chi_2) \equiv V(\|\phi+\chi\|) = \mu^2\chi_1^2 + \mu\sqrt{\lambda}(\chi_1^3 + \chi_1\chi_2^2) + \frac{\lambda}{4}(\chi_1^2+\chi_2^2)^2. \tag{3.18}$$

That is:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\chi_1)^2 + \frac{g^2\phi_0^2}{2}(A_\mu - \frac{1}{g\phi_0}\partial_\mu\chi_2)^2 \\ &+ gA^\mu(\partial_\mu\chi_1)\chi_2 + g^2\phi_0\left(A_\mu - \frac{1}{g\phi_0}\partial_\mu\chi_2\right)A^\mu\chi_1 \\ &+ \frac{1}{2}g^2A_\mu^2\chi_1^2 + \frac{1}{2}(gA_\mu\chi_2)^2 - f(\chi_1,\chi_2).\end{aligned} \tag{3.19}$$

Since we have used infinitesimal fields we focus only on terms of at most second order:

$$\mathcal{L}_\chi = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\chi_1)^2 + \frac{g^2\phi_0^2}{2}(A_\mu - \frac{1}{g\phi_0}\partial_\mu\chi_2)^2 - \mu^2\chi_1^2 \tag{3.20}$$

Redefining the vector field $B_\mu = A_\mu - \frac{\partial_\mu\chi_2}{g\phi_0}$ we get:

$$\mathcal{L}_\chi \equiv -\frac{1}{4}F_{B\,\mu\nu}F_B^{\mu\nu} + \frac{1}{2}(\partial_\mu\chi_1)^2 - \mu^2\chi_1^2 + \frac{g^2\phi_0^2}{2}(B_\mu)^2, \tag{3.21}$$

where $F_B^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F^{\mu\nu}$. This new Lagrangian consist of a real scalar field χ_1 and a massive boson vector B_μ. Note that the field χ_2 is not longer present explicitly in the Lagrangian, it is part of the new vector field. The new vector boson B_μ acquires then an additional degree of freedom given by the degree of freedom of the real scalar χ_2. So we have found that after SSB of a gauge symmetry we can retrieve a massive vector boson. This is the

BEH mechanism for the second order Lagrangian.

The degrees of freedom of the theory remain the same, four. Comparing the degrees of freedom of a $U(1)$ theory with a trivial vacuum and one with SSB we see that in the trivial case we have two real bosons each one with one degree of freedom and a massless gauge vector field with two degrees of freedom . In the SSB case we have only one boson χ_1 and the massive vector field B_μ with three degrees of freedom. Eq.(3.21) is the SSB Lagrangian in terms of the physical fields only until second degree.[2]

To see the full physical Lagrangian is better to use the unitary gauge . Using the radial parametrization:

$$\phi = (\phi_0 + \rho)e^{i\eta/\phi_0} \tag{3.22}$$

where the fields transform as:

$$\begin{aligned} \eta' &= \eta + \phi_0 \alpha(x) \\ \rho' &= \rho \\ A'_\mu &= A_\mu - \frac{1}{g}\partial_\mu \alpha \end{aligned} \tag{3.23}$$

We impose a subsidiary condition:

$$\phi = \phi^* \tag{3.24}$$

that implies $\eta = 0$. Condition (3.24) is fixing the unitary gauge. The remaining fields are then ρ and A_μ. Expanding the potential we arrive at the exact Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\phi_0 + \rho)^2 g^2 A_\mu^2 - \mu^2\rho^2 - \sqrt{\lambda}\mu\rho^3 - \frac{\lambda}{4}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.25}$$

that shows explicitly the dissapearance of the Goldstone bosons. In this gauge then, the theory is composed of only the physical fields and the original gauge boson acquires mass. This is the BEH mechanism.

3.2 Theories with non Abelian Symmetries

In the first Chapter we have shown all the elements needed to construct models with non-abelian gauge symmetry that include the Standard Model itself. Summarizing, the general principles are:

(a) Choose the gauge group G with d_G generators

(b) Add d_G vector fields (gauge bosons) of spin 1 that transform in the Adjoint Representation of G.

(c) Choose representations under G for the matter fields (elementary particles). These have to include chiral representation of G for the spin 1/2 fermions and complex or real Lorentz scalar representations under G for the spin 0 bosons.

[2] and without fixing the gauge

(d) Define adequate covariant derivatives for each non gauge field and write the most general renormalizable Lagrangian, gauge invariant under G. Matter-gauge interactions will be derived from the covariant derivative.

(e) In case we require massive vector fields, the Lagrangian has to include a renormalizable Higgs potential composed of scalar fields in order to spontaneously break the gauge symmetry G using a generalized BEH mechanism. If the gauge symmetry does not allow certain fermions to have a Dirac or Majorana mass term then we need to intruduce Yukawa coupling (trilinears) between the fermions and the scalars of the Higgs potential.

The skeleton of a Lagrangian will be:

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{Y} + \mathcal{L}_{\text{CPv}} \tag{3.26}$$

with:

$$\mathcal{L}_{gauge} = \sum_i -\frac{1}{4} F^{\mu\nu a}_{\;\;G_i} F^{a}_{\mu\nu\, G_i} \tag{3.27}$$

Where the sum over G_i is over each simple group (if there are more than one) of the gauge symmetry group $G = G_1 \times \cdots \times G_m$.[3] This sector gives the kinetic energy of the gauge fields and their self interactions.
The fermionic kinetic terms are:

$$\mathcal{L}_{\text{fermions}} = \sum_\psi \bar{\psi} \mathcal{D}_\mu \psi \tag{3.28}$$

Where the sum is over the different fermionic fields of the theory. From the covariant derivative we get interactions between the scalar and gauge fields.

$$\mathcal{L}_{\text{scalar}} = \sum_\phi \frac{1}{2} \|\mathcal{D}_\mu \phi\|^2 - V(\phi) \tag{3.29}$$

where $\|\mathcal{D}_\mu \phi\|^2$ denotes the kinetic term of the ϕ's and the $\|\|^2$ denotes the scalar product in the specified representation of ϕ. The Yukawa sector is:

$$\mathcal{L}_Y = V(\phi, \psi) \tag{3.30}$$

This sector will contain terms constructed between fermions and scalar bosons that are gauge invariant that can be. It is the generalization of the Yukawa sector and gives masses to the correspondent fermions after SSB.

In the next section we deal in the most general way with the spontaneous breaking of gauge symmetries of the orthogonal $O(n)$ and special unitary groups $SU(n)$. First we will write the most general potential invariant under G in each of the different representations. Then we minimize the potential to retrieve the vev. As we will see, there will be a more

[3]for example for the Standard Model with gauge symmetry $SU(3)_c \times SU(2)_L \times U(1)_Y$ for $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$

compact way to express the vev in each case using transformations that satisfy the symmetries of the potential, that as in the case of of the potential of the Standard Model, has larger symmetry than the symmetry of the Lagrangian. In order to calculate the dimension of the lower symmetry group (modulo homomorphisms) and see the different symmetry breaking patterns we need to know the number of unbroken generators of the lager symmetry group for the corresponding vev. Since the number of massive vector bosons is equal to the number of would be Goldstone bosons and these last ones are equal to the number of broken generators, inserting the vev on the kinetic term gives us the number of vector fields that acquire mass and consequently (doing a substraction) also the number of broken generators.

Another procedure to get the symmetry breaking patterns [32], that we won't follow is explicitly analyzing the action of the generators of G on the factorized vev $\langle\Sigma\rangle$. Using this approach we could find the broken and unbroken generators and retrieve the subalgebra of the unbroken generators by analyzing the specific relationships they have (i.e. if they are traceless, hermitian etc.). The usefulness of this approach is that we do not need to calculate the vector boson mass to get the symmetry breaking patterns. Even more, it is more general since it just assumes global symmetries tough in practice if we focus only in the SSB breaking of the potential that is irrelevant.

3.3 Spontaneous breaking of Symmetry in the $O(n)$ group

The matrix Lie Group $O(n)$ is defined as the set of real matrices $n \times n$, denoted by O, that satisfy:

$$O^T O = \mathbf{1}. \tag{3.31}$$

The group $SO(n)$ is the subgroup of $O(n)$ with the additional constrain:

$$\det[O] = 1. \tag{3.32}$$

Using $O = e^X$, [4] with X an element of $\mathfrak{o}(n)$ ($\mathfrak{so}(n)$), the Lie Algebra of $O(n)$ ($SO(n)$), we have the corresponding constraints:

$$X = -X^T \tag{3.33}$$
$$\mathrm{Tr}[X] = 0. \tag{3.34}$$

Equation (3.31) gives us $\frac{n(n+1)}{2}$ restrictions. Thus the initial n^2 parameters of $O \in O(n)$ are reduced to $\frac{n(n-1)}{2}$ independent parameters. In a similar way from Eq.(3.33) the same restriction is retrieved for any element of $\mathfrak{o}(n)$. For $SO(n)$, Eq. (3.31) gives us $\frac{n(n+1)}{2}$ restrictions and the Lie Algebra is the same as the one for $O(n)$ since the additional constraint Eq.(3.34) is included in Eq.(3.33) thus $\mathfrak{o}(n) = \mathfrak{so}(n)$.

A base for the $\frac{n(n-1)}{2}$ antisymmetric matrices X, elements of $\mathfrak{o}(n)$ are the set $\{L_{ab}\}$,

[4] This expression is valid globally for compact Lie Groups, that include all Classical groups

$a < b = 1, \cdots n$, real matrices with entries as:

$$[L_{ab}]_{ij} = (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}). \tag{3.35}$$

These satisfy the following rule:

$$[L_{ij}, L_{kl}] = \delta_{jk}L_{il} + \delta_{il}L_{jk} - \delta_{ik}L_{jl} - \delta_{jl}L_{ik} \qquad i, j, k, l = 1, \cdots, n \tag{3.36}$$

This is the matrix Lie Algebra $\mathfrak{o}(n)$, which spans a vector space, with scalar product given by the trace:

$$\mathrm{Tr}[X_{ij}X_{kl}] = 2(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{3.37}$$

The correspondence with another usual notation of $\mathfrak{o}(n)$ with a single symbol $\{T_a\}$ is straightforward. For $\mathfrak{so}(3)$ for example we have the correspondence:

$$T_1 = L_{23}, \qquad T_2 = L_{31}, \qquad T_3 = L_{12}. \tag{3.38}$$

Each couple of indices x, y in L_{xy} is interpreted as rotation in the plane given by x, y.

Using this notation, the fundamental representation of the Lie Group is defined as:

$$U(\epsilon_{ij}) = e^{\frac{1}{2}L_{ij}\epsilon_{ij}} \tag{3.39}$$

Then using Eq.(3.33) we see that ϵ_{ij} are $n(n-1)/2$ real antisymmetric parameters.
Since the algebra has $n(n-1)/2$ elements to construct a gauge theory $O(n)$ we need the same number of bosonic gauge bosons, $W_{\mu ij}$ such that the Yang Mills field is : $W_\mu = W_{\mu ij}L_{ij}$.
The $W_{\mu ij}$ are antisymmetric. In fact $W_\mu = W_{\mu ij}L_{ij} = -W_{ij}L_{ji}$, then projecting into the direction L_{ab} we have $W_{ab} = -W_{ba}$.
The gauge boson vectors transform as the adjoint representation [5] Eq.(1.70) (using a different normalization) under a gauge transformation:

$$W_\mu \to W'_\mu = e^{\frac{1}{2}L_{ij}\epsilon_{ij}}\left(W_{\mu ab}L_{ab} - \frac{2}{g}\partial_\mu\right)e^{-\frac{1}{2}L_{ij}\epsilon_{ij}} \tag{3.40}$$

Expanding near the identity we get:

$$\begin{aligned} W'_\mu =& W_\mu + \frac{1}{2}[L_{ij}\epsilon_{ij}, W_{\mu ab}L_{ab}] + \frac{1}{g}(\partial_\mu\epsilon_{ij})L_{ij} \\ =& W_\mu + \frac{1}{2}\epsilon_{ij}W_{\mu ab}\Big(\delta_{ib}L_{ja} + \delta_{ja}L_{ib} - \delta_{ia}L_{jb} - \delta_{jb}L_{ia}\Big) + \frac{1}{g}(\partial_\mu\epsilon_{ij})L_{ij} \\ =& W_\mu + \epsilon_{ik}W_{\mu kj}L_{ij} + \epsilon_{ik}W_{\mu jk}L_{ji} + \frac{1}{g}(\partial_\mu\epsilon_{ij})L_{ij}. \end{aligned} \tag{3.41}$$

[5]that is always real for classical Lie groups

Using Eq. (3.37) we can project onto the L_{ij} element, ($\mathrm{Tr}[W_\mu L_{ij}]$), to obtain:

$$W'_{\mu ij} = W_{\mu ij} + \epsilon_{ik} W_{\mu kj} + \epsilon_{jk} W_{\mu ik} + \frac{1}{g}(\partial_\mu \epsilon_{ij}). \tag{3.42}$$

To construct the kinetic part of the Lagrangian we need:

$$F^{ij}_{\mu\nu} = \partial_\mu W^{ij}_\nu - \partial_\nu W^{ij}_\mu + g(W^{ik}_\mu W^{kj}_\nu - W^{ik}_\nu W^{kj}_\mu) \tag{3.43}$$

The Lagrangians that we will use for the rest of the chapter have the following form:

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}[F^\dagger_{\mu\nu} F^{\mu\nu}] + \mathcal{L}_{\mathrm{kin}} - V = -\frac{1}{4} F^{ij}_{\mu\nu} F^{\mu\nu}_{ij} + \mathcal{L}_{\mathrm{kin}} - V \tag{3.44}$$

where $\mathcal{L}_{\mathrm{kin}}$ is the kinetic term of the scalar field and V is the potential. Both V and $\mathcal{L}_{\mathrm{kin}}$ will depend on the representation of the scalar field.

In the following subsections we will calculate the symmetry breaking patterns. In each subsection we derive the transformation law of the field in the specific representation and it's covariant derivative.

3.3.1 Spontaneous Breaking in the vector representation

A finite transformation in the fundamental representation is:

$$\phi' = U(\epsilon)\phi \tag{3.45}$$

with:

$$U(\epsilon_{ij}) = e^{\frac{1}{2} L_{ij}\epsilon_{ij}}. \tag{3.46}$$

For infinitesimal transformations, this implies:

$$\phi'_i = \phi_i + \epsilon_{ij}\phi_j. \tag{3.47}$$

The covariant derivative can be seen using Eq.(3.35) with $W_\mu = W_{\mu ab} L_{ab}$:

$$\begin{aligned}\mathcal{D}_\mu \phi_i =& \partial_\mu \phi_i - \frac{g}{2} W_{\mu ab} (L_{ab})_{ij} \phi_j = \partial_\mu \phi_i - \frac{g}{2} W_{\mu ab}(\delta_{ai}\delta_{bj} - \delta_{bi}\delta_{aj})\phi_j \\ =& \partial_\mu \phi_i - g W_{\mu ij}\phi_j.\end{aligned} \tag{3.48}$$

Using these results, we can construct the kinetic part of the scalar field in the fundamental representation:

$$\mathcal{L}_{\mathrm{kin}} = \frac{1}{2}(\mathcal{D}^\mu \phi)^\dagger \mathcal{D}_\mu \phi = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - g W_{\mu ij}\phi_j \partial^\mu \phi_i + \frac{1}{2} g^2 W_{\mu ik}\phi_k W^\mu_{ik'}\phi_{k'}. \tag{3.49}$$

Recall that the last term is responsible for giving masses to the gauge bosons W_μ.
The most general renormalizable potential in the vector representation is:

$$V(\phi) = -\frac{1}{2}\mu^2(\phi_i\phi_i) + \frac{1}{4}\lambda(\phi_i\phi_i)^2 \tag{3.50}$$

where $\lambda > 0$ so the potential is bounded below. To minimize the potential we do:

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i=\langle\phi_i\rangle} = \Big(-\mu^2 + \lambda\langle\phi_j\rangle\langle\phi_j\rangle\Big)\langle\phi_i\rangle = 0 \qquad i = 1, \cdots, n. \tag{3.51}$$

For $\mu^2 > 0$ the standard extremum $\langle\phi_i\rangle = 0, \quad i = 1, \cdots, n$ gives a local maximum.
The equation that defines the set of ground states is:

$$\langle\phi_i\rangle\langle\phi_i\rangle = \frac{\mu^2}{\lambda} \equiv v^2. \tag{3.52}$$

We select the vev as

$$\langle\phi_i\rangle = \delta_{in}v. \tag{3.53}$$

Remembering the discussion of Section (3.1) we need to fix the gauge to get a physical vev. We assume for this section and the next ones that from the kinetic term of the gauge vector bosons the vev of them is pure gauge i.e. $\langle W_{\mu\, ij}\rangle = \partial_\mu\alpha_{ij}$ and that from the minimization of the kinetic term of the scalar boson that gives :

$$\mathcal{D}_\mu\phi|_{\phi_i=\langle\phi_i\rangle, W_\mu=\langle W_\mu\rangle} = 0 \tag{3.54}$$

the gauge is fixed and so Eq.(3.53) is valid unambiguously.

Using Eq.(3.49), we have the term that gives masses to the gauge bosons:

$$\mathcal{L}_M = \frac{1}{2}g^2 W_{\mu ik}W^\mu_{il}\langle\phi_k\rangle\langle\phi_l\rangle = g^2\sum_{i=1}^{n-1} W_{\mu in}W^\mu_{in}\frac{v^2}{2}. \tag{3.55}$$

So we have $n-1$ bosons that acquire mass. Thus we have:

$$\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2} \tag{3.56}$$

massless vector bosons. The breaking is then:

$$O(n) \to O(n-1) \tag{3.57}$$

This implies that there are $\frac{(n-1)(n-2)}{2}$ generators that do not break the symmetry meaning that the residual symmetry (Little group of the vev $\langle\phi\rangle$) is $O(n-1)$ and there is an equal number of scalar bosons.

3.3.2 Spontaneous Breaking in the second rank Antisymmetric Representation

The demonstration on how to retrieve the covariant derivative and transformation law for these representations will be derived in Part 3.6. An infinitesimal transformation in this representation is:

$$\phi'_{ij} = \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik}. \tag{3.58}$$

The covariant derivative is:

$$\mathcal{D}_\mu\phi_{ij} = \partial_\mu\phi_{ij} - gW_{\mu ik}\phi_{kj} - gW_{\mu jk}\phi_{ik}. \tag{3.59}$$

The kinetic term for a second rank tensor is:

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& \frac{1}{2}\partial^\mu\phi_{ij}\partial_\mu\phi_{ij} - g(W^\mu_{ik}\phi_{kj} + W^\mu_{jk}\phi_{ik})\partial_\mu\phi_{ik} \\ &+ g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj}) - g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{li}).\end{aligned} \tag{3.60}$$

Note that the gauge boson masses will come from the terms proportional to g^2

$$\mathcal{L}_M = g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj}) + g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{il}). \tag{3.61}$$

The most general quartic potential for antisymmetric second order tensor representation of $O(n)$ is:

$$\begin{aligned}V(\phi) =& -\frac{1}{2}\mu^2\,\mathrm{Tr}[\phi^\dagger\phi] + \frac{1}{4}\lambda_1(\mathrm{Tr}[\phi^\dagger\phi])^2 + \frac{1}{4}\lambda_2\,\mathrm{Tr}[\phi^\dagger\phi\phi^\dagger\phi] \\ =& \frac{1}{2}\mu^2\,\mathrm{Tr}[\phi^2] + \frac{1}{4}\lambda_1(\mathrm{Tr}[\phi^2])^2 + \frac{1}{4}\lambda_2\,\mathrm{Tr}[\phi^4]\end{aligned} \tag{3.62}$$

or equivalently:

$$V(\phi) = -\frac{1}{2}\mu^2\phi_{ij}\phi_{ij} + \frac{1}{4}\lambda_1(\phi_{ij}\phi_{ij})^2 + \frac{1}{4}\lambda_2\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}. \tag{3.63}$$

Any real antisymmetric matrix can be expressed in "standard form" [33] using $\phi = O\Sigma O^T$. Using $n = 2L$ for n even and $n = 2L + 1$ for n odd, we have:

$$\Sigma = \begin{pmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & \ddots & \\ & 0 & & A_L \end{pmatrix}, \quad n = 2L \tag{3.64}$$

and

$$\Sigma = \begin{pmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \ddots & & \\ & & & A_L & \\ & 0 & & & 0 \end{pmatrix}, \quad n = 2L+1 \tag{3.65}$$

where:

$$A_i = a_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a_i \text{ real}, \tag{3.66}$$

so

$$A_i^2 = a_i^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_i^4 = a_i^4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.67}$$

Using $OO^T = 1$ and permutation propriety of factors inside the trace we can write Eq.(3.62) as:

$$V(\Sigma) = \frac{1}{2}\mu^2 \operatorname{Tr}[\Sigma^2] + \frac{1}{4}\lambda_1(\operatorname{Tr}[\Sigma^2])^2 + \frac{1}{4}\lambda_2 \operatorname{Tr}[\Sigma^4]. \tag{3.68}$$

Note that we have used the gauge invariance propriety of the Lagrangian both in the potential and in the kinetic term such that Σ is just a field redefinition.
Explicitly:

$$V(\phi) = -\mu^2 \sum_{i=1}^{L} a_i^2 + \lambda_1(\sum_{i=1}^{L} a_i^2)^2 + \frac{1}{2}\lambda_2(\sum_{i=1}^{L} a_i^4). \tag{3.69}$$

Taking the minimum we have:

$$\frac{\partial V}{\partial a_i}\bigg|_{a_i=\langle a_i\rangle} = 2\langle a_i\rangle \left[-\mu^2 + 2\lambda_1(\sum_{j=1}^{L}\langle a_j^2\rangle) + \lambda_2\langle a_i^2\rangle\right] = 0, \qquad i = 1, \cdots L. \tag{3.70}$$

As in the vector representation. if $\mu^2 > 0$ then $\langle a_i\rangle = 0$ for all i is a maximum of the potential. Thus, we look for solutions such that a component of the vev is different from zero ,$\langle a_i\rangle \neq 0$ for some i. This is the case where spontaneous symmetry breaking happens.

Suppose we have $K \leq L$ values different than zero. Then:

$$\left[-\mu^2 + 2\lambda_1(\sum_{j=1}^{L}\langle a_j^2\rangle) + \lambda_2\langle a_i^2\rangle\right] = 0, \qquad i = 1, \cdots, K. \tag{3.71}$$

We have K equations of the last type. If we now subtract two of them, for example for $r, s \leq K$, we find:

$$\langle a_r^2\rangle = \langle a_s^2\rangle \tag{3.72}$$

and this can be done for all equations so all squared coefficients are the same:

$$\langle a_i^2\rangle = \frac{\mu^2}{2\lambda_1 K + \lambda_2} \equiv a^2, \qquad i = 1, \cdots K \tag{3.73}$$

Since $\mu^2 > 0$ we obtain an important inequality:

$$2\lambda_1 K + \lambda_2 > 0 \tag{3.74}$$

This condition is associated to the stability of the potential. To see this, use the fact that all the components $\langle a_i \rangle$ of the vev $\langle \Sigma \rangle$ are the same, replace it in the potential, and derive. We have:

$$\frac{\partial V}{\partial a} = 2aK \left[-\mu^2 + 2\lambda_1 K a^2 + \lambda_2 a^2 \right]. \tag{3.75}$$

Deriving again and imposing that we are at the minimum, $\frac{d^2V}{da^2} > 0$ we have:

$$\frac{d^2V}{da^2} = -2K\mu^2 + 12K^2\lambda_1 a^2 + 6K\lambda_2 a^2 > 0. \tag{3.76}$$

This implies:

$$3a^2(2\lambda_1 K + \lambda_2) > \mu^2 > 0. \tag{3.77}$$

Showing consistency with (3.74) .
The potential at the minimum is dependent on the K entries different than zero:

$$V(K) = -\frac{1}{2}\frac{K\mu^4}{2\lambda_1 K + \lambda_2}. \tag{3.78}$$

We now want to know how many $\langle a_i \rangle$'s are different from zero such that the last equation is minimized, that is, to find K with $1 \leq K \leq L$ such that Eq.(3.78) is minimized. For this analysis we treat K as a continuous variable. The derivative with respect to K is:

$$\frac{\partial V}{\partial K} = -\frac{1}{4}\frac{\lambda_2 \mu^4}{(2\lambda_1 K + \lambda_2)^2}. \tag{3.79}$$

First note from last equation that the potential is monotonic since it does not have any extrema for K i.e. $\frac{\partial V}{\partial K} = 0$ does not have any finite solutions. We also see that the sign derivative depends only the sign of λ_2. We shall analyze two cases.

Vev in the case $\lambda_2 < 0$

When $\lambda_2 < 0$ we have $\frac{\partial V}{\partial K} > 0$ so the potential is monotonically increasing thus the minimum is at the smallest value for$K$, $K = 1$. So we have the solution:

$$\langle \Sigma \rangle = b \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ & & 0 \end{pmatrix} \tag{3.80}$$

where, using Eq.(3.73) with $K = 1$:

$$b = \sqrt{\frac{\mu^2}{2\lambda_1 + \lambda_2}}. \tag{3.81}$$

Vev in the case $\lambda_2 > 0$

The case when $\lambda_2 > 0$ we see that the potential monotically decreasing so the minimum is at the largest allowed value for K, $K = L$.

$$\langle \Sigma \rangle = a \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, n = 2L. \tag{3.82}$$

$$\langle \Sigma \rangle = a \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix}, n = 2L + 1. \tag{3.83}$$

where using Eq.(3.73) with $K = L$ we have:

$$a = \sqrt{\frac{\mu^2}{2L\lambda_1 + \lambda_2}}. \tag{3.84}$$

The precedent reasoning can be clearly visualized in Figure (3.1). To minimize V for $\lambda_2 > 0$ we need the largest possible K and for $\lambda_2 < 0$ the smallest.

Derivation of the symmetry breaking pattern

The calculation of the symmetry breaking pattern will depend on λ_2 but the procedure to calculate the number of massive gauge bosons is very similar in both cases. Since the fields Σ and ϕ are related by an orthogonal transformations both give the same kinetic term thus give masses to the gauge bosons in the same way thus is sufficent to use the vev of Σ. Then from the Higgs Mechanism each Goldstone Boson is converted into the longitudinal degree of freedom of a massive gauge vector and it is also seen the number of gauge bosons that remain massless, thus showing the new lower symmetry. This new symmetry group is the little group

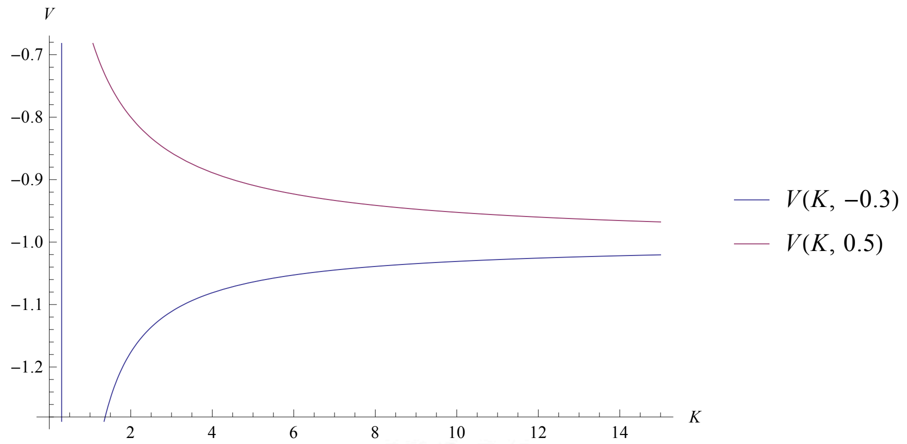


Figure 3.1 The Potential at the vev, $V(K,\lambda_2)$, for $\lambda_2 = -0.3$ and $\lambda_2 = 0.5$.

of $\langle\Sigma\rangle$. Then, as outlined before, to get the massive gauge bosons we insert the vev on the mass Lagrangian $\mathcal{L}_M$.
The vev of Σ is:

$$\langle\Sigma_{kj}\rangle = c\sum_{l=0}^{K-1}\left(\delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1}\right). \tag{3.85}$$

Where $K = 1$ and $c = b$ in the case $\lambda_2 < 0$ and $K = L$ and $c = a$ when $\lambda_2 > 0$, (see Eq.(3.80 and Eq.(3.82 respectively).
We start from Eq.(3.61). Lets consider the first term and insert the vev $\Sigma \to \langle\Sigma\rangle$:

$$\begin{aligned}\mathcal{L}_{M_1} =& g^2c^2W^{\mu}_{ik}(\sum_{l=0}^{K-1}\delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1})W_{\mu i\,k'}(\sum_{l'=0}^{K-1}\delta_{k'\,2l'+1}\delta_{j\,2l'+2} - \delta_{k'\,2l'+2}\delta_{j\,2l'+1})\\ =& g^2c^2W^{\mu}_{i\,k}W_{\mu i\,k'}c^2\sum_{l,l=0}^{K-1}[\delta_{k\,2l+1}\delta_{k'\,2l'+1}\delta_{2l+2\,2l'+2} + \delta_{k\,2l+2}\delta_{k'\,2l+2}\delta_{2l+1\,2l'+1}\\ &-\delta_{k\,2l+1}\delta_{k'\,2l'+2}\delta_{2l+2\,2l'+2} - \delta_{l\,2l+1}\delta_{k'\,2l+1}\delta_{2l+1\,2l'+2}].\end{aligned} \tag{3.86}$$

But since $\delta_{2l+1\,2l'+2}$ always vanishes since l, l' are integers, we have:

$$\mathcal{L}_{M_1} = g^2c^2\sum_{l=0}^{K-1}\left(W^{\mu}_{i\,2l+1}W_{\mu\,i\,2l+1} + W^{\mu}_{i\,2l+2}W_{\mu\,i\,2l+2}\right). \tag{3.87}$$

For the second part we have:

$$\begin{aligned}\mathcal{L}_{M_2} =& g^2 W^{\mu}_{ik} c^2 (\sum_{l=0}^{K-1} \delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1}) W_{\mu j\,k'} (\sum_{l'=0}^{K-1} \delta_{k'\,2l'+1}\delta_{i\,2l'+2} - \delta_{k'\,2l'+2}\delta_{i\,2l'+1}) \\ =& - g^2c^2 \sum_{l,l'=0}^{K-1} (W^{\mu}_{2l'+2\,2l+1} W_{\mu\,2l+2\,2l'+1} + W^{\mu}_{2l'+1\,2l+2} W_{\mu\,2l+1\,2l'+2} - W^{\mu}_{2l'+1\,2l+1} W_{\mu\,2l+2\,2l'+2} \\ & - W^{\mu}_{2l'+2\,2l+2} W_{\mu\,2l+1\,2l'+1}) \\ =& 2g^2c^2 \sum_{l,l'=0}^{K-1} \left(W^{\mu}_{2l+1\,2l'+1} W_{\mu\,2l'+2\,2l+2} - W^{\mu}_{2l+1\,2l'+2} W_{\mu\,2l'+1\,2l+2} \right) .\end{aligned} \tag{3.88}$$

Example with $O(4)$

We first start with an example then show the general way to derive the symmetry breaking pattern for both cases. Let's start first with the case where the initial symmetry is $O(4)$. This group has 6 generators, so 6 gauge bosons.

If we have $\lambda_2 < 0$, then $K = 1$ and $c = b$. Thus:

$$\mathcal{L}_{M_1} = g^2b^2 \left(W^{\mu}_{i\,1} W_{\mu\,i\,1} + W^{\mu}_{i\,2} W_{\mu\,i\,2} \right) = g^2b^2 (2W^2_{12} + W^2_{13} + W^2_{14} + W^2_{23} + W^2_{24}) \tag{3.89}$$

and

$$\mathcal{L}_{M_2} = -2g^2b^2 W^{\mu}_{1\,2} W_{\mu\,1\,2}. \tag{3.90}$$

So:

$$\mathcal{L}_M = g^2b^2 (W^2_{13} + W^2_{14} + W^2_{23} + W^2_{24}). \tag{3.91}$$

Four vectors acquire mass, so we have $6 - 4 = 2$ massless gauge bosons. Since there is not a simple algebra of dimension two the unbroken generators have to satisfy the algebra of $O(2) \times SO(2)$. Note that since $SO(2) \backsim O(2)/\mathbb{Z}_2$ we can't have $O(2) \times O(2)$ or $SO(2) \times SO(2)$ as the stability group for parity reasons (we have just one $\mathbb{Z}_2$) although all those groups have the same Lie Algebra. So, the symmetry breaking pattern is $O(4) \to O(2) \times SO(2)$.

If we have $\lambda_2 > 0$ then $K = 2$ and $c = a$. Then:

$$\begin{aligned}\mathcal{L}_{M_1} =& g^2a^2 \sum_{l=0}^{1} \left(W^{\mu}_{i\,2l+1} W_{\mu\,i\,2l+1} + W^{\mu}_{i\,2l+2} W_{\mu\,i\,2l+2} \right) \\ =& 2g^2a^2 \left(W^2_{12} + W^2_{13} + W^2_{14} + W^2_{23} + W^2_{24} + W^2_{34} \right).\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}\mathcal{L}_{M_2} =& 2g^2a^2 \sum_{l,l'=0}^{1} \left(W^{\mu}_{2l+1\,2l'+1} W_{\mu\,2l'+2\,2l+2} - W^{\mu}_{2l+1\,2l'+2} W_{\mu\,2l'+1\,2l+2} \right) \\ =& 2g^2a^2 \left(2W^{\mu}_{14} W_{\mu\,23} - 2W^{\mu}_{13} W_{\mu\,24} - W^2_{12} - W^2_{34} \right) .\end{aligned} \tag{3.93}$$

Then

$$\begin{aligned}\mathcal{L}_M =&2g^2a^2\left(W_{13}^2+W_{14}^2+W_{23}^2+W_{24}^2+2W_{14}^{\mu}W_{\mu\,23}-2W_{13}^{\mu}W_{\mu\,24}\right)\\ =&2g^2a^2\left((W_{\mu 13}-W_{\mu\,24})^2+(W_{\mu 14}^2+W_{\mu 23})^2\right)\end{aligned} \tag{3.94}$$

We see that only two linear combinations of gauge bosons acquire mass. So we have $6-2=4$ massless gauge bosons. Then, the stability group has 4 generators and corresponds to $U(2)$. The symmetry breaking pattern is then:

$$O(4)\to U(2) \tag{3.95}$$

General Case for $\lambda_2<0$

In the general case when $\lambda_2<0$ we have:

$$\begin{aligned}\mathcal{L}_M =&g^2b^2\left(W_{i\,1}^{\mu}W_{\mu\,i\,1}+W_{i\,2}^{\mu}W_{\mu\,i\,2}-2W_{1\,2}^{\mu}W_{\mu\,1\,2}\right)\\ =&g^2b^2\sum_{i=3}^{n}\left(W_{i\,1}^{\mu}W_{\mu\,i\,1}+W_{i\,2}^{\mu}W_{\mu\,i\,2}\right).\end{aligned} \tag{3.96}$$

And so we have $n-2+(n-2)=2n-4$ bosons that acquire mass thus the number of massless bosons is:

$$\frac{n(n-1)}{2}-(2n-4)=n^2-\frac{5}{2}n+4=\frac{(n-2)(n-3)}{2}+1. \tag{3.97}$$

So we have the breaking pattern:

$$O(n)\to O(n-2)\times SO(2). \tag{3.98}$$

General Case for $\lambda_2>0$

When $\lambda_2>0$ we have $K=L$:

$$\begin{aligned}\mathcal{L}_M =&g^2c^2\sum_{l=0}^{L-1}\left(W_{i\,2l+1}^{\mu}W_{\mu\,i\,2l+1}+W_{i\,2l+2}^{\mu}W_{\mu\,i\,2l+2}\right)+2g^2c^2\sum_{l,l'=0}^{L-1}\Big(W_{2l+1\,2l'+1}^{\mu}W_{\mu\,2l'+2\,2l+2}\\ &-W_{2l+1\,2l'+2}^{\mu}W_{\mu\,2l'+1\,2l+2}\Big)\end{aligned} \tag{3.99}$$

$$\begin{aligned}\mathcal{L}_M =& g^2a^2 \sum_{l,l'=0}^{L-1} \Big(W^{\mu}_{2l'+1\,2l+1} W_{\mu\,2l'+1\,2l+1} + W^{\mu}_{2l'+1\,2l+2} W_{\mu\,2l'+1\,2l+2} + W^{\mu}_{2l'+2\,2l+1} W_{\mu\,2l'+2\,2l+1} \\ & + W^{\mu}_{2l'+2\,2l+2} W_{\mu\,2l'+2\,2l+2} \Big) + 2g^2a^2 \sum_{l,l'=0}^{L-1} \Big(W^{\mu}_{2l+1\,2l'+1} W_{\mu\,2l'+2\,2l+2} - W^{\mu}_{2l+1\,2l'+2} W_{\mu\,2l'+1\,2l+2} \Big) \\ & + g^2a^2\delta_{n\;2L+1} \sum_{l=0}^{L-1} \Big(W^{\mu}_{2L+1\,2l+1} W_{\mu 2L+1\,2l+1} + W^{\mu}_{2L+1\,2l+2} W_{\mu\,2L+1\,2l+2} \Big) \\ =& g^2a^2 \sum_{l,l'=0}^{L-1} \Big[(W^{\mu}_{2l'+1\,2l+1} + W^{\mu}_{2l'+2\,2l+2})^2 + (W^{\mu}_{2l'+1\,2l+2} - W^{\mu}_{2l'+2\,2l+1})^2 \Big] \\ & + g^2a^2\delta_{n\;2L+1} \sum_{l=0}^{L-1} \Big(W^{\mu}_{2L+1\,2l+1} W_{\mu 2L+1\,2l+1} + W^{\mu}_{2L+1\,2l+2} W_{\mu\,2L+1\,2l+2} \Big) .\end{aligned} \tag{3.100}$$

First let's suppose we are in the even case: $n = 2L$, then last term is null. Remembering that each gauge vector with same indexes is null, $W_{\mu ii} = 0$, we see that in the first term there are $\frac{L(L-1)}{2}$ linear combinations of gauge bosons that acquire mass and in the second another set with the same number. So we have in total $L(L-1)$ massive vectors. The number of massless gauge bosons is then:

$$\frac{(2L)(2L-1)}{2} - L(L-1) = L^2. \tag{3.101}$$

So the symmetry breaking pattern is:

$$O(2L) \to U(L). \tag{3.102}$$

In the case $n = 2L+1$ we acquire additional $2L$ massive vectors from the last term of Eq.(3.99) thus:

$$\frac{(2L+1)(2L+1-1)}{2} - L(L-1) - 2L = L^2. \tag{3.103}$$

So the symmetry breaking pattern is still:

$$O(2L+1) \to U(L). \tag{3.104}$$

3.3.3 Spontaneous Breaking in the second rank Symmetric Representation

Again, we delay the demonstration on how to retrieve the covariant derivative and transformation law for this representation until section 3.6.
The symmetric representation ϕ_{ij} has the property:

$$\phi_{ij} = \phi_{ji}. \tag{3.105}$$

An infinitesimal transformation in this representation is:

$$\phi_{ij} \to \phi'_{ij} = \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik}. \tag{3.106}$$

The covariant derivative is:

$$\mathcal{D}_\mu \phi_{ij} = \partial_\mu \phi_{ij} - gW^\mu_{ik}\phi_{kj} - gW_{\mu jk}\phi_{ik}. \tag{3.107}$$

The kinetic term is:

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& \frac{1}{2}\partial^\mu \phi_{ij}\partial_\mu \phi_{ij} - g(W^\mu_{ik}\phi_{kj} + W^\mu_{jk}\phi_{ik})\partial_\mu \phi_{ik} \\ &+ g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj}) + g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{li}).\end{aligned} \tag{3.108}$$

Note that the vector boson masses will come from the terms proportional to g^2

$$\mathcal{L}_M = g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj}) + g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{li}). \tag{3.109}$$

The most general quartic potential for a traceless symmetric second order tensor representation of $O(n)$, ϕ' is:

$$\begin{aligned}V(\phi') =& -\frac{1}{2}\mu^2 \operatorname{Tr}[\phi'^\dagger\phi'] + \frac{1}{4}\lambda_1(\operatorname{Tr}[\phi'^\dagger\phi'])^2 + \frac{1}{4}\lambda_2 \operatorname{Tr}[\phi'^\dagger\phi'\phi'^\dagger\phi'] + \frac{1}{3}\lambda_3 \operatorname{Tr}[\phi'^\dagger\phi'\phi'^\dagger] \\ =& -\frac{1}{2}\mu^2 \operatorname{Tr}[\phi'^2] + \frac{1}{4}\lambda_1(\operatorname{Tr}[\phi'^2])^2 + \frac{1}{4}\lambda_2 \operatorname{Tr}[\phi'^4] + \frac{1}{3}\lambda_3 \operatorname{Tr}[\phi'^3].\end{aligned} \tag{3.110}$$

Ulteriorly, the condition $\operatorname{Tr}[\phi'] = 0$ is added as a Lagrange multiplier otherwise we don't have an irreducible symmetric representation [4].
It is very well known that a real and symmetric matrix can be diagonalized such that $\phi' = O\phi O^T$ with $\phi_{ij} = \delta_{ij}\phi_i$ [33], then we have:

$$V(\phi) = -\frac{1}{2}\mu^2\sum_{i=1}^n \phi_i^2 + \frac{1}{4}\lambda_1(\sum_{i=1}^n \phi_i^2)^2 + \frac{1}{4}\lambda_2\sum_{i=1}^n \phi_i^4 + \frac{1}{3}\lambda_3\sum_{i=1}^n \phi_i^3 - \lambda_0\sum_{i=1}^n \phi_i. \tag{3.111}$$

For simplicity we impose an additional restriction, parity symmetry in the potential. To satisfy this symmetry, terms proportional to λ_3 are taken away. Since adding a Lagrangian multiplier has a different origin, is a restriction on the representation, and is imposed after the definition of the symmetry of the potential,the restriction of parity symmetry does not apply to this term. We need to find the minimum of this potential with respect to ϕ_i and λ_0 .
The zero trace condition is given by:

$$\frac{\partial V}{\partial \lambda_0} = 0. \tag{3.112}$$

For the other variables ϕ_i we have:

$$\frac{\partial V}{\partial \phi_i}\Big|_{\phi_i=\langle\phi_i\rangle} = -\mu^2\langle\phi_i\rangle + \lambda_1(\sum_{j=1}^{n}\langle\phi_j\rangle^2)\langle\phi_i\rangle + \lambda_2\langle\phi_i\rangle^3 - \lambda_0 = 0, \qquad i = 1,\cdots,n \quad (3.113)$$

We now show that there are at most three different values of ϕ_i's, $\phi_1 \neq \phi_2 \neq \phi_3$. Let's suppose this is true then:

$$-\mu^2\phi_1 + \lambda_1(\sum_{j=1}^{n}\langle\phi_j\rangle^2)\phi_1 + \lambda_2\phi_1^3 - \lambda_0 = 0 \quad (3.114)$$

$$-\mu^2\phi_2 + \lambda_1(\sum_{j=1}^{n}\langle\phi_j\rangle^2)\phi_2 + \lambda_2\phi_2^3 - \lambda_0 = 0 \quad (3.115)$$

$$-\mu^2\phi_3 + \lambda_1(\sum_{j=1}^{n}\langle\phi_j\rangle^2)\phi_3 + \lambda_2\phi_3^3 - \lambda_0 = 0. \quad (3.116)$$

We subtract Eq.(3.115) from Eq.(3.114) arriving at:

$$\left[-\mu^2 + \lambda_1(\sum_{j=1}^{n}\langle\phi_j\rangle^2 + \lambda_2(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right](\phi_1 - \phi_2) = 0. \quad (3.117)$$

Since $\phi_1 \neq \phi_2$ we have:

$$\left[-\mu^2 + \lambda_1\sum_{j=1}^{n}\langle\phi_j\rangle^2 + \lambda_2(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right] = 0. \quad (3.118)$$

From the subtraction of Eq.(3.116) from Eq.(3.114) we have:

$$\left[-\mu^2 + \lambda_1\sum_{j=1}^{n}\langle\phi_j\rangle^2 + \lambda_2(\phi_1^2 + \phi_1\phi_3 + \phi_3^2)\right] = 0. \quad (3.119)$$

Then subtracting last two equations we have:

$$(\phi_1 + \phi_2 + \phi_3)(\phi_2 - \phi_3) = 0, \quad (3.120)$$

that implies:

$$\phi_1 + \phi_2 + \phi_3 = 0. \quad (3.121)$$

Assuming there exists another value ϕ_4 we have following the procedure of before:

$$\phi_4 + \phi_2 + \phi_3 = 0. \quad (3.122)$$

Then from these last equations $\phi_1 = \phi_4$. So we have shown that the vev of ϕ is composed of three different values ϕ_1, ϕ_2, ϕ_3 each n_1, n_2, n_3 times present.

The vev of ϕ is:

$$\langle\phi\rangle = \begin{pmatrix} \phi_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & \phi_1 & & & & & & \\ & & & \phi_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \phi_2 & & & \\ & & 0 & & & & \phi_3 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \phi_3 \end{pmatrix}. \tag{3.123}$$

However, these three ϕ_i are not independent. Summarizing we have the following equations:

$$\begin{aligned} n_1 + n_2 + n_3 =& n \\ \phi_1 + \phi_2 + \phi_3 =& 0 \\ n_1\phi_1 + n_2\phi_2 + n_3\phi_3 =& 0 \qquad (\text{ Trace condition}). \end{aligned} \tag{3.124}$$

Solving ϕ_2 and ϕ_3 in terms of ϕ_1 we have:

$$\phi_2 = \frac{n_3 - n_1}{n_2 - n_3}\phi_1 \equiv \rho_2\phi_1 \tag{3.125}$$

$$\phi_3 = \frac{n_2 - n_1}{n_3 - n_2}\phi_1 \equiv \rho_3\phi_1. \tag{3.126}$$

Notice that from the trace condition doesn't allow $n_2 - n_3 = 0$. Inserting the vev in the potential we have in terms of ϕ_1 :

$$V_m = -a\phi_1^2 + b\phi_1^4. \tag{3.127}$$

Defining:

$$Y_k(n_1, n_2, n_3) = n_1(n_2 - n_3)^k + n_2(n_3 - n_1)^k + n_3(n_1 - n_2)^k. \tag{3.128}$$

For k even this function is positive and invariant under permutations of the n_i's. If we also define:

$$X = (n_2 - n_3)^2. \tag{3.129}$$

We have:

$$a = \frac{\mu^2}{2}\frac{Y_2}{X} > 0. \tag{3.130}$$

$$b = \frac{1}{4}\frac{\left[\lambda_1(Y_2)^2 + \lambda_2 Y_4\right]}{X^2}. \tag{3.131}$$

Since Eq.(3.127) is the lowest value of the potential, thus a minimum, we have:

$$\frac{\partial V}{\partial \phi_1} = \phi_1(-2a + 4b\phi_1^2) = 0. \tag{3.132}$$

The solution is:

$$\phi_1^2 = \frac{a}{2b}. \tag{3.133}$$

Then, making explicit the fact that we are dealing with a vev:

$$\langle\phi_1\rangle^2 = \frac{\mu^2(n_2 - n_3)^2}{\lambda_1 Y_2 + \lambda_2 \frac{Y_4}{Y_2}} \tag{3.134}$$

We now have the vev as a function of the parameters of the Lagrangian $(\lambda_1, \lambda_2, \mu)$ and the three n_i's. At this point we can get the stability condition, imposing $\frac{\partial^2 V_m}{\partial \phi_1^2} > 0$. This gives $6b > a$ that will give an inequality relating the λ's and the n's. Explicitly, from the vev, expressing it in function of ϕ_1 and inserting in the potential we have:

$$V(\phi_1) = -\frac{1}{2}\mu^2(n_1 + n_2\rho_2^2 + n_3\rho_3^2)\phi_1^2 + \frac{1}{4}\lambda_1(n_1 + n_2\rho_2^2 + n_3\rho_3^2)^2\phi_1^4 + \frac{1}{4}\lambda_2(n_1 + n_2\rho_2^4 + n_3\rho_3^4)\phi_1^4 \tag{3.135}$$

Imposing the minimum condition $\frac{\partial^2 V}{\partial \phi_1^2} > 0$ we arrive at the stability condition:

$$\phi_1^2\left((n_1 + n_2\rho_2^2 + n_3\rho_3^2)\lambda_1 + \frac{(n_1 + n_2\rho_2^4 + n_3\rho_3^4)}{(n_1 + n_2\rho_2^2 + n_3\rho_3^2)}\lambda_2\right) > \frac{1}{3}\mu^2 > 0 \tag{3.136}$$

We can now proceed to obtain the values of n_i. The value of the potential at the vev using Eq. (3.133) is:

$$\begin{aligned} V_m(n_1, n_2, n_3) &= -\frac{a^2}{4b} = -\frac{\mu^4}{4}\frac{1}{\lambda_1 + \lambda_2 \frac{Y_4}{(Y_2)^2}} \\ &= -\frac{\mu^4}{4}\frac{1}{\lambda_1 + \lambda_2 f(n_1, n_2, n_3)} \end{aligned} \tag{3.137}$$

with $f \equiv \frac{Y_4}{(Y_2)^2} > 0$ since is a ratio of positive functions and is monotonic and such that $f \sim 1/n$. To start with the analysis we fix $\lambda_1 > 0$ otherwise the results we get are symmetrical with respect to the sign of λ_2. We need to calculate configurations (n_1, n_2, n_3) such that the potential is minimal. This is equivalent to minimizing V over the $\mathbb{R}^3$ space of the n's with the constrain condition $n_1 + n_2 + n_3 = n$ thus is in fact minimizing the potential in two independent variables. Using f as an independent variable we derive:

$$\frac{\partial V_m}{\partial f} = \frac{\mu^4}{4}\frac{\lambda_2}{[\lambda_1 + \lambda_2 f(n_1, n_2, n_3)]^2} \tag{3.138}$$

Then, when $\lambda_2 > 0$, V_m is monotonically increasing with respect to f so the smallest value of V_m is when f is minimum. If $\lambda_2 < 0$ the V_m is monotonically decreasing and smallest is when f is maximum. Since $f \sim 1/n$ we retrieve a similar plot as 3.1 with n as K before, but in this case the only thing that this analysis shows is what to do in each case of the sign of λ_2 since we still to find the configurations of n_1 and n_2 to get the extremum.
Using the following identity:

$$Y_4 = \frac{1}{2}Y_2\left((n_2-n_3)^2+(n_1-n_2)^2+(n_1-n_3)^2\right). \tag{3.139}$$

We have:

$$f = \frac{1}{2Y_2}\left((n_2-n_3)^2+(n_1-n_2)^2+(n_1-n_3)^2\right). \tag{3.140}$$

Note that f is invariant under permutations of the n_i's. We then change variables:

$$x = n_1 + n_2 \tag{3.141}$$
$$y = n_1 - n_2 \tag{3.142}$$
$$n_3 = n - x. \tag{3.143}$$

So the result is:

$$f(x,y) = \frac{3y^2+(3x-2n)^2}{(8n-9x)y^2+x(3x-2n)^2}, \tag{3.144}$$

with domain:

$$0 \leq x \leq n \tag{3.145}$$
$$-n \leq y \leq n. \tag{3.146}$$

Note that f is even in y so it is enough to study when:

$$0 \leq y \leq n \tag{3.147}$$

Also, using last equation and Eq.(3.145) we have:

$$0 < x - y \leq n \tag{3.148}$$

Note that this implies that the possible values of $g(c) = x - y = c$ are discrete ($c = 0, \cdots n$).

In order to understand how to maximize or minimize f, we take the partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{8y(3x-2n)^3}{\left(y^2(8n-9x)+x(2n-3x)^2\right)^2} \tag{3.149}$$

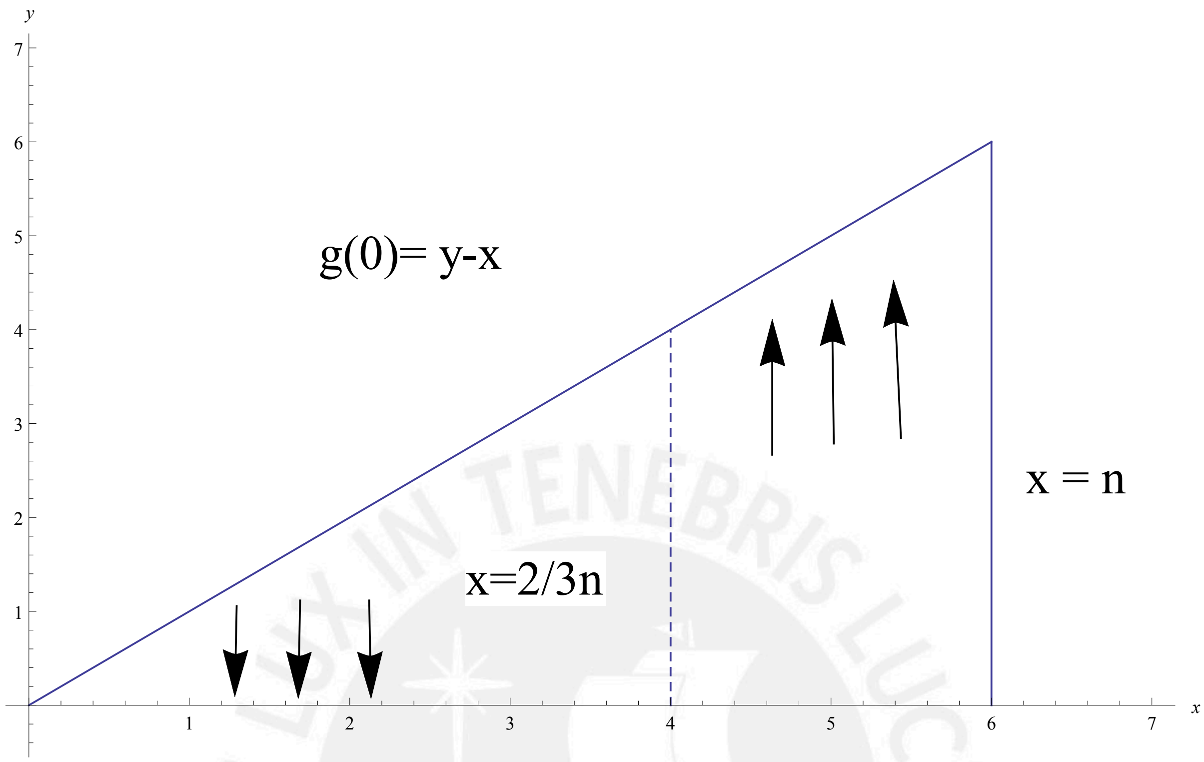


Figure 3.2 Domain of the function $f(x,y)$ in Eq.(3.144) with the arrows denoting the direction of the monotonical increments of $f(x,y)$ in the y axis. In the plot $n = 6$

From Eq.(3.149) we have that:

$$\begin{aligned} \frac{\partial f}{\partial y} < 0 \quad \text{for} \quad x < \frac{2n}{3} \\ \frac{\partial f}{\partial y} > 0 \quad \text{for} \quad x > \frac{2n}{3} \end{aligned} \tag{3.150}$$

Thus for all possible cases we have the following table:

	Maximize f $(\lambda_2 < 0)$	Minimize f $(\lambda_2 > 0)$
$\frac{\partial f}{\partial y} < 0$	Max y	Min y
$\frac{\partial f}{\partial y} > 0$	Min y	Max y

Table 3.1 Cases to minimize the Symmetric Orthogonal Potential at the vev, V_m

So when we want to minimize y the solutions have to satisfy the condition $y = n_1 - n_2 = 0$. When we want to maximize y the largest value can't be n (equivalently $n_1 = n$) for boundary reasons thus have to be in the boundary defined by Eq.(3.148) that is when $y = x$ or $n_2 = 0$.

Case $\lambda_2 > 0$

We want to minimize f. Following Table (3.1) when $x < \frac{2n}{3}$ (equivalently $\frac{\partial f}{\partial y} < 0$) the minimum has to be on the upper boundary, that is, has to satisfy Eq.(3.148) with $c = 0$ thus the minimum will satisfy the equation $y = x$. We still need to get the value x of the minimum. Expressing the values of f restricted to this line we have:

$$f(x,x) = \frac{3x^2 - 3nx + n^2}{xn(n-x)}. \tag{3.151}$$

The minimum is the solution of:

$$\frac{\partial f}{\partial x} = n^3 \frac{(2x-n)}{(xn(n-x))^2} = 0. \tag{3.152}$$

Solving last equation, the minimum is when:

$$x = \frac{n}{2} = y. \tag{3.153}$$

So:

$$n_1 = \frac{n}{2}, \quad n_2 = 0, \quad n_3 = \frac{n}{2}. \tag{3.154}$$

When $x > \frac{2n}{3}$ from Table (3.1) we have $y = 0$ and $\frac{2}{3} < x < n$. Thus we have:

$$f(x,0) = \frac{1}{x} \tag{3.155}$$

Since $f(x,0)$ is monotonically decreasing with respect to x it is lowest at the largest value of x, so the minimum is at:

$$x = n, \qquad y = 0 \tag{3.156}$$

Thus we have:

$$n_1 = n_2 = \frac{n}{2}, \quad n_3 = 0 \tag{3.157}$$

Since $f(n_1, n_2, n_3)$ is symmetric under permutation of the variables we have that the points given in Eq.(3.154) and Eq.(3.157) that minimize each region give the same value of f and so are equivalent. So the global minimum is then:

$$n_1 = \frac{n}{2}, \quad n_2 = \frac{n}{2} \quad n_3 = 0 \qquad n \text{ even} \tag{3.158}$$

In the case where n is odd since it is not permitted to have a minimum at $n_1 = \frac{n}{2}, n_2 = \frac{n}{2}$, the minimum is then in the nearest allowed point:

$$n_1 = \frac{n+1}{2}, \quad n_2 = \frac{n-1}{2}, \quad n_3 = 0 \qquad n \text{ odd} \tag{3.159}$$

which is a global minimum in the odd case.

Case $\lambda_2 < 0$

When $\lambda_2 < 0$ we need to find set of values such that $f(x, y)$ is maximum. From Table (3.1) and from the precedent analysis we need to find them in the neighborhood of $x = y = n$ when $x > \frac{2n}{3}$ and $x = y = 0$ for $x < \frac{2n}{3}$. Using the analytic form of $f(x, y)$ the values of $f(0, 0)$ and $f(n, n)$ diverge. The next maximum value is $f(n, n-1)$ but we can't use it since it gives non integers values for the n_1, n_2 i.e. $n_1 = \frac{2n-1}{2}$. Thus the following allowed point is:

$$x = n, \quad y = n - 2 \tag{3.160}$$

and this is the maximum. This can be written as:

$$n_1 = n - 1, \quad n_2 = 1, \quad n_3 = 0 \tag{3.161}$$

We can try another point $x = y = n - 1$, which gives $n_1 = n - 1, n_2 = 0, n_3 = 1$ and since $f(n_1, n_2, n_3)$ is invariant under permutations of the n_i's it gives the same result as $f(n, n-2)$.

Derivation of the symmetry breaking pattern

To calculate the exact pattern we need to use Eq.(3.49) with the vev:

$$\phi = \begin{pmatrix} \phi_1 & & & & & 0 \\ & \ddots & & & & \\ & & \phi_1 & & & \\ & & & \phi_2 & & \\ & & & & \ddots & \\ & & & & & \phi_2 \end{pmatrix} \tag{3.162}$$

Where there is a diagonal submatrix of dimension n_1 with ϕ_1 and another submatrix of dimension $n - n_1$ with ϕ_2. n_1 will depend on the sign of λ_2. Equivalently:

$$\langle \phi_{kj} \rangle = \phi_1 \sum_{l=1}^{n_1} \delta_{kl}\delta_{jl} + \phi_2 \sum_{l=n_1+1}^{n} \delta_{kl}\delta_{jl}. \tag{3.163}$$

Using Eq.(3.134) we have:

$$\phi_1^2 = \frac{\mu^2(n - n_1)^2}{n^2(\lambda_1 n_1 + \lambda_2) - n n_1(\lambda_1 n_1 + 3\lambda_2) + 3\lambda_2 n_1^2}. \tag{3.164}$$

From Eq.(3.109) we have:

$$\mathcal{L}_{M1} = g^2 W^{\mu}_{ik} \left(\phi_1 \sum_{l=1}^{n_1} \delta_{kl}\delta_{jl} + \phi_2 \sum_{l=n_1+1}^{n} \delta_{kl}\delta_{jl} \right) W_{\mu ik'} \left(\phi_1 \sum_{l=1}^{n_1} \delta_{k'l'}\delta_{jl'} + \phi_2 \sum_{l'=n_1+1}^{n} \delta_{k'l'}\delta_{jl'} \right). \tag{3.165}$$

Since:

$$\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n}\delta_{kl}\delta_{jl}\delta_{k'l'}\delta_{jl'}=\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n}\delta_{kl}\delta_{k'l'}\delta_{ll'},\tag{3.166}$$

we see from the set of indices in the summation that $\delta_{ll'}=0$ always so terms proportional to $\phi_1\phi_2$ in Eq.(3.165) are null. This leaves us with:

$$\begin{aligned}\mathcal{L}_{M1}=&g^2W^\mu_{ik}\langle\phi_{kj}\rangle W_{\mu ik'}\langle\phi_{k'j}\rangle\\=&g^2W^\mu_{ik}W_{\mu ik'}\left(\phi_1^2\sum_{l,l'=1}^{n_1}\delta_{kl}\delta_{jl}\delta_{k'l'}\delta_{jl'}+\phi_2^2\sum_{l,l'=n_1+1}^{n}\delta_{kl}\delta_{jl}\delta_{k'l'}\delta_{jl'}\right)\\=&g^2\phi_1^2\sum_{l=1}^{n_1}W^\mu_{il}W_{\mu il}+g^2\phi_2^2\sum_{l=n_1+1}^{n}W^\mu_{il}W_{\mu il}.\end{aligned}\tag{3.167}$$

For the other term we have:

$$\begin{aligned}\mathcal{L}_{M_2}=&g^2W^\mu_{ik}\langle\phi_{kj}\rangle W_{\mu jk'}\langle\phi_{k'i}\rangle\\=&g^2W^\mu_{ik}W_{\mu jk'}(\sum_{l,l'=1}^{n_1}\delta_{kl}\delta_{jl}\delta_{il'}\delta_{k'l'}\phi_1^2+\sum_{l,l'=n_1+1}^{n}\delta_{kl}\delta_{jl}\delta_{il'}\delta_{k'l'}\phi_2^2\\&+2\phi_1\phi_2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n}\delta_{kl}\delta_{jl}\delta_{il'}\delta_{k'l'})\\=&-g^2\phi_1^2\sum_{l,l'=1}^{n_1}W^\mu_{ll'}W_{\mu ll'}-g^2\phi_2^2\sum_{l,l'=n_1+1}^{n}W^\mu_{ll'}W_{\mu ll'}-2g^2\phi_1\phi_2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n}W^\mu_{ll'}W_{\mu ll'}\end{aligned}\tag{3.168}$$

First we give an example of the breaking of $O(4)$. We see that in the case with $\lambda_2>0$ we have that $n_1=\frac{n}{2}$. The vev is:

$$\langle\phi\rangle=\begin{pmatrix}\phi_1&&0&\\&\phi_1&&\\&0&\phi_2&\\&&&\phi_2\end{pmatrix}\tag{3.169}$$

So we have:

$$\begin{aligned}\mathcal{L}_M=&g^2\phi_1^2\sum_{l=1}^{2}W^\mu_{il}W_{\mu il}+g^2\phi_2^2\sum_{l=3}^{4}W^\mu_{il}W_{\mu il}-g^2\phi_1^2\sum_{l,l'=1}^{2}W^\mu_{ll'}W_{\mu ll'}\\&+g^2\phi_2^2\sum_{l,l'=3}^{4}W^\mu_{ll'}W_{\mu ll'}-g^2\phi_1\phi_2\sum_{l=1}^{2}\sum_{l'=3}^{4}W^\mu_{ll'}W_{\mu ll'}\\=&g^2(\phi_1-\phi_2)^2(W_{14}^2+W_{24}^2+W_{13}^2+W_{23}^2)\end{aligned}\tag{3.170}$$

Thus 4 gauge bosons acquire mass. The remaining symmetry has only two generators and the subgroup of $O(4)$ that has this number of generators is the product $O(2)\times O(2)$.

If $\lambda_2 < 0$ then $n_1 = n - 1$. The vev is:

$$\langle\phi\rangle = \begin{pmatrix} \phi_1 & & 0 & \\ & \phi_1 & & \\ 0 & & \phi_1 & \\ & & & \phi_2 \end{pmatrix} \tag{3.171}$$

So we have:

$$\begin{aligned} \mathcal{L}_M =& g^2\phi_1^2\sum_{l=1}^{3} W^{\mu}_{il}W_{\mu\, i\, l} + g^2\phi_2^2 W^{\mu}_{i4}W_{\mu\, i\, 4} - g^2\phi_1^2\sum_{l,l'=1}^{3} W^{\mu}_{ll'}W_{\mu\, l\, l'} \\ &+ g^2\phi_2^2 W^{\mu}_{44}W_{\mu\, 4\, 4} - g^2\phi_1\phi_2\sum_{l=1}^{3} W^{\mu}_{l4}W_{\mu\, l\, 4} \\ =& g^2(\phi_1-\phi_2)^2(W_{14}^2 + W_{24}^2 + W_{34}^2) \end{aligned} \tag{3.172}$$

We have that 3 gauge bosons acquire mass so the unbroken subgroup has 3 generators and thus is the group $O(3)$[6].
In general we have:

$$\begin{aligned} \mathcal{L}_M =& g^2\phi_1^2\sum_{l=1}^{n_1} W^{\mu}_{il}W_{\mu\, i\, l} + g^2\phi_2^2\sum_{l=n_1+1}^{n} W^{\mu}_{il}W_{\mu\, i\, l} - g^2\phi_1^2\sum_{l,l'=1}^{n_1} W^{\mu}_{ll'}W_{\mu\, l\, l'} - g^2\phi_2^2\sum_{l,l'=n_1+1}^{n} W^{\mu}_{ll'}W_{\mu\, l\, l'} \\ &- g^2\phi_1\phi_2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n} W^{\mu}_{ll'}W_{\mu\, l\, l'} \\ =& \phi_1^2\sum_{l=n_1+1}^{n}\sum_{l'=1}^{n_1} W^{\mu}_{ll'}W_{\mu\, l\, l'} + g^2\phi_2^2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n} W^{\mu}_{ll'}W_{\mu\, l\, l'} - g^2\phi_1\phi_2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n} W^{\mu}_{ll'}W_{\mu\, l\, l'} \\ =& (\phi_1-\phi_2)^2\sum_{l=1}^{n_1}\sum_{l'=n_1+1}^{n} W^{\mu}_{ll'}W_{\mu\, l\, l'} \end{aligned} \tag{3.173}$$

From the last equation we have $n_1(n-n_1)$ gauge bosons that acquire mass. Thus the number of unbroken generators is:

$$\frac{n(n-1)}{2} - n_1(n-n_1) = \frac{(n-n_1)(n-n_1-1)}{2} + \frac{(n_1-1)(n_1)}{2} \tag{3.174}$$

In all cases the breaking is $O(n) \to O(n-n_1)\times O(n_1)$.

[6] Note that it cannot be $U(1)\times U(1)\times U(1)$ this last group is of rank 3 and $O(3)$ is of rank 1

Summarizing we have the breaking patterns:

$$O(n) \to O(n_1) \times O(n-n_1), \quad \lambda_1 > 0, \lambda_2 > 0 \\ \text{with } n_1 = \frac{n}{2} \text{ if } n \text{ even} \\ \text{with } n_1 = \frac{n+1}{2} \text{ if } n \text{ even} \tag{3.175}$$

and

$$O(n) \to O(n-1), \quad \lambda_1 > 0, \lambda_2 < 0 \tag{3.176}$$

Here, using Eq.(3.164), we have for $\lambda_2 > 0$ in the even case:

$$\phi_1^2 = \frac{\mu^2}{n\lambda_1 + \lambda_2} \tag{3.177}$$

and in the odd case:

$$\phi_1^2 = \frac{(n-1)^2\mu^2}{n(n^2-1)\lambda_1 + (n^2+3)\lambda_2} \tag{3.178}$$

On the other hand, when $\lambda_2 < 0$:

$$\phi_1^2 = \frac{\mu^2}{(n^2-n)\lambda_1 + (3+n^2-3n)\lambda_2} \tag{3.179}$$

3.3.4 The Spinor Representation

The irreducible representations of $O(n)$ can be classified into two categories, single-valued and double-valued representations. In the previous subsections, we already have seen single valued representations up to rank 2: the fundamental, the antisymmetric and symmetric traceless representations. A characteristic of this representations is that are real thus not suitable for chiral fermions (see Section .(5.1)).

On the other hand, the double-valued representations, called also spinor representations since the fields in this representations transform like spinors in a n-dimensional coordinate space, are complex representations thus suitable for chiral fermions. In addition, using spinor fields we can retrieve new spontaneous symmetry breaking patterns for a Lagrangian with gauge symmetry $O(n)$; in fact the $SO(10)$ GUT model uses some of these representation in the Higgs sector ($\mathbf{16}, \mathbf{120}$ and $\mathbf{126}$).

From the definition of the $O(n)$ groups, the condition Eq.(3.31) is equivalent to invariance of the norm squared of an n-dimensional real vector under rotations i.e. that the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2$ is left invariant. An interesting fact happens if we write this quadratic form as the square of a linear form of x_i's.

$$x_1^2 + x_2^2 + \ldots + x_n^2 = (\gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n)^2\,, \tag{3.180}$$

To satisfy this equation the γ's necessarily have to be matrices which have to satisfy the property:

$$\{\gamma_i, \gamma_j\} = \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbf{1}_{N\times N} \qquad i,j = 1,\cdots n \tag{3.181}$$

Last equation is a $N \times N$ matrix equation. [7]In the literature, last equation is referred as the Clifford Algebra of dimension n, $Cl_n(\mathbb{R})$.

In order to show the connection of the γ's with the Lie Group $O(n)$ we define the matrix:

$$s_{ij} = \frac{1}{2}[\gamma_i, \gamma_j] \tag{3.182}$$

From Eq.(3.181) we obtain:

$$[s_{ij}, \gamma_k] = \delta_{jk}\gamma_i - \delta_{ik}\gamma_j. \tag{3.183}$$

So the s_{ij} satisfy the Lie Algebra $\mathfrak{so}(n)$ (see Eq.(3.36)) in the representation of $N\times N$ matrices:

$$[s_{ij}, s_{kl}] = \delta_{jk}s_{il} + \delta_{il}s_{jk} - \delta_{ik}s_{jl} - \delta_{jl}s_{ik} \tag{3.184}$$

This means that the matrices s_{ij}, constructed using the γ's, can be used as generators of $O(n)$.

Additionally, using the s_{ij} as generators we define the Lie Group $Spin(n)$ as the set of $N \times N$ matrices:

$$Spin(n) := \{e^{\frac{1}{2}\epsilon_{ij}s_{ij}} : \epsilon_{ij} = -\epsilon_{ji}\} \tag{3.185}$$

Note that both $O(n)$ and $Spin(n)$ have the same Lie Algebra $\mathfrak{so}(n)$ and depend on the same number of parameters $\{\epsilon\}$; in fact there exists an homomorphism between the two. It can be shown that this group is the double covering group of $O(n)$ [28]; for example for $n = 3$ we have $Spin(3) \simeq SU(2)$ and is well known that $SU(2)$ is the double cover of $SO(3)$.
To see how $Spin(n)$ induces a transformation on the γ's let's do a rotation in the fundamental representation $x' = Ox = e^{\epsilon_{ij}L_{ij}}x$ with O an element of $O(n)$.
Then the linear form 3.180 transforms as:

$$\gamma \cdot x \to \gamma \cdot x' = \gamma_i O_{ik} x_k = \gamma'_k x_k = \gamma' \cdot x \tag{3.186}$$

Correspondingly doing a transformation of $\gamma \cdot x$, as a matrix, i.e. using the same set of $\{\epsilon\}$'s for the element $S(O) = e^{\frac{1}{2}\epsilon_{ij}s_{ij}}$ of the $Spin(n)$ group, we have:

$$S(O)\gamma \cdot x S^{-1}(O) = \gamma \cdot x' \tag{3.187}$$

Infinitesimally, since $x'_i = x_i + \epsilon_{ik}x_k$ we arrive at:

$$\frac{1}{2}\epsilon_{ij}[s_{ij}, \gamma_k]x_k = \epsilon_{ab}x_b\gamma_a = \frac{1}{2}(\epsilon_{ab} - \epsilon_{ba})x_b\gamma_a = \frac{1}{2}\epsilon_{ij}x_k(\delta_{kj}\gamma_i - \delta_{ki}\gamma_j) \tag{3.188}$$

retrieving again equation (3.182).

Notice that the anticommutation relations remain unchanged, i.e.

$$\{\gamma'_i, \gamma'_j\} = O_{ik}O_{jl}\{\gamma_k, \gamma_l\} = 2O_{ik}O_{jk} = 2\delta_{ij}\,. \tag{3.189}$$

[7]It can be shown that the dimension of the γ matrices must be even.

The correspondence $O \to S(O)$ serves as a N-dimensional representation of $O(n)$ and so is called the spinor representation of $O(n)$. The objects that transform under this N dimensional representation are then called spinors. Equivalently, the spinors are defined as the Ndimensional vectors ψ_i that tranforms as the fundamental representation of $Spin(n)$. In any case the explicit transformation is :

$$\psi \to \psi' = S(O)_{ij}\psi_j \tag{3.190}$$

and these objects are called covariant spinors. Their complex conjugate , ψ_i^* , has the transformation property:

$$\psi^* \to \psi_j'^* S^{-1}(O)_{ji} \tag{3.191}$$

and are called controvariant spinors.

The dimensionality is easily seen for $SO(3)$. In this case the Clifford algebra is simply given by the three Pauli matrices, and a finite transformation looks like

$$S(O(\varphi)) = e^{\frac{i}{2}\sigma_i\varphi_i} = \cos\frac{|\varphi|}{2} + i\frac{\sigma_i\varphi_i}{|\varphi|}\sin\frac{|\varphi|}{2}\,, \tag{3.192}$$

where we have defined $\epsilon_{23} \equiv -\varphi_1$, $\epsilon_{13} \equiv -\varphi_2$, $\epsilon_{12} \equiv -\varphi_3$ and $|\varphi| = \sqrt{\varphi_1^2+\varphi_2^2+\varphi_3^2}$, showing then that for $n = 3$ the Lie groups $Spin(3)$ is conformed of 2×2 matrices.
To see the relation between N, the size of γ's matrices, and n, we start building explicit representations of the γ's inductively. We start with $n = 2K = 2$. Since the Pauli matrices satisfy the Clifford algebra

$$\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}\,, \tag{3.193}$$

we can choose two of them as the γ's.

$$\gamma_1^{(1)} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma_2^{(1)} = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.194}$$

thus showing that $N = 2 \equiv f(1)$ for $n = 2$ with $f(K)$ a function relating the the dimensionality of the group $n = 2K$ with the size of the square matrix, N. Then the γ matrices for $K > 1$ are constructed by recursion. The iteration from $2K$ to $2(K+1)$ is defined by the matrices:

$$\gamma_i^{(K+1)} = \begin{pmatrix} \gamma_i^{(K)} & 0 \\ 0 & -\gamma_i^{(K)} \end{pmatrix} \qquad \text{for} \qquad i = 1, 2, \ldots, 2K = n\,, \tag{3.195}$$

$$\gamma_{2K+1}^{(K+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_{2K+2}^{(K+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.196}$$

Starting with $n = 2$ using Eq.(3.194) in Eq.(3.195) for $n = 4$ we have $N = 4 = f(1) \times f(1) = 2^1$. For $n = 6$ we have $N = 8 = f(1) \times f(1) \times f(1) = 2^2$ so for $n = 2K$ we have $N = f(1)^K = 2^K$ and so on as can be seen using induction and the explicit formula.

Given the fact that the $\gamma_i^{(K)}$ matrices satisfy the Clifford algebra $Cl_n(\mathbb{R})$ we show that the set of $\gamma_i^{(K+1)}$ satisfy the Clifford Algebra $Cl_{n+2}(\mathbb{R})$ as well,

$$\left\{\gamma_i^{(K+1)},\gamma_j^{(K+1)}\right\} = \begin{pmatrix} \left\{\gamma_i^{(K)},\gamma_j^{(K)}\right\} & 0 \\ 0 & \left\{\gamma_j^{(K)},\gamma_i^{(K)}\right\} \end{pmatrix} = \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & 2\delta_{ij} \end{pmatrix} = 2\delta_{ij}\,, \tag{3.197}$$

$$\left\{\gamma_i^{(K+1)},\gamma_{2K+1}^{(K+1)}\right\} = \begin{pmatrix} 0 & \gamma_i^{(K)} \\ -\gamma_i^{(K)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma_i^{(K)} \\ \gamma_i^{(K)} & 0 \end{pmatrix} = 0\,, \tag{3.198}$$

$$\left(\gamma_{2K+1}^{(K+1)}\right)^2 = 1\,. \tag{3.199}$$

Analogously one finds

$$\left\{\gamma_i^{(K+1)},\gamma_{2K+2}^{(K+1)}\right\} = 2\delta_{ij}\,, \qquad \left\{\gamma_{2K+1}^{(K+1)},\gamma_{2K+2}^{(K+1)}\right\} = 0\,, \qquad \left(\gamma_{2K+2}^{(K+1)}\right)^2 = 1\,. \tag{3.200}$$

A similar construction can be done for $n = 2K+1$ odd. The size of the n matrices $\gamma_i^{(K)}$ is 2^K. The representation space of Cl_n (equivalently of $S(O)$) for n odd is irreducible.

However for $SO(n)$ groups with n even the representation $S(O)$ is not irreducible. To see this we construct the chiral projector γ_f defined by

$$\gamma_f = (-i)^n \gamma_1\gamma_2\cdots\gamma_{2n}\,. \tag{3.201}$$

γ_f anticommutes with γ_i since $2n$ is even[8] and consequently we get $\left[\gamma_f, s_{kl}\right] = 0$. Thus if ψ transforms as $\psi_i' = S(O)_{ij}\psi_j$, the positive and negative chiral components

$$\psi^+ \equiv \frac{1}{2}\left(1+\gamma_f\right)\psi \qquad \text{and} \qquad \psi^- \equiv \frac{1}{2}\left(1-\gamma_f\right)\psi \tag{3.202}$$

transform separately. In other words ψ^+ and ψ^- form two irreducible spinor representations of dimension 2^{n-1}. The relationship is analogous to the usual left - right spinors under the Lorentz Algebra.

[8]Notice that this would not be the case for $SO(2N+1)$ groups.

3.4 Spontaneous breaking of Symmetry in the $SU(n)$ group

The Lie group $SU(n)$ is defined as the $n \times n$ complex matrices that satisfy:

$$MM^{\dagger} = I \tag{3.203}$$
$$\det[M] = 1. \tag{3.204}$$

The $n \times n$ complex matrices that satisfy only Eq.(3.203) form the Lie Group $U(n)$ that contains $SU(n)$.
For an element of $SU(n)$ using $M = e^{iX}$ we have that the matrices of the Lie Algebra $\mathfrak{su}(n)$, the Lie algebra of $SU(n)$, have to be hermitian $X = X^{\dagger}$ and traceless $\text{Tr}[X] = 0$.
From the conditions Eq.(3.203) and Eq. (3.204) we see that there are only $n^2 - 1$ independent generators for $\mathfrak{su}(n)$. This shows the dimension of $\mathfrak{su}(n)$ and correspondingly of $SU(n)$.
To construct a basis for $\mathfrak{su}(n)$ we define the following n^2 real matrices $(n \times n)$

$$[X_i^j]_{mn} = \delta_{im}\delta_{jn}. \tag{3.205}$$

These n^2 matrices satisfy the following relationship:

$$[X_i^j, X_k^l] = \delta_k^j X_i^l - \delta_i^l X_k^j \qquad i, j, k, l = 1, \cdots, n. \tag{3.206}$$

This is the Lie Algebra $\mathfrak{u}(n)$. These matrices form a vector space with scalar product:

$$\langle X_i^j, X_k^l \rangle = \text{Tr}[X_i^j X_k^l] = \delta_{ik}\delta_{jl}. \tag{3.207}$$

Thus to express any element of $SU(n)$ as

$$M = e^{i\epsilon_i^j X_i^j}. \tag{3.208}$$

we need to impose certain restrictions on the complex parameters ϵ_i^j. In general any matrix that belongs to the subspace of the real (complex) matrix vector space $M_n(\mathbb{R})(M_n(\mathbb{C}))$, for example all Lie algebras of classical groups, can be generated using the set Eq.(3.205) and complex parameters with specific restrictions derived from the definition of the groups.
Without any restriction, the set of ϵ represents $2n^2$ independent real parameters. Using Eq.(3.203) we have that:

$$\epsilon_i^j = (\epsilon_j^i)^*. \tag{3.209}$$

This reduces the independent parameters to n^2 in total with $n(n-1)$ with distinct indexes and n with equal indexes. From Eq.(3.204), using $\det[M] = e^{i\,\text{Tr}(\epsilon_i^j X_i^j)} = 1$ we see that:

$$\text{Tr}[\epsilon_i^j X_i^j] = \epsilon_i^j \,\text{Tr}[X_i^j] = \epsilon_i^i \,\text{Tr}[X_i^i] = \epsilon_1^1 + \cdots + \epsilon_n^n = 0. \tag{3.210}$$

This implies that we have only $n-1$ independent parameters ϵ with equal indexes.
The fundamental representation of $SU(n)$ is then defined as:

$$U(\epsilon) = e^{i\epsilon_i^j X_i^j}. \tag{3.211}$$

with the restrictions on the ϵ outlined above. Since the algebra has dimension $n^2 - 1$ to add the Yang-Mills field we need the same number of bosonic gauge bosons, $W^j_{\mu i}$. The vector W^μ expressed in the base of the Lie Algebra Eq.(3.206) is:

$$W^\mu = W^{\mu j}_i X^j_i. \tag{3.212}$$

Contrary to the orthogonal case these are complex. They satisfy the same restrictions as the ϵ since they are constructed in the same way:

$$W^j_{\mu i} = (W^i_{\mu j})^* \tag{3.213}$$

$$\sum_{i=1}^{n} W^i_{\mu\, i} = 0. \tag{3.214}$$

W_μ transforms in the adjoint representation under a gauge transformation Eq.(1.70):

$$W'_\mu = e^{i\epsilon^j_i X^j_i}[W^l_{\mu\, k} X^l_k + \frac{i}{g}\partial_\mu]e^{-i\epsilon^j_i X^j_i}. \tag{3.215}$$

Expanding near the identity :

$$W'_\mu = W_\mu + i[\epsilon^j_i X^j_i, W^l_{\mu k} X^l_k] + \frac{1}{g}\partial_\mu \epsilon^j_i X^j_i = W_\mu + i\epsilon^k_i W^l_{\mu k} X^l_i - \epsilon^j_i W^i_{\mu k} X^j_k + \frac{1}{g}\partial_\mu \epsilon^j_i X^j_i \tag{3.216}$$

where we have used the Algebra (3.206). The diagonal summation between ϵ and W is implied. Projecting onto the component W^j_i, $\mathrm{Tr}[X^j_i W_\mu]$ and using Eq.(3.207), we have the transformation law:

$$W'^j_{\mu i} = W^j_{\mu\, i} + i\epsilon^k_i W^j_{\mu\, k} - i\epsilon^j_k W^k_{\mu\, i} + \frac{1}{g}\partial_\mu \epsilon^j_i. \tag{3.217}$$

Some W's are complex which are interpreted as charged bosons. However to properly count massive gauge bosons we need to find their decomposition in real boson gauge bosons. An example on how we get real bosons from complex ones will be done with the Yang Mills field of $SU(2)$. A general complex boson is:

$$W^j_{\mu\, i} = Re(W^j_{\mu\, i}) + iIm(W^j_{\mu\, i}). \tag{3.218}$$

With this formalism we have in $SU(2)$:

$$W_\mu = W^j_{\mu\, i} X^j_i = W^2_{\mu\, 1} X^2_1 + W^1_{\mu\, 2} X^1_2 + W^1_{\mu\, 1} X^1_1 + W^2_{\mu\, 2} X^2_2. \tag{3.219}$$

In principle the Yangs Mills field would depend on 8 real gauge bosons but from the conditions we have using Eq.(3.213):

$$W^1_{\mu\, 1}, W^2_{\mu\, 2} \quad \text{real}, \qquad\qquad W^2_{\mu\, 1} = W^{*1}_{\mu\, 2}, \tag{3.220}$$

and using Eq.(3.214)

$$W^1_{\mu\, 1} = -W^2_{\mu\, 2} \tag{3.221}$$

so we have only 3 independent gauge bosons. Suppose that $W^2_{\mu 1}$ gets mass we have then:

$$m^2 W^2_{\mu 1} W^{*\,\mu\,2}_1 = m^2 \left(Re(W^2_{\mu 1})^2 + Im(W^2_{\mu 1})^2\right), \tag{3.222}$$

so we have two real vectors that acquire mass.
To construct the kinetic part of the Lagrangian we need:

$$F^j_{\mu\nu i} = \partial_\mu W^j_{\nu i} - \partial_\nu W^j_{\mu\, i} + ig(W^k_{\mu i} W^j_{\nu k} - W^k_{\nu i} W^j_{\mu k}). \tag{3.223}$$

The Lagrangians that we will use have the following form:

$$\mathcal{L} = -\frac{1}{4} F^j_{\mu\nu i} F^{\mu\nu i}_j + \mathcal{L}_{\mathrm{k}} - V(\phi) \tag{3.224}$$

with $\mathcal{L}_k$ the kinetic Lagrangian of the scalar field. Both the explicit form of $\mathcal{L}_k$ and $V(\phi)$will depend on the representation of the scalar field and so also the symmetry breaking patterns.

In the following sections we perform the generalized BEH mechanism for different representations of the $SU(n)$ group. We start deriving the law transformations in the specific representation and the covariant derivative. Then use factorization procedures we will retrieve the corresponding vev for different values of the parameters of the potential.

3.4.1 Spontaneous Breaking in the vector Representation

In $SU(n)$ we only have single valued irreducible representations. A finite transformation in the fundamental representation is:

$$\psi' = U(\epsilon)\psi \tag{3.225}$$

Where ψ is a complex n dimensional vector. We define the equivalence:

$$\psi^*_i = \psi^i \tag{3.226}$$

such that the norm squared of a vector is:

$$\psi^*_i \psi_i = \psi^i \psi_i. \tag{3.227}$$

For infinitesimal transformations we have:

$$\psi'_i = (1 + \epsilon^l_k X^l_k)_{ij} \psi_j = \psi_i + i\epsilon^l_k \psi_j \delta_{ki} \delta_{lj} = \psi_i + i\epsilon^j_i \phi_j. \tag{3.228}$$

The covariant derivative is:

$$\mathcal{D}_\mu \psi_i = \partial_\mu \psi_i - igW_{\mu a} \Gamma(t_a)_{ij} \psi_j = \partial_\mu \psi_i - igW^b_{\mu a} (X^b_a)_{ij} \psi_j = \partial_\mu \psi_i - igW^j_{\mu i} \psi_j. \tag{3.229}$$

In this way we can construct the kinetic part of the scalar field in the fundamental representation:

$$\mathcal{L}_{\rm k} = \frac{1}{2}(\mathcal{D}^\mu\psi)^\dagger\mathcal{D}_\mu\psi = \frac{1}{2}\partial^\mu\psi^i\partial_\mu\psi_i - igW^j_{\mu i}\psi_j\partial^\mu\psi^i + \frac{1}{2}g^2W^k_{\mu i}\psi_kW^{*\mu k'}_i\psi_{k'}. \tag{3.230}$$

The potential is:

$$V(\psi) = -\frac{1}{2}\mu^2\psi_i\psi^i + \frac{1}{4}\lambda(\psi_i\psi^i)^2, \qquad \mu^2, \lambda \text{ real.} \tag{3.231}$$

To retrieve fields that minimize the potential we do:

$$\left.\frac{\partial V}{\partial\psi_i}\right|_{\psi_i=\langle\psi_i\rangle} = \left(-\mu^2 + \lambda\langle\psi_i\rangle\langle\psi^i\rangle\right)\langle\psi_i\rangle = 0, \qquad i = 1,\cdots,n. \tag{3.232}$$

If $\mu^2 > 0$ (after phase transition) one can show that the condition :

$$\langle\psi_i\rangle = 0 \quad i = 1,\cdots,n \tag{3.233}$$

give a local maximum. This is consistent with the fact that we are in the the spontaneously broken case since the order parameter φ(number of grounds states) is not null. Then at least we need to have $\neq 0$ for some i. The set of solutions in fact are defined by the following equation:

$$\langle\psi_i\rangle\langle\psi^i\rangle = \frac{\mu^2}{\lambda} = v \tag{3.234}$$

where v is real. [9] We select a particular direction:

$$\langle\psi_i\rangle = v\delta_{in}. \tag{3.235}$$

The mass term using Eq.(3.230) is:

$$\mathcal{L}_M = \frac{1}{2}g^2W^{*\,\mu j}_i W^{j'}_{\mu i}\langle\psi_j\rangle^*\langle\psi_{j'}\rangle = \frac{1}{2}g^2v^2W^{*\mu n}_iW^n_{\mu i}, \qquad \text{no sum over } n. \tag{3.236}$$

Thus we have $2n-1$ boson vectors that acquire mass. To see this, note from the sum that each $W^{*\mu n}_iW^n_{\mu i}$ for $i \neq n$ gives two real gauge vector bosons and in addition there is the single W^n_n. Thus we have $n^2 - 1 - (2n-1) = (n-1)^2 - 1$ vector bosons that are massless. The symmetry breaking is then:

$$SU(n) \to SU(n-1). \tag{3.237}$$

A clear example is the case of $SU(2)$. Selecting $n = 2$, we have:

$$\mathcal{L}_M = \frac{1}{2}g^2v^2\left(W^{*\mu 2}_1W^2_{\mu 1} + (W^2_2)^2\right) \tag{3.238}$$

[9]it has the same value as the one of the Standard Model

From the fact that W_2^2 is real and W_2^1 is complex we have 3 real gauge bosons that acquire the same mass. Doing the same calculation with the usual fundamental representations:

$$T_a = \frac{\tau_a}{2} \qquad a = 1, 2, 3 \tag{3.239}$$

we see that the mass term is:

$$\begin{aligned}\mathcal{L}_M =& \frac{1}{2}\frac{g^2v^2}{4}\left((A_{\mu1})^2 + (A_{\mu2})^2 + (A_{\mu3})^2\right) = \frac{1}{2}\frac{g^2v^2}{4}(A_{\mu3})^2 + \frac{1}{4}g^2v^2\frac{1}{2}(A_{\mu1} + iA_{\mu2})(A_{\mu1} - iA_{\mu2})\\ =& \frac{1}{2}\frac{g^2v^2}{4}(A_{\mu3})^2 + \frac{1}{4}g^2v^2W^\dagger W^-.\end{aligned} \tag{3.240}$$

The correspondence between the two notations is then clear:

$$\begin{aligned} W^2_{\mu1} &= \frac{1}{2}(A_{\mu1} + iA_{\mu2}) = \frac{1}{\sqrt{2}}W^+ \\ W^{*2}_{\mu1} &= \frac{1}{2}(A_{\mu1} - iA_{\mu2}) = \frac{1}{\sqrt{2}}W^- \\ W^2_{\mu2} &= -\frac{1}{2}A_{\mu3} \end{aligned} \tag{3.241}$$

and it can be readily generalized for any gauge vector of $SU(n)$ in any representation constructed through the fundamental representation.
It can be shown that when having two sets of vectors the symmetry is reduced as $SU(n) \to SU(n-2)$. Inductively using m vectors the breaking pattern is $SU(n) \to SU(n-m)$.

3.4.2 Spontaneous Breaking of $SU(n) \times U(1) \cdots U(1)$ in the Vector Representation

In this section we deal with a simple extension of the last case. Let's consider the symmetry group $G = SU(n) \times U(1) \times \cdots \times U(1)$ where M is the number of $U(1)$ symmetry groups. As it is well known all the one dimensional irreducible representations of each $U(1)$ are labeled by the value of the "charge" Y that can in principle be any number. [10] Denoting β as the parameter of $U(1)_Y$, a transformation in the specific representation of ψ with hypercharge Y_ψ is:

$$\psi \to e^{i\beta Y_\psi}\psi \tag{3.242}$$

Since the fundamental representation of $U(1)$ is one dimensional, the product representation of the fundamental $SU(n) \times U(1)$ is still a vectorial one, and a so a gauge transformation of $U(1)$ is just a phase transformation. Since the generators of the abelian subgroups of the gauge group commute with the ones $SU(n)$ they have to be proportional to $\mathbf{1}$ when expressed in the fundamental representation of $SU(n)$ that is the only important representation.Then

[10] In other words Y is not quantized. A suitable solution to this is embedding $U(1)_Y$ into a large simple group, as is well known for the $SU(5)$ GUT.

the Abelian components of the Yang Mills field are just a diagonal matrix in the fundamental representation of $SU(n)$. We then have that the covariant derivative is:

$$\mathcal{D}_\mu \psi_i = \partial_\mu \psi_i - ig_1 W^j_{\mu\,i}\psi_j - \sum_{\alpha=1}^{M} ig'_\alpha \frac{Y_\alpha}{2} B_{\alpha\mu}\psi_i \qquad \alpha = 1, \cdots, M \tag{3.243}$$

where M is the number of $U(1)$ symmetries. As usual the kinetic term is gauge invariant. We use the same potential as in last section since the $SU(n)$ symmetry is the dominant one and it includes all the possible $U(1)$ symmetry groups. We then get the same vev: $\langle\psi_i\rangle = \delta_{in}v$. Then to get the mass term we have:

$$\begin{aligned}\mathcal{L}_M =& \frac{g^2}{2} W^j_{\mu i} W^{*\mu k}_i \langle\psi_j\rangle \langle\psi^*_k\rangle + \frac{g}{2}\left(W^{*j}_{\mu i}\langle\psi^*_j\rangle (\sum_{\alpha=1}^{M}\frac{Y_\alpha}{2}\langle\psi_i\rangle g'_\alpha B^\mu_\alpha) + W^k_{\mu i}\langle\psi_k\rangle (\sum_{\alpha=1}^{M}\frac{Y_\alpha}{2} g'_\alpha \langle\psi^*_i\rangle B^\mu_\alpha)\right)\\ &+ \frac{1}{2}\left(\sum_{\alpha=1}^{M}\frac{Y_\alpha}{2} g'_\alpha \langle\psi^*_i\rangle B^\mu_\alpha\right)\left(\sum_{\beta=1}^{M}\frac{Y_\beta}{2} g'_\beta \langle\psi_i\rangle B_{\mu\beta}\right)\\ =& \frac{g^2v^2}{2} W^n_{\mu i} W^{*\mu n}_i + \frac{gv^2}{2}\left(W^{*n}_{\mu n} + W^n_{\mu n}\right)(\sum_{\alpha=1}^{M}\frac{Y_\alpha}{2} g'_\alpha B^\mu_\alpha) + \frac{v^2}{2}\left(\sum_{\alpha=1}^{M}\frac{Y_\alpha}{2} g'_\alpha B^\mu_\alpha\right)\left(\sum_{\beta=1}^{M}\frac{Y_\beta}{2} g'_\beta B_{\mu\beta}\right)\end{aligned} \tag{3.244}$$

From the second term, note that the real gauge boson $W^n_{\mu n}$ is the only on that mixes with the $U(1)$ gauge bosons. Then denoting $\frac{\bar{Y}_\beta}{2}\bar{g}\bar{B}_\mu = \sum_{\beta=1}^{M}\frac{Y_\beta}{2} g'_\beta B_{\mu\beta}$ we get that:

$$\begin{aligned}\mathcal{L}_M =& \frac{g^2v^2}{2}\sum_{i=1}^{n-1} W^n_{\mu i} W^{*\mu n}_i + \frac{v^2}{2}(\bar{g}\frac{\bar{Y}_\beta}{2}\bar{B}_\mu + gW^n_{\mu n})(\frac{\bar{Y}_\beta}{2}\bar{g}\bar{B}^\mu + gW^{*\mu n}_n)\\ =& \frac{g^2v^2}{2}\sum_{i=1}^{n-1} W^n_{\mu i} W^{*\mu n}_i + \frac{v^2(g^2+\bar{g}^2)}{2}(\sin\theta\frac{\bar{Y}_\beta}{2}\bar{B}_\mu + \cos\theta W^n_{\mu n})(\sin\theta\frac{\bar{Y}_\beta}{2}\bar{B}^\mu + \cos\theta W^{\mu n}_n)\end{aligned} \tag{3.245}$$

Where we define:

$$\sin\theta = \frac{\bar{g}}{\sqrt{g^2+\bar{g}^2}} \tag{3.246}$$

as an angle with an obvious remembrance of the Weinberg Angle. The second term in Eq. (3.245) can be seen as a mass matrix which gives one non-zero mass gauge boson as a combination pf $B_{\mu\beta}$ and $W^n_{\mu\,n}$. Thus we see that we have $n-1$ complex gauge bosons that acquired mass $m = gv$ and one real boson that acquired mass $\sqrt{g^2+\bar{g}^2}\frac{v}{2}$. Thus the symmetry breaking is:

$$SU(n)\times U(1)_{Q_1}\times\cdots\times U(1)_{Q_M} \to SU(n-1)\times U(1)_{Q'_1}\times\cdots\times U(1)_{Q'_{M-1}} \tag{3.247}$$

3.4.3 Spontaneous Breaking in the second rank Symmetric Representation

The symmetric tensor has the properties:

$$\psi_{ij} = \psi_{ji} = (\psi^{ij})^* \tag{3.248}$$

The derivation of transformation rule and covariant derivative will be done in Section (3.5.2). We state the results here.
The transformation rule is:

$$\psi_{ij} \to \psi_{ij} + i\epsilon_i^k \psi_{kj} + i\epsilon_j^k \psi_{ik} \tag{3.249}$$

The covariant derivative:

$$\mathcal{D}_\mu \psi_{ij} = \partial_\mu \psi_{ij} - igW^l_{\mu i}\psi_{lj} - igW^l_{\mu j}\psi_{il} \tag{3.250}$$

The masses will come from:

$$\mathcal{L}_M = g^2(W^k_{\mu i}\psi_{kj}W_i^{*\mu k'}\psi^*_{k'j}) + g^2(W^k_{\mu i}\psi^*_{kj}W_j^{\mu k' *}\psi^*_{ik'}) \tag{3.251}$$

The most general potential in this case is [11]:

$$\begin{aligned} V(\psi) &= -\frac{1}{2}\mu^2 \operatorname{Tr}[\psi^\dagger \psi] + \frac{1}{4}\lambda_1(\operatorname{Tr}[\psi^\dagger\psi])^2 + \frac{1}{4}\lambda_2 \operatorname{Tr}[\psi^\dagger\psi\psi^\dagger\psi] \\ &= -\frac{1}{2}\mu^2\psi_{ij}\psi^{ij} + \frac{1}{4}\lambda_1(\psi_{ij}\psi^{ij})^2 + \frac{1}{4}\lambda_2(\psi_{ij}\psi^{jk}\psi_{kl}\psi^{li}) \end{aligned} \tag{3.252}$$

To calculate the minimum, we have [12]:

$$\frac{\partial V}{\partial \psi_{ij}}\bigg|_{\psi_{ij}=\langle\psi_{ij}\rangle} = -\frac{1}{2}\mu^2\psi^{ij} + \frac{1}{2}\lambda_1(\psi_{lm}\psi^{lm})\psi^{ij} + \frac{1}{2}\lambda_2(\psi^{jk}\psi_{kl}\psi^{li}) = 0 \quad i,j = 1\cdots,n \tag{3.253}$$

We define a new matrix as:

$$\Sigma_l^k = \psi_{lm}\psi^{mk} \tag{3.254}$$

which is hermitian since:

$$(\Sigma^\dagger)_l^k = (\Sigma_k^l)^* = (\psi_{km}\psi^{ml})^* = \psi^{km}\psi_{ml} \underset{eq.3.248}{=} \psi_{lm}\psi^{mk} = \Sigma_l^k \tag{3.255}$$

Then the potential is:

$$V(\psi) = -\frac{1}{2}\mu^2 \operatorname{Tr}[\Sigma] + \frac{1}{4}\lambda_1(\operatorname{Tr}[\Sigma])^2 + \frac{1}{4}\lambda_2 Tr[\Sigma\Sigma] \tag{3.256}$$

[11] Note that we cannot possible add a cubic term: $\operatorname{Tr}[\psi^\dagger\psi\psi^\dagger]$ since it not gauge invariant under the transformation $\psi \to U\psi U^T$

[12] we omit the bracket parenthesis in the following parts but we are working with the vev components of ψ and Σ

So Eq.(3.253) is: [13]

$$-\mu^2\psi^{ij} + \lambda_1(\mathrm{Tr}[\Sigma])\psi^{ij} + \lambda_2\Sigma^j_l\psi^{li} = 0 \tag{3.257}$$

Since Σ is hermitian we can diagonalize it via a unitary transformation:

$$\Sigma \to \Sigma' = U\Sigma U^\dagger = U\psi\psi^* U^\dagger = U\psi U^T U^* \psi^* U^\dagger = U\psi U^T (U\psi U^T)^* = \tilde{\psi}\tilde{\psi}^* \tag{3.258}$$

Where we have used $U^T U^* = 1$. This is just a change of basis on the ψ space as shown above, and does not change the shape of the potential. Then Σ' has the form:

$$\Sigma' = \begin{pmatrix} \sigma'_1 & & & \\ & \sigma'_2 & & \\ & & \ddots & \\ & & & \sigma'_n \end{pmatrix} \tag{3.259}$$

where the set σ''s are real and not all of them zero. Using $\Sigma = U^\dagger\Sigma' U$ we have $\mathrm{Tr}(\Sigma) = \mathrm{Tr}[U^\dagger\Sigma' U] = \mathrm{Tr}[\Sigma'] = \sum_{i=1}^n \sigma'_i$. Rewriting Eq.(3.257) hiding the tildes on ψ_{ij} and the primes on Σ and σ we have:

$$\left(-\mu^2 + \lambda_1(\sum_{k=1}^n \sigma_k)\right)\psi^{ij} + \lambda_2\sigma_j\delta^j_l\psi^{li} = \left(-\mu^2 + \lambda_1(\sum_{k=1}^n \sigma_k) + \lambda_2\sigma_j\right)\psi^{ij} = 0 \quad j = 1, \cdots, n \tag{3.260}$$

Since we are in the spontaneous symmetry breaking case there will be some $\psi_{ij} \neq 0$. Let's suppose we have K non zero σ's[14], then the equation defining the set of vevs is:

$$\left(-\mu^2 + \lambda_1(\sum_{k=1}^K \sigma_k) + \lambda_2\sigma_j\right) = 0, \qquad j = 1, \cdots, n \tag{3.261}$$

This equation has the same structure as equation Eq.(3.71) that appeared in the minimization of the potential of the antisymmetric second rank tensor representation of $O(n)$. As before the calculation of the vev of ψ_{ij} will depend on the number K of non zero σ's . Using the same procedure as before (Eq.(3.71) and below), the σ's that are non zero are all equal to each other, thus from Eq. (3.261) we have:

$$\sigma = \frac{\mu^2}{K\lambda_1 + \lambda_2} \tag{3.262}$$

We can also derive the correspondent stability condition. This is done in analogy with Eq.(3.74) with the result:

$$K\lambda_1 + \lambda_2 > 0 \tag{3.263}$$

[13] omitting parenthesis in the componentes of the vev of ψ_{ij}

[14] the order is unimportant since we can always use transformations that change order leave the potential invariant

Case $\lambda_2 > 0$

Using the same arguments of Part (3.82), if $\lambda_2 > 0$ we have that $K = n$ thus $\Sigma = c^2\mathbf{1}$ with:

$$c^2 = \frac{\mu^2}{(n\lambda_1 + \lambda_2)}, \qquad n\lambda_1 + \lambda_2 > 0 \tag{3.264}$$

The next step is to retrieve $\langle\psi\rangle$ from Σ .
The defining equations of ψ are:

$$\psi\psi^* = \Sigma = c^2\mathbf{1} \tag{3.265}$$

$$\psi^T = \psi \tag{3.266}$$

Using $\psi = A + iB$ with A and B real $n \times n$ matrices, from Eq.(3.266) we see that A and B have to be symmetric. From Eq.(3.265) we have that:

$$A^2 + B^2 = c^2\mathbf{1} \tag{3.267}$$

$$AB = BA \tag{3.268}$$

Since A and B are symmetric and commute with each other, we can diagonalize them together using an orthogonal transformation:

$$A = OA'O^T \tag{3.269}$$

$$iB = iOB'O^T \tag{3.270}$$

with:

$$A' = \begin{pmatrix} a_1 & & 0 & \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ & 0 & & a_n \end{pmatrix} \tag{3.271}$$

$$B' = \begin{pmatrix} b_1 & & 0 & \\ 0 & b_2 & & \\ \vdots & & \ddots & \\ & 0 & & b_n \end{pmatrix} \tag{3.272}$$

From Eq.(3.267) and the orthogonality of O:

$$Oc^2\mathbf{1}O^T = c^2\mathbf{1} = A^2 + B^2 = OA'O^TOA'O^T + OB'O^TOB'O^T = O(A'^2 + B'^2)O^T \tag{3.273}$$

so we have n equations:

$$a_k^2 + b_k^2 = c^2, \qquad k = 1, \cdots n \tag{3.274}$$

Then the matrix ψ' from $\psi = O\psi' O^T$ is:

$$\begin{aligned}\psi' = A' + iB' &= \begin{pmatrix} a_1 + ib_1 & & & 0 \\ 0 & a_2 + ib_2 & & \\ & & \ddots & \\ & 0 & & a_n + ib_n \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{a_1^2 + b_1^2} e^{i\alpha_1} & & & 0 \\ 0 & \sqrt{a_2^2 + b_2^2} e^{i\alpha_2} & & \\ & & \ddots & \\ & 0 & & \sqrt{a_n^2 + b_n^2} e^{i\alpha_n} \end{pmatrix} \\ &= c \begin{pmatrix} e^{i\alpha_1} & & & 0 \\ 0 & e^{i\alpha_2} & & \\ & & \ddots & \\ & 0 & & e^{i\alpha_n} \end{pmatrix}\end{aligned} \tag{3.275}$$

This determines the structure of the vev of ψ up to n phases. However, we can factorize ψ':

$$\psi' = U\psi'' U^T \tag{3.276}$$

such that:

$$\psi'' = c\mathbf{1} \tag{3.277}$$

and:

$$U = \begin{pmatrix} e^{i\alpha_1/2} & & & 0 \\ 0 & e^{i\alpha_2/2} & & \\ & & \ddots & \\ & 0 & & e^{i\alpha_n/2} \end{pmatrix} \tag{3.278}$$

by doing this, the potential remains invariant meaning that the phase are not physical and can be removed.
Summarizing we have:

$$\psi = OU\psi''(OU)^T \tag{3.279}$$

with, explicitely expressing ψ'' as a vev:

$$\langle \psi'' \rangle = c\mathbf{1} \equiv c \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix} \tag{3.280}$$

Having the vev we now need to calculate the number of massive gauge bosons. We insert the vevs in the Lagrangian and remove the primes so $\psi \rightarrow \psi''$. Then calculate the masses. The vev is:

$$\langle \psi''_{ij} \rangle = c\delta_{ij} \tag{3.281}$$

then:

$$\begin{aligned}\mathcal{L}_M =& g^2 (W^k_{\mu i} \langle \psi_{kj} \rangle W^{*\mu\, k'}_i \langle \psi^*_{k'j} \rangle) + g^2 (W^{*k}_{\mu i} \langle \psi^*_{kj} \rangle W^{\mu k'}_j \langle \psi_{i\, k'} \rangle)\\ =& g^2 c^2 (W^k_{\mu i} \delta_{kj} W^{*\mu\, k'}_i \delta_{k'j}) + g^2 c^2 W^i_{\mu\, k} \delta_{jk} W^{\mu k'}_j \delta_{i\, k'}\\ =& g^2 c^2 W^{*j}_{\mu i} W^{\mu\, j}_i + g^2 c^2 W^{*j}_{\mu\, i} W^{\mu i}_j\\ =& 2g^2 c^2 W^{*i}_{\mu i} W^{\mu\, i}_i + 2g^2 c^2 \sum_{i<j} W^{*j}_{\mu i} (W^{\mu\, j}_i + W^{*\mu\, j}_i)\end{aligned} \tag{3.282}$$

Then we see that $n-1$ diagonal gauge bosons acquire mass and $\frac{n(n-1)}{2}$ non diagonal real gauge bosons acquire mass thus we have:

$$n^2 - 1 - (\frac{n^2}{2} + \frac{n}{2} - 1) = \frac{n(n-1)}{2} \tag{3.283}$$

massless bosons. Thus the symmetry breaking pattern is:

$$SU(n) \to O(n) \tag{3.284}$$

Case $\lambda_2 < 0$

For the case $\lambda_2 < 0$ we have that $K = 1$ as in the antisymmetric $O(n)$ case, as when we arrived at Eq.(3.80). Using Eq.(3.262) and Eq.(3.263), the vev of Σ is

$$\langle \Sigma \rangle = d^2 \begin{pmatrix} 1 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \\ & & & & 0 \end{pmatrix}, \qquad d^2 = \frac{\mu^2}{\lambda_1 + \lambda_2}, \quad \lambda_1 + \lambda_2 > 0 \tag{3.285}$$

As before $\psi = A + iB$ with all the properties shown for the case $\lambda_2 > 0$. The difference is that we only have one equation:

$$a_1^2 + b_1^2 = d^2 \tag{3.286}$$

$$\psi' = A' + iB' = \begin{pmatrix} a_1 + ib_1 & & & 0 \\ 0 & 0 & & \\ & & \ddots & \\ & 0 & & 0 \end{pmatrix} = c \begin{pmatrix} e^{i\alpha_1} & & & 0 \\ 0 & 0 & & \\ & & \ddots & \\ & 0 & & 0 \end{pmatrix} \tag{3.287}$$

Doing the corresponding redefinition $U =$ diag$(e^{-i\alpha_1/2}, \cdots, 0)$ we arrive at:

$$\langle \psi'' \rangle \equiv d \begin{pmatrix} 1 & & & & \\ & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \\ & & & & 0 \end{pmatrix} \tag{3.288}$$

To calculate the masses we do as the preceding section with vev:

$$\langle \psi''_{ij} \rangle = d\delta_{1i}\delta_{1j} \tag{3.289}$$

$$\mathcal{L}_M = g^2(W^k_{\mu i}\psi_{kj}W_i^{*\mu\, k'}\psi^*_{k'j}) + g^2(W^{*k}_{\mu i}\psi^*_{kj}W_j^{\mu k'}\psi_{i\,k'}) = g^2d^2(W^1_{\mu i}W_i^{*\mu\, 1}) + g^2d^2(W^{*1}_{\mu 1}W_1^{\mu 1}) \tag{3.290}$$

For example for $SU(3)$ with 8 generators, we have:

$$\begin{aligned} \mathcal{L}_M =& g^2d^2\left(W^1_{\mu 1}W_1^{\mu\, 1} + W_1^{\mu\, 2}W^{*\,2}_{\mu\, 1} + W_1^{\mu\, 3}W^{*\,3}_{\mu\, 1}\right) + g^2d^2\left(W^1_{\mu\, 1}W_1^{\mu 1}\right) \\ =& 2g^2d^2W^1_{\mu\, 1}W_1^{\mu 1} + g^2d^2\left(W_1^{\mu\, 2}W^{*\,2}_{\mu\, 1} + W_1^{\mu\, 3}W^{*\,3}_{\mu\, 1}\right) \end{aligned} \tag{3.291}$$

Note that the gauge vector $W^3_{\mu 2}$ is not present this gives two massless gauge bosons. Also note that the real gauge bosons $W^2_{\mu 2}$ and $W^3_{\mu 3}$ are not present but $W^1_{\mu 1}$ is. Since one of them is not independent we have one massless vector in the subspace of $W^2_{\mu 2}$ and $W^3_{\mu 3}$ and $W^1_{\mu 1}$ as a real massive gauge vector. A diagonalization in the complex gauge subspace $(W^3_{\mu 1}, W^2_{\mu 1})$ gives the real masses to each independent gauge vector thus we have 4 real massive gauge bosons. In total we have 5 massive gauge vector thus the remaining symmetry has 3 generators corresponding to the 3 massless gauge bosons so the unbroken group is $SU(2)$.

In the general case we have:

$$\mathcal{L}_M = g^2d^2\sum_{i=2}^{n}(W^1_{\mu i}W_i^{*\mu\, 1}) + 2g^2d^2(W^{*1}_{\mu 1}W_1^{\mu 1}) \tag{3.292}$$

The first factor gives $2n-2$ real massive bosons and the second gives one so in total we have $2n-1$. Since:

$$n^2 - 1 - (2n-1) = (n-1)^2 - 1 \tag{3.293}$$

The breaking pattern is:

$$SU(n) \to SU(n-1) \tag{3.294}$$

As another example, for $SU(5)$ we have:

$$\mathcal{L}_M = g^2d^2\sum_{i=1}^{4}(W^1_{\mu i}W_i^{*\mu\, 1}) + 2g^2d^2(W^{*1}_{\mu 1}W_1^{\mu 1}) \tag{3.295}$$

The first factor has 8 real vector bosons and the second one in total we have 9 massive gauge bosons. So $24 - 9 = 15$ equal to the number of generators of $SU(4)$.

3.4.4 Spontaneous Breaking in the second rank Antisymmetric Representation

The antisymmetric representation has the properties:

$$\psi_{ij} = -\psi_{ji} = (\psi^{ij})^*. \tag{3.296}$$

The following proprieties will be derived in Section (3.5.2). The transformation rule is:

$$\psi_{ij} \to \psi_{ij} + i\epsilon_i^k \psi_{kj} + i\epsilon_j^k \psi_{ik}. \tag{3.297}$$

The covariant derivative:

$$\mathcal{D}_\mu \psi_{ij} = \partial_\mu \psi_{ij} - igW^l_{\mu i}\psi_{lj} - igW^l_{\mu j}\psi_{il}. \tag{3.298}$$

The masses will come from:

$$\mathcal{L}_M = g^2(W^k_{\mu i}\psi_{kj}W_i^{*\mu\, k'}\psi^*_{k'j}) + g^2(W^{*k}_{\mu i}\psi^*_{kj}W_j^{\mu k'}\psi_{i\,k'}). \tag{3.299}$$

The potential has the same form as the potential for the symmetric representation Eq.(3.252):

$$\begin{aligned} V(\psi) &= -\frac{1}{2}\mu^2\,\mathrm{Tr}[\psi^\dagger\psi] + \frac{1}{4}\lambda_1(\mathrm{Tr}[\psi^\dagger\psi])^2 + \frac{1}{4}\lambda_2\,\mathrm{Tr}[\psi^\dagger\psi\psi^\dagger\psi] \\ &= -\frac{1}{2}\mu^2\psi_{ij}\psi^{ij} + \frac{1}{4}\lambda_1(\psi_{ij}\psi^{ij})^2 + \frac{1}{4}\lambda_2(\psi_{ij}\psi^{jk}\psi_{kl}\psi^{li}) \end{aligned} \tag{3.300}$$

As the solution before, we define : $\Sigma = \psi\psi^*$. In components: $\Sigma^l_k = \psi_{km}\psi^{ml}$. It is also hermitian. The procedure of minimization is the same as the previous section. The results are the same, the value of σ is given by Eq.(3.262) and the stability condition is Eq. (3.263).

Case $\lambda_2 > 0$

If $\lambda_2 > 0$ we have

$$\langle\Sigma\rangle = c^2\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix}, \qquad c^2 = \frac{\mu^2}{n\lambda_1 + \lambda_2}, \quad n\lambda_1 + \lambda_2 > 0 \tag{3.301}$$

that is valid for n odd or even.
Thus:

$$\psi\psi^* = c^2\mathbf{1} \tag{3.302}$$

Let's first focus on the even case $n = 2L$. To retrieve ψ from Eq.(3.302) note that ψ can be written as:

$$\psi = A + iB \tag{3.303}$$

with A, B antisymmetric. The defining conditions are the same as before, from Eq.(3.302):

$$A^2 + B^2 = c^2\mathbf{1} \tag{3.304}$$

$$AB = BA \tag{3.305}$$

Since iA and iB are hermitian and commute with each other they can diagonalized with the same unitary transformation. This is just an application of the Spectral Theorem. The eigenvalues are real and occur in pairs. Let λ be an eigenvalue, using $A = -A^T$ we have :

$$\det(\lambda\mathbf{1} - iA) = \det(\lambda\mathbf{1} + iA^T) = \det(\lambda\mathbf{1}^T + iA^T) = \det((\lambda\mathbf{1} + iA)^T) = \det(\lambda\mathbf{1} + iA) \tag{3.306}$$

Thus both λ and$-\lambda$ give solution to the characteristic polynomial. Then:

$$iA = UiA_dU^\dagger \tag{3.307}$$

$$iB = UiB_dU^\dagger \tag{3.308}$$

with:

$$A_d = \begin{pmatrix} ia_1 & & & & & & \\ & -ia_1 & & & 0 & & \\ & & ia_2 & & & & \\ & & & -ia_2 & & & \\ & & & & \ddots & & \\ & 0 & & & & ia_L & \\ & & & & & & -ia_L \end{pmatrix} \tag{3.309}$$

$$B_d = \begin{pmatrix} ib_1 & & & & & & \\ & -ib_1 & & & 0 & & \\ & & ib_2 & & & & \\ & & & -ib_2 & & & \\ & & & & \ddots & & \\ & 0 & & & & ib_L & \\ & & & & & & -ib_L \end{pmatrix} \tag{3.310}$$

This can be simplified. Using the unitary matrix:

$$k = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{3.311}$$

We see that:

$$k\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}k^\dagger = \frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \tag{3.312}$$

Defining:

$$K = \begin{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} & & 0\\ & \ddots & \\ 0 & & \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\end{pmatrix} \tag{3.313}$$

We see that $A_d = KA_SK^\dagger$ with:

$$A_S = \begin{pmatrix}a_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} & & \\ & \ddots & \\ & & a_L\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\end{pmatrix} \tag{3.314}$$

also $B_d = KB_SK^\dagger$:

$$B_S = \begin{pmatrix}b_1\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} & & \\ & \ddots & \\ & & b_L\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\end{pmatrix} \tag{3.315}$$

that is K transforms A_d and B_d into standard antisymmetric form.
Summarizing:

$$\psi = A + iB = UKA_S(UK)^\dagger + iUKB_S(UK)^\dagger = UK(A_S + iB_S)(UK)^\dagger = UK\psi'(UK)^\dagger \tag{3.316}$$

But on the other hand it is known that for any real antisymmetric matrix (A, B) a transformation into standard antisymmetric form (A_S, B_S) has to be real and orthogonal so $UK \equiv O$ is a real and orthogonal matrix, thus $(UK)^\dagger = (UK)^T$. From $\psi\psi^* = \mathbf{1}c^2$ we have using Eq.(3.316).

$$O\mathbf{1}c^2O^T = \mathbf{1}c^2 = \psi\psi^* = O\psi'\psi'^*O^T = O(A_S^2 + B_S^2)O^T \tag{3.317}$$

Since O belongs to $O(n)$ is included also in the symmetry group $SU(n)$ of the Lagrangian, thus the factorization $\psi \to \psi'$ leaves the Lagrangian invariant. Equation (3.317) implies:

$$a_i^2 + b_i^2 = c^2, \qquad i = 1, \cdots, L \tag{3.318}$$

We have then:

$$\begin{aligned} \psi' = A_S + iB_S &= \begin{pmatrix} (a_1+ib_1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & (a_L+ib_L)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \\ &= c \begin{pmatrix} e^{i\alpha_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & e^{i\alpha_L}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \end{aligned} \tag{3.319}$$

As before we do another factorization using V to get rid of the phases:

$$\psi' = V\psi''(V)^T \tag{3.320}$$

with the diagonal matrix:

$$V = \begin{pmatrix} e^{i\alpha_1/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & 0 \\ 0 & e^{i\alpha_2/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \\ & & \ddots \\ & 0 & e^{i\alpha_L/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{3.321}$$

which again leaves the Lagrangian invariant. [15] Then ψ'' explicitly shown as a vev is:

$$\langle\psi''\rangle = c\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, n = 2L \tag{3.322}$$

For the case $n = 2L + 1$ the procedure analogous, we just add a 0 in the diagonal.

$$\langle\psi''\rangle = c\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix}, n = 2L+1 \tag{3.323}$$

Case $\lambda_2 < 0$

For the case $\lambda_2 < 0$ the vev of Σ is analogous to Eq.(3.285)

$$\langle\Sigma\rangle = d^2\begin{pmatrix} 1 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \\ & & & & 0 \end{pmatrix}, \quad d^2 = \frac{\mu^2}{\lambda_1 + \lambda_2}, \quad \lambda_1 + \lambda_2 > 0 \tag{3.324}$$

For the vev of ψ we can proceed as in last section and arrive to:

$$\langle\psi''\rangle = d\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ & & 0 \end{pmatrix} \tag{3.325}$$

And this is valid for n odd or even.

[15] note that the covariant derivative goes like $\mathrm{Tr}[(\mathcal{D}_\mu\psi)^\dagger\mathcal{D}^\mu\phi]$ thus V cancels with it's conjugate.

Calculation of the symmetry breaking patterns

In both cases the vev is :

$$\langle \psi''_{kj} \rangle = c \sum_{l=0}^{K-1} \left(\delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1} \right) \tag{3.326}$$

where $K = 1$ for $\lambda_2 < 0$ and $K = L$ for $\lambda_2 > 0$. Using:

$$\mathcal{L}_M = g^2 (W^i_{\mu k} \psi_{kj} W^{\mu\,k'}_i \psi^*_{k'j}) + g^2 (W^i_{\mu\,k} \psi^*_{kj} W^{\mu k'}_j \psi_{i\,k'}) \tag{3.327}$$

we have:

$$\begin{aligned}
\mathcal{L}_{M_1} =& g^2 c^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} \left(W^i_{\mu k} W^{\mu\,k'}_i \left(\delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1} \right) \left(\delta_{k'\,2l'+1}\delta_{j\,2l'+2} - \delta_{k'\,2l'+2}\delta_{j\,2l'+1} \right) \right) \\
=& g^2 c^2 \sum_{l=0}^{K-1} (W^i_{\mu 2l+1} W^{\mu\,2l+1}_i + W^i_{\mu 2l+2} W^{\mu\,2l+2}_i)
\end{aligned} \tag{3.328}$$

and

$$\begin{aligned}
\mathcal{L}_{M_2} =& -g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} \left(W^i_{\mu\,k} W^{\mu k'}_j \left(\delta_{k\,2l+1}\delta_{j\,2l+2} - \delta_{k\,2l+2}\delta_{j\,2l+1} \right) (\delta_{k'\,2l'+1}\delta_{i\,2l'+2} - \delta_{k'\,2l'+2}\delta_{i\,2l'+1}) \right) \\
=& -g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} \left(W^{2l'+2}_{\mu\,2l+1} W^{\mu 2l'+1}_{2l+2} + W^{2l'+1}_{\mu\,2l+2} W^{\mu 2l'+2}_{2l+1} - W^{2l'+1}_{\mu\,2l+1} W^{\mu 2l'+2}_{2l+2} - W^{2l'+2}_{\mu\,2l+2} W^{\mu 2l'+1}_{2l+1} \right) \\
=& -2g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} W^{*2l+1}_{\mu 2l'+2} W^{\mu 2l'+1}_{2l+2} + 2g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} W^{*\,2l+1}_{\mu 2l'+1} W^{\mu 2l'+2}_{2l+2}
\end{aligned} \tag{3.329}$$

Thus:

$$\begin{aligned}
\mathcal{L}_M =& g^2 c^2 \sum_{l=0}^{K-1} (W^i_{\mu 2l+1} W^{\mu\,2l+1}_i + W^i_{\mu 2l+2} W^{\mu\,2l+2}_i) - 2g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} W^{*2l'+2}_{\mu 2l+1} W^{\mu 2l'+1}_{2l+2} \\
&+ 2g^2 \sum_{l=0}^{K-1} \sum_{l'=0}^{K-1} W^{*\,2l+1}_{\mu 2l'+1} W^{\mu 2l'+2}_{2l+2}
\end{aligned} \tag{3.330}$$

Case $\lambda_2 < 0$

In the case $\lambda_2 < 0$ we have $K = 1$. Last equation is:

$$\begin{aligned}\mathcal{L}_M =& g^2c^2(W^{*1}_{\mu i}W^{\mu\,1}_i + W^{*2}_{\mu i}W^{\mu\,2}_i) - 2g^2c^2W^{*1}_{\mu 2}W^{\mu 1}_2 + 2g^2W^{*\,1}_{\mu 1}W^{\mu 2}_2\\ =& g^2c^2\sum_{i=3}^{n}(W^{*1}_{\mu i}W^{\mu\,1}_i + W^{*2}_{\mu i}W^{\mu\,2}_i) + g^2c^2(W^{*1}_{\mu 1}W^{\mu\,1}_1 + W^{*2}_{\mu 1}W^{\mu\,2}_1 + W^{*1}_{\mu 2}W^{\mu\,1}_2 + W^{*2}_{\mu 2}W^{\mu\,2}_2)\\ &- 2g^2c^2W^{*1}_{\mu 2}W^{\mu 1}_2 + g^2c^2\Big(W^{1}_{\mu\,1}W^{\mu 2}_2 + W^{2}_{\mu\,2}W^{\mu 1}_1\Big)\\ =& g^2c^2\sum_{i=3}^{n}(W^{*1}_{\mu i}W^{\mu\,1}_i + W^{*2}_{\mu i}W^{\mu\,2}_i) + g^2c^2(W^{*1}_{\mu 1} + W^{*2}_{\mu 2})(W^{\mu 1}_1 + W^{\mu 2}_2)\end{aligned} \tag{3.331}$$

Then in the first term we have $2(n-2)$ complex gauge bosons that acquire mass thus $4(n-2)$ real vectors acquire mass and in the second term we have that a linear combination of $W^{\mu 1}_1, W^{\mu 2}_2$ acquire mass. Both are real so we have another gauge boson with mass. In total we have $4(n-2)+1$ gauge bosons that acquire mass. The masless vectors are then:

$$n^2 - 1 - (4(n-2) - 1) = \Big((n-2)^2 - 1\Big) + 3 \tag{3.332}$$

So the symmetry breaking pattern is:

$$SU(n) \to SU(n-2) \times SU(2) \tag{3.333}$$

Case $\lambda_2 > 0$

In the case $\lambda_2 > 0$ we have $K = L$. First let's see an example with $SU(4)$ to see the pattern of factorization. We have $L = 2$ so:

$$\begin{aligned}\mathcal{L}_{M_1} =& g^2c^2(W^{*1}_{\mu i}W^{\mu\,1}_i + W^{*2}_{\mu i}W^{\mu\,2}_i + W^{*3}_{\mu i}W^{\mu\,3}_i + W^{*4}_{\mu i}W^{\mu\,4}_i)\\ =& g^2c^2(W^{*1}_{\mu 1}W^{\mu\,1}_1 + W^{*2}_{\mu 2}W^{\mu\,2}_2 + W^{*3}_{\mu 3}W^{\mu\,3}_3 + W^{*4}_{\mu 4}W^{\mu\,4}_4 + 2W^{*2}_{\mu 1}W^{\mu\,2}_1 + 2W^{*3}_{\mu 1}W^{\mu\,3}_1\\ &+ 2W^{*4}_{\mu 1}W^{\mu\,4}_1 + 2W^{*3}_{\mu 2}W^{\mu\,3}_2 + 2W^{*4}_{\mu 2}W^{\mu\,4}_2 + 2W^{*4}_{\mu 3}W^{\mu\,4}_3)\end{aligned} \tag{3.334}$$

$$\begin{aligned}\mathcal{L}_{M_2} =& -2g^2c^2(W^{*1}_{\mu 2}W^{\mu 1}_2 + W^{*3}_{\mu 2}W^{\mu 4}_1 + W^{*1}_{\mu 4}W^{\mu 3}_2 + W^{*3}_{\mu 4}W^{\mu 3}_4) + 2g^2c^2(W^{*1}_{\mu\,1}W^{\mu\,2}_2\\ &+ W^{*3}_{\mu\,1}W^{\mu\,2}_4 + W^{*1}_{\mu\,3}W^{\mu\,4}_2 + W^{*3}_{\mu\,3}W^{\mu\,4}_4)\end{aligned} \tag{3.335}$$

Note that the terms $W^{*1}_{\mu 2}W^{\mu 1}_2$ and $W^{*3}_{\mu 4}W^{\mu 3}_4$ cancel, and factorizing we have:

$$\begin{aligned}\mathcal{L}_M =& g^2c^2(W^{*1}_{\mu 1} + W^{*2}_{\mu 2})(W^{\mu 1}_1 + W^{\mu 2}_2) + g^2c^2(W^{*3}_{\mu 3} + W^{*4}_{\mu 4})(W^{\mu 3}_3 + W^{\mu 4}_4)\\ &+ 2g^2c^2(W^{*1}_{\mu 4} - W^{*3}_{\mu 2})(W^{\mu 1}_4 - W^{\mu 3}_2) + 2g^2c^2(W^{*1}_{\mu 3} + W^{*4}_{\mu 2})(W^{\mu 1}_3 + W^{\mu 4}_2)\end{aligned} \tag{3.336}$$

The first two terms each gives one real bosons and the other two give two complex ones so $1+1+2+2=6$. The number of masless bosons is $15-6=9$. We find there is not a Lie algebra with this number of generators. The closer one is the algebra of $Sp(2n)$ with $n=2$ that is the intersection between the unitary group $U(2n)$ and the sympletic group $Sp(2n,\mathbb{C})$

$$Sp(2n) = U(2n) \cap Sp(2n,\mathbb{C}) \tag{3.337}$$

An element of the sympletic group $Sp(2n,\mathbb{C})$ is an M such that:

$$M^T J M = J \tag{3.338}$$

were J the $2n \times 2n$ matrix:

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \tag{3.339}$$

Since J is antisymmetric, condition Eq.(3.338) provides $\frac{2n(2n-1)}{2}$ restrictions thus the dimension of this group is:

$$2(2n)^2 - \frac{2n(2n-1)}{2} = n(6n+1) \tag{3.340}$$

If M belongs to $Sp(2n)$ then M has to satisfy Eq. (3.338) and also the condition $MM^\dagger = \mathbf{1}$. The dimension of $Sp(2n)$ is then:

$$n(6n+1) - (2n)^2 = 2n^2 + n = \frac{2n}{2}(2n+1) \tag{3.341}$$

Thus the dimension of $Sp(N)$ with N even is $\frac{N}{2}(N+1)$. $Sp(4)$ has dimension 10. Assuming instead that the initial symmetry of the Lagrangian is $U(4)$ (in practice throwing away condition $W_i^{\mu i} = 0$) we have $16-6=10$ massless gauge bosons thus the symmetry breaking is $U(4) \to Sp(4)$. Things change when we have odd number. For $SU(5)$ we have that still $L=2$ but the summation on i in Eq.(3.330) goes until 5, we have:

$$\begin{aligned} \mathcal{L}_M &= g^2c^2(W^{*1}_{\mu 1} + W^{*2}_{\mu 2})(W_1^{\mu 1} + W_2^{\mu 2}) + g^2c^2(W^{*3}_{\mu 3} + W^{*4}_{\mu 4})(W_3^{\mu 3} + W_3^{\mu 3}) \\ &+ 2g^2c^2(W^{*1}_{\mu 4} - W^{*3}_{\mu 2})(W_4^{\mu 1} - W_2^{\mu 3}) + 2g^2c^2(W^{*1}_{\mu 3} + W^{*4}_{\mu 2})(W_3^{\mu 1} + W_2^{\mu 4}) \\ &+ g^2c^2(W^{*1}_{\mu 5}W_5^{\mu 1} + W^{*2}_{\mu 5}W_5^{\mu 2} + W^{*3}_{\mu 5}W_5^{\mu 3} + W^{*4}_{\mu 5}W_5^{\mu 4}) \end{aligned} \tag{3.342}$$

The last line adds 8 real gauge bosons so the number of masless generators is : $24-14=10$. So the symmetry breaking is $SU(5) \to Sp(4)$, exactly.

Generalizing the procedure before, we have the factorization:

$$\mathcal{L}_M = g^2c^2 \sum_{l,l'=0}^{L-1} (W^{*2l'+1}_{\mu 2l+2} - W^{*2l+1}_{\mu 2l'+2})(W^{\mu 2l'+1}_{2l+2} - W^{\mu 2l+1}_{2l'+2})$$
$$+ g^2c^2 \sum_{l,l'=0}^{L-1} (W^{*2l+1}_{\mu 2l'+1} + W^{*2l'+2}_{\mu 2l+2})(W^{\mu 2l+1}_{2l'+1} + W^{\mu 2l'+2}_{2l+2}) + g^2c^2\delta_{n\,2L+1} \sum_{i=1}^{2L} W^{*i}_{\mu\,2L+1} W^{\mu i}_{2L+1} \tag{3.343}$$

Remember that $SU(n)$ we can have n odd or even, $n = 2L+1$ or $n = 2L$. The first term contributes with $\frac{L}{2}(L-1)$ complex vector bosons , the second with $\frac{L}{2}(L-1)$ complex vector bosons and L real vector bosons and the last term is different from zero when $n = 2L+1$, i.e. $SU(n)$ is odd, and contributes with $2L$ gauge boson bosons. Thus when n is even we have $2L(L-1)+L$ massive real vector bosons thus the massless vectors are using and "approximate" $SU(n)$ symmetry:

$$(2L)^2 - 2L(L-1) - L = L(2L+1) \tag{3.344}$$

This is the dimension of the algebra of $Sp(n)$ with $n = 2L$. Thus the approximate symmetry breaking is

$$U(n) \to Sp(n) \qquad \text{n even} \tag{3.345}$$

In the odd case we have:

$$(2L+1)^2 - 1 - 2L(L-1) - L - 4L = 4L^2 - 2L(L-\frac{1}{2}) = L(2L+1) \tag{3.346}$$

So the symmetry breaking is

$$SU(2L+1) \to Sp(2L) \qquad \text{n=2L+1 odd} \tag{3.347}$$

3.4.5 Spontaneous Breaking in the Adjoint Representation

A scalar ψ in the adjoint representation can be written as:

$$\psi = \psi^j_i X^j_i \tag{3.348}$$

A similar calculation that lead to Eq.(3.213) shows that it has the same properties as the gauge bosons, which means:

$$\psi^j_i = (\psi^i_j)^* \tag{3.349}$$

$$\sum_{i=1}^{n} \psi^i_i = 0 \tag{3.350}$$

From Eq.(3.349) we have that:

$$(\psi^\dagger)^j_i = \psi^{*i}_j = \psi^j_i \tag{3.351}$$

Thus ψ is hermitian.
The infinitesimal transformation of ψ_i^j is:

$$\psi_i'^j = \psi_i^j + i\epsilon_i^k\psi_k^j - i\epsilon_k^j\psi_i^k \tag{3.352}$$

Since ψ lives in the adjoint representation the covariant derivative is:

$$\mathcal{D}_\mu\psi = \partial_\mu\psi - ig[W_\mu, \psi] = \partial_\mu\psi - igW_{\mu i}^m\psi_m^l X_i^l + igW_{\mu m}^j\psi_k^m X_k^j \tag{3.353}$$

So projecting:

$$\mathcal{D}_\mu\psi_{ij} = \partial_\mu\psi_{ij} - igW_{\mu i}^l\psi_l^j + igW_{\mu l}^j\psi_i^l \tag{3.354}$$

The kinetic Lagrangian is:

$$\mathcal{L}_{kin} = \frac{1}{2}\,\mathrm{Tr}[(\mathcal{D}_\mu\psi)^\dagger\mathcal{D}_\mu\psi] = \frac{1}{2}(\partial_\mu\psi_{ij}^* + igW_{\mu i}^{*l}\psi_l^j - igW_{\mu l}^j\psi_i^l)(\partial^\mu\psi_{ij} - igW_i^{\mu l}\psi_l^j + igW_l^{\mu j}\psi_i^l) \tag{3.355}$$

From this we can extract the mass terms:

$$\mathcal{L}_{M_1} = g^2\left(W_{\mu i}^k\psi_k^j W_i^{\mu * k'}\psi_{k'}^j + W_{\mu k}^j\psi_i^k W_{k'}^{\mu * j}\psi_i^{k'}\right) \tag{3.356}$$

$$\mathcal{L}_{M_2} = -g^2\left(W_{\mu k}^j\psi_i^k W_i^{\mu * k'}\psi_{k'}^j + W_{\mu i}^k\psi_k^j W_{k'}^{\mu * j}\psi_i^{k'}\right) \tag{3.357}$$

The most general invariant potential (imposing $\psi \to -\psi$ symmetry) is:

$$V(\psi) = -\frac{1}{2}\mu^2\psi_i^j\psi_j^i + \frac{1}{4}\lambda_1(\psi_i^j\psi_j^i)^2 + \frac{1}{4}\lambda_2(\psi_i^j\psi_j^k\psi_k^l\psi_l^i) \tag{3.358}$$

where the crossed index summation is implied. Since ψ is hermitian it can be diagonalized via a unitary matrix:

$$\psi_i^j = (U^\dagger)_k^j\phi_k U_i^k = \delta_i^j\phi_j, \qquad \phi_i \text{ real} \tag{3.359}$$

The potential Eq.(3.348) then can be rewritten as:

$$V = -\frac{1}{2}\mu^2\sum_{i=1}^n\phi_i^2 + \frac{1}{4}\lambda_1\left(\sum_{i=1}^n\phi_i^2\right)^2 + \frac{1}{4}\lambda_2\left(\sum_{i=1}^n\phi_i^4\right) - g\sum_{i=1}^n\phi_i \tag{3.360}$$

where we have added the restriction Eq.(3.350) as a Lagrange multiplier. This equation has the same structure as the one used to calculate the minimum in the symmetric tensor in the

$O(n)$ group as is shown in Eq.(3.113). Following these results, the vev is:

$$\langle\psi\rangle = \begin{pmatrix} \phi_1 & & & & & \\ & \ddots & & & & \\ & & \phi_1 & & & \\ & & & \phi_2 & & \\ & & & & \ddots & \\ & & & & & \phi_2 \end{pmatrix} \tag{3.361}$$

Where we have n_1 (n_2) entries with value ϕ_1 (ϕ_2). The stability condition is the same as Eq.(3.136) with $n_3 = 0$

In the case where $\lambda_2 > 0$ we have that n_1 is:

$$\begin{aligned} n_1 =& \frac{1}{2} n \text{ even} \\ n_1 =& \frac{1}{2}(n+1) \text{ odd} \end{aligned} \tag{3.362}$$

If $\lambda_2 < 0$ we have that $n_1 = n - 1$.

Replacing the vev:

$$\langle\psi\rangle_{ij} = \sum_{l=1}^{n_1} \phi_1 \delta_{i\,l}\delta_{j\,l} + \sum_{l=n_1}^{n} \phi_2 \delta_{i\,l}\delta_{j\,l} \tag{3.363}$$

From the kinetic term inserting the vev $\psi \to \langle\psi\rangle$ we have that the mass terms in the Lagrangian are:

$$\begin{aligned} \mathcal{L}_{M_1} =& g^2 \left(W^k_{\mu i} \langle\psi^j_k\rangle W^{\mu * k'}_i \langle\psi^j_{k'}\rangle + W^j_{\mu k} \langle\psi^k_i\rangle W^{\mu * j}_{k'} \langle\psi^{k'}_i\rangle \right) \\ =& g^2 \left(\phi_1^2 \sum_{l=1}^{n_1} W^l_{\mu i} W^{\mu * l}_i + \phi_2^2 \sum_{l=n_1+1}^{n} W^l_{\mu i} W^{\mu * l}_i + \phi_1^2 \sum_{l=1}^{n_1} W^j_{\mu l} W^{\mu * j}_l + \phi_2^2 \sum_{l=n_1+1}^{n} W^j_{\mu l} W^{\mu * j}_l \right) \end{aligned} \tag{3.364}$$

$$\begin{aligned} \mathcal{L}_{M_2} =& -g^2 \left(W^j_{\mu k} \langle\psi^k_i\rangle W^{\mu * k'}_i \langle\psi^j_{k'}\rangle + W^k_{\mu i} \langle\psi^j_k\rangle W^{\mu * j}_{k'} \langle\psi^{k'}_i\rangle \right) \\ =& -2g^2 \left(\phi_1^2 \sum_{l,l'=1}^{n_1} W^{l'}_{\mu l} W^{\mu * l'}_l + \phi_2^2 \sum_{l,l'=n_1+1}^{n} W^{l'}_{\mu l} W^{\mu * l'}_l \right) \\ & - 2g^2 \phi_1 \phi_2 \sum_{l'=1}^{n_1} \sum_{l=n_1+1}^{n} (W^{l'}_{\mu l} W^{\mu * l'}_l + W^l_{\mu l'} W^{\mu * l}_{l'}) \end{aligned} \tag{3.365}$$

Thus we have:

$$\mathcal{L}_M = 2(\phi_1 - \phi_2)^2 \sum_{l'=1}^{n_1} \sum_{l=n_1+1}^{n} W^{l'}_{\mu l} W^{* \mu l'}_l \tag{3.366}$$

The number of gauge bosons that acquire mass is all the possible combination of ψ with the first index less than n_1 (n_1 indices) and the other more than n_1 ($n - n_1$ indices). Thus we have $n_1(n - n_1)$ possible combinations. Since these are alternate terms, gauge bosons are complex thus we have $2n_1(n - n_1)$ real bosons that acquire mass.
Since:

$$n^2 - 1 - 2n_1(n - n_1) = (n - n_1)^2 - 1 + (n_1)^2 - 1 + 1 \tag{3.367}$$

We see that the symmetry breaking is:

$$SU(n) \to SU(n - n_1) \times SU(n_1) \times U(1) \tag{3.368}$$

In the case $\lambda_2 > 0$ we have:

$$\begin{aligned} n_1 &= \frac{1}{2} n \text{ even} \\ n_1 &= \frac{1}{2}(n+1) \text{ odd} \end{aligned} \tag{3.369}$$

In the case $\lambda_2 < 0$, we have $n_1 = n - 1$ so the symmetry breaking is:

$$SU(n) \to SU(n-1) \times U(1) \tag{3.370}$$

As an example we deal with the $SU(3)$. In the case $\lambda_2 < 0$, we have $n_1 = 2$. Using eq. Eq.(3.366) :

$$\mathcal{L}_M = 2(\phi_1 - \phi_2)^2 \sum_{l'=1}^{3} W^{l'}_{\mu 3} W_3^{\mu * l'} = 2(\phi_1 - \phi_2)^2 \left(W^1_{\mu 3} W_3^{*\mu 1} + W^2_{\mu 3} W_3^{\mu * 2} \right) \tag{3.371}$$

Clearly 4 bosons acquire mass. Thus we have 4 massless bosons. The symmetry breaking pattern is $SU(3) \to SU(2) \times U(1)$.When $\lambda_2 > 0$ we have $n_1 = 2$ the symmetry breaking pattern is the same.
Another example can be found with $SU(5)$. In the case $\lambda_2 > 0$, we have $n_1 = 3$. Using eq. 3.366 :

$$\begin{aligned} \mathcal{L}_M = 2(\phi_1 - \phi_2)^2 \sum_{l'=1}^{3} \sum_{l=4}^{5} W^{l'}_{\mu l} W_l^{\mu * l'} = 2(\phi_1 - \phi_2)^2 \Big(& W^1_{\mu 4} W_4^{*\mu 1} + W^1_{\mu 5} W_5^{*\mu 1} + W^2_{\mu 4} W_4^{*\mu 2} \\ & + W^2_{\mu 5} W_5^{*\mu 2} + W^3_{\mu 4} W_4^{*\mu 3} + W^3_{\mu 5} W_5^{*\mu 3} \Big) \end{aligned} \tag{3.372}$$

Thus 12 vectors acquire mass then we have 24-12 = 12 massless gauge bosons. These are the 8 generators of $SU(3)$, 3 of $SU(2)$ and 1 generator for $U(1)$. This is a very important fact since it is the the breaking of the grand unification model using $SU(5)$:

$$SU(5) \to SU(3) \times SU(2) \times U(1) \tag{3.373}$$

In the case where $\lambda_2 < 0$ we have that $n_1 = 4$, so:

$$\mathcal{L}_M = 2(\phi_1 - \phi_2)^2 \sum_{l'=1}^{4} W^{l'}_{\mu 5} W^{\mu * l'}_{5} = 2(\phi_1 - \phi_2)^2 \Big(W^1_{\mu 5} W^{*\mu 1}_5 + W^2_{\mu 5} W^{*\mu 2}_5 + W^3_{\mu 5} W^{*\mu 3}_5 + W^4_{\mu 5} W^{*\mu 4}_5 \Big) \tag{3.374}$$

We have 8 bosons that acquire mass so 24-8 =16 massless bosons. These are equal to 15 bosons of $SU(4)$ and a single boson for $U(1)$.
The symmetry breaking pattern is: $SU(5) \to SU(4) \times U(1)$.

3.5 Spontaneous Breaking of products of Simple Groups

In the following section we develop the transformations of representations of product Groups i.e. representations of $G_1 \times G_2$. This is important since in the case where $G_1 = G_2$ and $g_1 = g_2$ we have that the fundamental representation of tensor product $G_1 \times G_2$ as a gauge group is isomorphic (transforms the same way both scalar boson and gauge boson) to the second tensor reducible irrep of G_1 that we have used in the chapter. In this way for the orthogonal and special unitary symmetries we retrieve the laws of transformations and the covariant derivative for the second rank antisymmetric and symmetric representations.

First let's continue the procedure of Section (1.5) and see in general how tensor irreps of a compact group G transform. For a ψ transforming in the second rank symmetric traceless or antisymmetric representation the kinetic term is:

$$\mathcal{L}_\psi = i\,\mathrm{Tr}[\bar{\psi}\partial_\mu \psi] = i\bar{\psi}_{ij}\partial_\mu \psi_{ij} \qquad i,j = 1,\cdots,d_G \tag{3.375}$$

and for a ϕ:

$$\|\partial_\mu \phi\|^2 = \mathrm{Tr}[(\partial_\mu \phi)^\dagger \partial^\mu \phi] = \partial_\mu \phi^*_{ij} \partial^\mu \phi_{ij} \qquad i,j = 1,\cdots,d_G \tag{3.376}$$

and in general for a K tensorial (K indices) irreducible representation:

$$\mathcal{L}_\psi = i\bar{\psi}_{i_1\cdots i_K}\partial_\mu \psi_{i_1\cdots i_K} \qquad i_1,\cdots,i_K = 1,\cdots,d_G \tag{3.377}$$

$$\|\partial_\mu \phi\|^2 = \partial_\mu \phi^*_{i_1\cdots i_K}\partial^\mu \phi_{i_1\cdots i_K} \qquad i_1,\cdots,i_K = 1,\cdots,d_G \tag{3.378}$$

since for each index we have both a fundamental and antifundamental transformation that cancel each other since representations with different indices commute since live in different tensor spaces. For example, for the transformation:

$$\psi_{i_1\cdots i_K} \to \psi'_{i_1\cdots i_K} = U_{i_1 i'_1}\cdots U_{i_K i'_K}\psi_{i'_1\cdots i'_K} \equiv [U\psi]_{i_1\cdots i_K} \tag{3.379}$$

we have:

$$\begin{aligned} i\bar{\psi}'_{i_1\cdots i_K}\partial_\mu\psi'_{i_1\cdots i_K} =& i\bar{\psi}_{i''_1\cdots i''_K}U^*_{i_1 i''_1}\cdots U^*_{i_K i''_K}U_{i_1 i'_1}\cdots U_{i_K i'_K}\partial_\mu\psi_{i'_1\cdots 1'_K} = i\bar{\psi}_{i'_1\cdots 1'_K}\partial_\mu\psi_{i''\cdots i''_K}\delta_{i'_1 i''_1}\cdots\delta_{i'_K 1''_K} \\ =& i\bar{\psi}_{i\cdots j}\partial_\mu\psi_{i\cdots j} \end{aligned} \tag{3.380}$$

Even more we can create tensor with both fundamental and antifundamental indices. In order to use these transformations in a gauged theory we need to know how the covariant derivative transform. We need then:

$$\mathcal{D}_\mu\Phi \to \mathcal{D}'_\mu\Phi' = U(\alpha)\mathcal{D}_\mu\Phi \tag{3.381}$$

with explicitely $U(\alpha) = U(\alpha)\otimes\cdots\otimes U(\alpha_K)$ is the tensor representation, where the $U(\alpha)$'s are fundamental or antifundamental. So the Yang Mills has to transforms in the same way:

$$A'_\mu = U(\alpha)\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^{-1}(\alpha) \tag{3.382}$$

The difficulty is now to calculate explicitly form of the transformation. In the next part we focus on second rank tensor representations for the orthogonal and special unitary groups.

3.5.1 Spontaneous Breaking of $O(n)\times O(m)$

Let the gauge symmetry be $G_1\times G_2$ with $G_1 = O(n), G_2 = O(m)$ where for convenience (otherwise we have a symmetric in the indices argument) we have $n\geq m$. The transformation on each fundamental representation is:

$$\phi^{(1)\prime}_i = \phi^{(1)}_i + \epsilon_{1ij}\phi^{(1)}_j \qquad i,j = 1,\cdots,n \tag{3.383}$$

$$\phi^{(2)\prime}_\alpha = \phi^{(2)}_\alpha + \epsilon_{2\,\alpha\beta}\phi^{(2)}_\beta \qquad \alpha,\beta = 1,\cdots,m \tag{3.384}$$

with their respective Yang Mills field transforming as Eq.(3.42):

$$W^{(1)}_{\mu\,ij} = W^{(1)}_{\mu ij} + \epsilon_{1\,ik}W^{(1)}_{\mu kj} + \epsilon_{1jk}W^{(1)}_{\mu ik} + \frac{1}{g_1}(\partial_\mu\epsilon_{1\,ij}) \qquad i,j = 1,\cdots,n \tag{3.385}$$

$$W^{(2)}_{\mu\,\alpha\beta} = W^{(2)}_{\mu\alpha\beta} + \epsilon_{2\,\alpha\gamma}W^{(2)}_{\mu\gamma\beta} + \epsilon_{2\beta\gamma}W^{(1)}_{\mu\alpha\gamma} + \frac{1}{g_2}(\partial_\mu\epsilon_{2\,\alpha\beta}) \qquad \alpha,\beta = 1,\cdots,m \tag{3.386}$$

Remember that this group has only real representations so there are no proper antifundamental representations (see Eq.(5.1)) so we only focus on fundamentals. In general we can construct N- rank tensor representations such that M indexes transforms as the fundamental representations of $O(n)$ and $N-M$ as the fundamental representation of $O(m)$:

$$\begin{gathered}\phi_{ik\cdots\alpha\gamma} \to \phi'_{ik\cdots\alpha\gamma} = \underbrace{O_{1\,ij}\cdots O_{1\,kl}}_{M\text{terms}}\underbrace{O_{2\,\alpha\beta}\cdots O_{2\,\gamma\delta}}_{N-M\text{terms}}\phi_{j\cdots l\beta\cdots\delta} \\ i,j,k,l=1,\cdots,n;\alpha,\beta,\gamma,\delta=1,\cdots,m\end{gathered} \tag{3.387}$$

Where we denote O_k the fundamental matrix representation of the group k with $k=1,2$. These representations are not necessarily irreducible.

Now let's focus on the second rank tensor representations:

$$\phi_{i\alpha} = \phi_i^{(1)}\phi_\alpha^{(2)} \tag{3.388}$$

When the second rank tensor representation is of type $\mathbf{n}\otimes\mathbf{1}$, that is the tensor transforms as an n-dimensional vector (fundamental representation) under $O(n)$ but as a singlet under $O(m)$, we have:

$$\phi'_{i\alpha} = (\phi_i^{(1)} + \epsilon_{1ij}\phi_j^{(1)})\phi_\alpha^{(2)} = \phi_{i\alpha} + \epsilon_{1ij}\phi_{j\alpha} \qquad i,j=1,\cdots,n \tag{3.389}$$

In this case the most general potential will have the same form as the one from Section(3.3.1) under $O(n)$ symmetry. The vev will be then $\langle\phi_{i\alpha}\rangle == v\delta_{in}$. The symmetry breaking pattern is $O(n)\times O(m)\to O(n-1)\times O(m)$. This means that the symmetry breaking pattern is not disturbed by other symmetries if the scalar is a singlet under other symmetries. Similarly, when $(1,m)$ the symmetry breaking is $O(n)\times O(m)\to O(n)\times O(m-1)$. Then, the most simple non trivial representation is where each $\phi^{(1)},\phi^{(2)}$ transform simultaneously under the fundamental representation of the orthogonal groups, (n,m). Infinitesimally:

$$\phi'_{i\alpha} = \phi_{ia} + \epsilon_{1\,ij}\phi_{j\alpha} + \epsilon_{2\,\alpha\beta}\phi_{i\beta} \qquad i,j=1,\cdots,n; \quad \alpha,\beta=1,\cdots,m. \tag{3.390}$$

or finitely as [16]:

$$\phi'_{i\alpha} = O_{1\,ij}O_{2\,\alpha\beta}\phi_{j\beta} = [O_1\phi O_2^T]_{i\alpha} \tag{3.391}$$

This is an irreducible representation.
The usual kinetic term is invariant under global transformations:

$$2\mathcal{L}_{\text{kin}} = \text{Tr}\left[(\partial_\mu\phi)^\dagger\partial^\mu\phi\right] = \text{Tr}\left[(\partial_\mu\phi')^\dagger\partial^\mu\phi'\right] \tag{3.392}$$

In fact :

$$\begin{aligned}\text{Tr}\left[(\partial_\mu\phi)'^\dagger\partial^\mu\phi'\right] &= \text{Tr}\left[(\partial_\mu O_1\phi O_2^T)^\dagger\partial^\mu(O_1\phi O_2^\dagger)\right] \\ &= \text{Tr}\left[O_2(\partial_\mu\phi)^\dagger O_1^\dagger O_1(\partial^\mu\phi)O_2^\dagger\right] = \text{Tr}\left[\partial_\mu\phi^\dagger\partial^\mu\phi\right]\end{aligned} \tag{3.393}$$

Where we have used the fact that for orthogonal representations we have: $O_k^\dagger = O_k^T$.
If we do a gauge transformation the terms that depend on $\partial_\mu\epsilon$ do not cancel.

[16] We simplify notation: $O_k \equiv O(\epsilon_k)$, an element of the representation of $SO(n_k)$ in the fundamental representation with $n_1=n, n_2=m$

As before we need to define an appropriate covariant derivative such that:

$$\mathcal{D}_\mu\phi \to \mathcal{D}'_\mu\phi' = O_1(\mathcal{D}_\mu\phi)O_2^T \tag{3.394}$$

in such way that under a trace property the kinetic product is invariant. So, we define the covariant derivative as:

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - \frac{1}{2}g_1 W_\mu^{(1)}\phi - \frac{1}{2}g_2\phi W_\mu^{(2)T} \tag{3.395}$$

where each field W transforms globally as Eq.(3.40) . The transformation is:

$$\begin{aligned}(\mathcal{D}_\mu\phi)' =&\partial_\mu\phi' - \frac{g_1}{2}W_\mu^{(1)\prime}\phi' - \frac{g_2}{2}\phi' W_\mu^{(2)\dagger\prime} = \partial_\mu(O_1\phi O_2^\dagger) - \frac{g_1}{2}O_1\left((W_\mu^{(1)} - \frac{2}{g_1}\partial_\mu)O_1^{-1}\right)O_1\phi O_2^\dagger\\ &- \frac{g_2}{2}(O_1\phi O_2^\dagger)\left(O_2(W_\mu^{(2)} - \frac{2}{g_2}\partial_\mu)O_2^{-1}\right)^\dagger\\ =&O_1(\mathcal{D}_\mu\phi)O_2^\dagger + \partial_\mu(O_1)\phi O_2^\dagger + O_1\phi\partial_\mu O_2^\dagger + O_1\partial_\mu(O_1^{-1})O_1\phi O_2^\dagger + O_1\phi O_2^\dagger(O_2\partial_\mu O_2^{-1})^\dagger\\ =&O_1(\mathcal{D}_\mu\phi)O_2^\dagger + \partial_\mu(O_1)\phi O_2^\dagger + O_1\phi\partial_\mu O_2^\dagger - \partial_\mu(O_1)\phi O_2^\dagger - O_1\phi\partial_\mu O_2^\dagger\\ =&O_1(\mathcal{D}_\mu\phi)O_2^\dagger\end{aligned} \tag{3.396}$$

Then the new kinetic term that is gauge invariant is:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\,\text{Tr}[(\mathcal{D}^\mu\phi)^\dagger\mathcal{D}_\mu\phi] \tag{3.397}$$

Using Eq.(3.35) and Eq.(3.395) we have explicitely:

$$\begin{aligned}\mathcal{D}_\mu\phi_{i\alpha} =&\partial_\mu\phi_{i\alpha} - \frac{g_1}{2}W_{\mu ab}^{(1)}(L_{ab})_{ik}\phi_{k\alpha} - \frac{g_2}{2}\phi_{i\beta}(W_{\mu ab}^{(2)}L_{ab})^\dagger_{\beta\alpha}\\ =&\partial_\mu\phi_{i\alpha} - \frac{g_1}{2}(W_{\mu ik}^{(1)} - W_{\mu\, ki}^{(1)})\phi_{k\alpha} - \frac{g_2}{2}\phi_{i\beta}W_{\mu ab}(L_{ab})_{\alpha\beta}\\ =&\partial_\mu\phi_{i\beta} - g_1 W_{\mu ik}^{(1)}\phi_{k\alpha} - g_2 W_{\mu\alpha\beta}^{(2)}\phi_{i\beta}\end{aligned} \tag{3.398}$$

The kinetic term for a second rank tensor is then:

$$\begin{aligned}\mathcal{L}_{\text{kin}} =&\frac{1}{2}\,\text{Tr}[(\mathcal{D}^\mu\phi)^\dagger\mathcal{D}_\mu\phi] = \frac{1}{2}(\mathcal{D}^\mu\phi)^\dagger_{\alpha i}(\mathcal{D}_\mu\phi)_{i\alpha} = \frac{1}{2}(\mathcal{D}^\mu\phi)_{i\alpha}(\mathcal{D}_\mu\phi)_{i\alpha}\\ =&\frac{1}{2}\partial^\mu\phi_{i\alpha}\partial_\mu\phi_{i\alpha} - (g_1 W_{ik}^{(1)\mu}\phi_{k\alpha} + g_2 W_{\alpha\beta}^{(2)\mu}\phi_{i\beta})\partial_\mu\phi_{i\alpha} + \frac{1}{2}(g_1^2 W_{ik}^{(1)\mu}\phi_{k\alpha}W_{\mu il}^{(1)}\phi_{l\alpha}\\ &+ g_2^2 W_{\alpha\beta}^{(2)\mu}\phi_{i\beta}W_{\mu\alpha\gamma}^{(2)}\phi_{i\gamma}) + g_1 g_2 W_{ik}^{(1)\,\mu}\phi_{k\alpha}W_{\mu\,\alpha\beta}^{(2)}\phi_{i\beta}\end{aligned} \tag{3.399}$$

Note that the gauge boson masses will come from the terms of second order in the couplings:

$$\mathcal{L}_M = \frac{1}{2}(g_1^2 W_{ik}^{(1)\mu}\phi_{k\alpha}W_{\mu il}^{(1)}\phi_{l\alpha} + g_2^2 W_{\alpha\beta}^{(2)\mu}\phi_{i\beta}W_{\mu\alpha\gamma}^{(2)}\phi_{i\gamma}) + g_1 g_2 W_{ik}^{(1)\,\mu}\phi_{k\alpha}W_{\mu\,\alpha\beta}^{(2)}\phi_{i\beta} \tag{3.400}$$

Minimization of the Potential

The invariant potential is:

$$V = -\frac{1}{2}\mu^2\phi_{i\alpha}\phi_{i\alpha} + \frac{1}{4}\lambda_1(\phi_{i\alpha}\phi_{i\alpha})^2 + \frac{1}{4}\lambda_2(\phi_{i\alpha}\phi_{i\beta})(\phi_{j\alpha}\phi_{j\beta}) \tag{3.401}$$

The set of minima satisfy:

$$\left.\frac{\partial V}{\partial \phi_{i\alpha}}\right|_{\phi=\langle\phi\rangle} = -\mu^2\langle\phi_{i\alpha}\rangle + \lambda_1(\langle\phi_{j\beta}\rangle\langle\phi_{j\beta}\rangle)\langle\phi_{i\alpha}\rangle + \lambda_2\langle\phi_{i\beta}\rangle(\langle\phi_{j\alpha}\rangle\langle\phi_{j\beta}\rangle) = 0 \tag{3.402}$$

Introducing:

$$X_{ij} = \sum_{\beta=1}^{m}\langle\phi_{i\beta}\rangle\langle\phi_{j\beta}\rangle = (\langle\phi\rangle\langle\phi\rangle^T)_{ij} \tag{3.403}$$

This is a real and symmetric $n \times n$ matrix and can be diagonalized by an orthogonal transformation then $X_{ij} = \delta_{ij}X_j$. Then Eq.(3.402) is:

$$\left(-\mu^2 + \lambda_1\sum_{j=1}^{n}X_j + \lambda_2 X_i\right)\langle\phi_{i\alpha}\rangle = 0 \tag{3.404}$$

This equation has the same structure as Eq.(3.71). Using the same argument we have:

$$X_i = \frac{\mu^2}{\lambda_1 K + \lambda_2}, \qquad i = 1,\cdots,K \tag{3.405}$$

$$X_i = 0, \quad i = k+1,\cdots,n \tag{3.406}$$

and

$$V = \frac{K\mu^4}{\lambda_1 K + \lambda_2} \tag{3.407}$$

The potential is monotonically increasing for $\lambda_2 < 0$. The minimum is at $K = 1$:

$$X = b\begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, b = \frac{\mu^2}{\lambda_1 + \lambda_2} \tag{3.408}$$

For $\lambda_2 > 0$ the potential is monotonically decreasing function of K hence the minimum is at the largest allowed value of K. This in principle could be:

$$X = c^2\mathbf{1}_n = c^2\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{3.409}$$

but since $n > m$ and considering each row as an m dimensional vector, we see that this not make sense since we at most can have m orthogonal m dimensional vectors. Thus X is:

$$X = c^2 \begin{pmatrix} \mathbf{1}_m & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad c^2 = \frac{\mu^2}{n\lambda_1 + \lambda_2} \tag{3.410}$$

i.e. a matrix with entries different than zero only in a block $m \times m$.

Symmetry Breaking Patterns

The vev when $\lambda_2 < 0$ is the $n \times m$ matrix:

$$\phi = \begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, b = \frac{\mu^2}{\lambda_1 + \lambda_2} \tag{3.411}$$

The vev when $\lambda_2 > 0$ is:

$$\phi = c \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, c = \frac{\mu^2}{m\lambda_1 + \lambda_2} \tag{3.412}$$

The vev in both case is represented as:

$$\langle \phi \rangle_{i\alpha} = d \sum_{l=1}^{K} \delta_{il}\delta_{l\alpha} \tag{3.413}$$

with $d = b(c)$ and $K = 1(m)$ for λ_2 positive (negative).

Case $\lambda_2 < 0$

Inserting the vev:

$$\begin{aligned}\mathcal{L}_M =&\frac{1}{2}\left(g_1^2 W_{ik}^{(1)\mu}\left\langle\phi\right\rangle_{k\alpha} W_{\mu il}^{(1)}\left\langle\phi\right\rangle_{l\alpha} + g_2^2 W_{\alpha\beta}^{(2)\mu}\left\langle\phi\right\rangle_{i\beta} W_{\mu\alpha\gamma}^{(2)}\left\langle\phi\right\rangle_{i\gamma}\right) + g_1 g_2 W_{ik}^{(1)\,\mu}\left\langle\phi\right\rangle_{k\alpha} W_{\mu\,\alpha\beta}^{(2)}\left\langle\phi\right\rangle_{i\beta}\\ =&\frac{1}{2}(g_1^2 b^2 W_{i1}^{(1)\mu} W_{\mu i1}^{(1)} + g_2^2 b^2 W_{\alpha 1}^{(2)\mu} W_{\mu\alpha 1}^{(2)}) + g_1 g_2 d^2 W_{11}^{(1)\,\mu} W_{\mu\,11}^{(2)}\\ =&\frac{1}{2}(g_1^2 b^2 W_{i1}^{(1)\mu} W_{\mu i1}^{(1)} + g_2^2 b^2 W_{\alpha 1}^{(2)\mu} W_{\mu\alpha 1}^{(2)})\end{aligned} \tag{3.414}$$

Where we used the fact that $W_{11}^{(1)\mu} = 0$ and $W_{11}^{(2)\mu} = 0$. So we have $n-1$ gauge bosons that acquire mass $g_1 b$ and $m-1$ gauge bosons with mass $g_2 b$ thus the symmetry breaking is:

$$O(n)\times O(m) \to O(n-1)\times O(m-1) \tag{3.414}$$

Case $\lambda_2 > 0$

Inserting the vev we have:

$$\begin{aligned}\mathcal{L}_M =&\frac{1}{2}\left(g_1^2 W_{ik}^{(1)\mu}\left\langle\phi\right\rangle_{k\alpha} W_{\mu il}^{(1)}\left\langle\phi\right\rangle_{l\alpha} + g_2^2 W_{\alpha\beta}^{(2)\mu}\left\langle\phi\right\rangle_{i\beta} W_{\mu\alpha\gamma}^{(2)}\left\langle\phi\right\rangle_{i\gamma}\right)\\ &+ g_1 g_2 W_{ik}^{(1)\,\mu}\left\langle\phi\right\rangle_{k\alpha} W_{\mu\,\alpha\beta}^{(2)}\left\langle\phi\right\rangle_{i\beta}\\ =&\frac{1}{2}\left(g_1^2 c^2\sum_{l=1}^{m} W_{il}^{(1)\mu} W_{\mu il}^{(1)} + g_2^2 c^2\sum_{l=1}^{m} W_{\alpha l}^{(2)\mu} W_{\mu\alpha\gamma}^{(2)}\right) + \sum_{l,l'=1}^{m} g_1 g_2 d^2 W_{ll'}^{(1)\,\mu} W_{\mu\,l'l}^{(2)}\\ =&\frac{1}{2}d^2\sum_{l,l'=1}^{m}\left(g_1 W_{ll'}^{(1)\mu} W_{\mu ll'}^{(1)} - g_2 W_{ll'}^{(2)\mu} W_{\mu ll'}^{(2)}\right)^2 + \frac{1}{2}g_1^2 d^2\sum_{l=1}^{m}\sum_{l=m+1}^{n} W_{ll'}^{(1)\mu} W_{\mu ll'}^{(1)}\\ =&\frac{1}{2}d^2(g_1^2+g_2^2)\sum_{l,l'=1}^{m}(\cos\theta W_{ll'}^{(1)\mu} W_{\mu ll'}^{(1)} - \sin\theta W_{ll'}^{(2)\mu} W_{\mu ll'}^{(2)})^2\\ &+\frac{1}{2}g_1^2 d^2\sum_{l=1}^{m}\sum_{l=m+1}^{n} W_{ll'}^{(1)\mu} W_{\mu ll'}^{(1)}\end{aligned} \tag{3.415}$$

with $\sin\theta = \frac{g_2}{\sqrt{(g_1^2+g_2^2}}$. Thus we have $\frac{m(m-1)}{2}$ gauge bosons with mass $\sqrt{g_1^2+g_2^2}d$ and $n(n-m)$ gauge bosons with mass $g_1 d$. Then the massless bosons are:

$$\frac{n(n-1)}{2} + \frac{m(m-1)}{2} - \frac{m(m-1)}{2} - \frac{2m(n-m)}{2} = \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} \tag{3.416}$$

Thus the symmetry breaking is:

$$O(n)\times O(m) \to O(m)\times O(n-m) \tag{3.417}$$

Case $O(n) = O(m)$

First of all assuming that even if $n = m$ we have that each transformation O_1 is independent of O_2, we get an irreducible representation and the preceding results are still valid. Now asuming that both transformations are simultanoeus we effectively see that the tensor $\phi_{ij} = \phi_{1i}\phi_{2j}$ is isomorphic to the full second rank representation of $O(n)$ since both transforms in the same way. This usual second rank tensor product Eq.(3.390) is not an irreducible representation. It can be shown that the second rank tensor decomposes into irreducible representation which are [17]: a second rank traceless symmetric, second rank antisymmetric and a scalar, with dimensions $\frac{n(n-1)}{2} - 1$, $\frac{n(n+1)}{2}$ and 1 respectively. Explicitly, for the tensors, we have:

$$\phi_{ij} \equiv \phi_{1i}\phi_{2j} \pm \phi_{2j}\phi_{1i} \tag{3.418}$$

where ϕ_{1i} and ϕ_{2j} transform in the fundamental representation of $O(n)$ and $+(-)$ gives the symmetric (antisymmetric) representation. In addition to make the symmetric representation irreducible we impose the traceless condition $\mathrm{Tr}[\phi] = 0$. A finite local or global transformation follows:

$$\begin{aligned}\phi'_{ij} = O_{ik}\phi_{1k}O_{jk'}\phi_{2k'} \pm O_{2jk'}\phi_{2k'}O_{1ik}\phi_{1k} &= O_{1ik}\phi_{kk'}O_{2jk'}\\ &= (O_1\phi O_2^T)_{ij}\end{aligned} \tag{3.419}$$

Then the covariant derivative constructed in the precedent paragraphs is still valid with some obvious replacements. This leads to the results in the various sections of the chapter. To preserve gauge invariance we do noot need that $g_1 = g_2$ to preserve gauge invariance at the level of the Lagrangian.

$$\begin{aligned}\mathcal{D}_\mu\phi_{i\alpha} =&\partial_\mu\phi - \frac{g_1}{2}W^{(1)}_{\mu ab}(L_{ab})_{ik}\phi_{k\alpha} - \frac{g_2}{2}\phi_{i\beta}(W^{(2)}_{\mu ab}L_{ab})^\dagger_{\beta\alpha} = \partial_\mu\phi_{i\beta} - \frac{g_1}{2}(W^{(1)}_{\mu ik} - W^{(1)}_{\mu\, ki})\phi_{k\beta}\\ &- \frac{g_2}{2}\phi_{i\beta}W_{\mu ab}(L_{ab})_{\alpha\beta}\\ =&\partial_\mu\phi_{i\beta} - g_1W^{(1)}_{\mu ik}\phi_{k\alpha} - g_2W^{(2)}_{\mu\alpha\beta}\phi_{i\beta}\end{aligned} \tag{3.420}$$

And if $g_1 \neq g_2$ we arrive at the result of Eq.(3.399). Using $g_1 = g_2$ and $W^{(1)}_\mu = W^{(2)}_\mu$ as we have done in the chapter to arrive at the following basic result.

$$\begin{aligned}\mathcal{L}_{\text{kin}} =&\frac{1}{2}\mathrm{Tr}[(\mathcal{D}^\mu\phi)^\dagger\mathcal{D}_\mu\phi] = \frac{1}{2}(\mathcal{D}^\mu\phi)_{ij}(\mathcal{D}_\mu\phi)_{ij}\\ =&\frac{1}{2}\partial^\mu\phi_{ij}\partial_\mu\phi_{ij} - \frac{1}{2}g(W^\mu_{ik}\phi_{kj} + W^\mu_{jk}\phi_{ik})\partial_\mu\phi_{ik} + g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj} + W^\mu_{ik}\phi_{kj}W_{\mu\, jl}\phi_{il})\\ =&\frac{1}{2}\partial^\mu\phi_{ij}\partial_\mu\phi_{ij} - g(W^\mu_{ik}\phi_{kj} + W^\mu_{jk}\phi_{ik})\partial_\mu\phi_{ik} + g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj})\\ &\pm g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{li})\end{aligned} \tag{3.421}$$

[17] for example following a procedure similar to the Young Diagrams of Chapter 5

Note that the gauge boson masses will come from the terms proportional to g^2

$$\mathcal{L}_M = g^2(W^\mu_{ik}\phi_{kj}W_{\mu il}\phi_{lj}) \pm g^2(W^\mu_{ik}\phi_{kj}W_{\mu jl}\phi_{li}) \tag{3.422}$$

3.5.2 Spontaneous Breaking of $SU(n) \times SU(m)$

Transformations

Let the gauge symmetry be $G_1 \times G_2$ with $G_1 = SU(n), G_2 = SU(m)$.[18] The transformation on each fundamental representation is:

$$\phi^{(1)\prime}_i = \phi^{(1)}_i + \epsilon^j_{1i}\phi^{(1)}_j \qquad i,j = 1,\cdots,n \tag{3.423}$$

$$\phi^{(2)\prime}_\alpha = \phi^{(2)}_\alpha + \epsilon^{\beta}_{2\,\alpha}\phi^{(2)}_\beta \qquad \alpha,\beta = 1,\cdots,m \tag{3.424}$$

with their respective Yang Mills fied transforming as:

$$W^{(1)\,j}_{\mu\,i} = W^{(1)\,j}_{\mu i} + i\epsilon^k_{1\,i}W^{(1)}_{\mu kj} - i\epsilon^j_{1k}W^{(1)\,k}_{\mu\ \ \ i} + \frac{1}{g_1}(\partial_\mu \epsilon^j_{1\,i}) \qquad i,j = 1,\cdots,n \tag{3.425}$$

$$W^{(2)\,\beta}_{\mu\,\alpha} = W^{(2)\,\beta}_{\mu\alpha} + i\epsilon^\gamma_{2\,\alpha}W^{(2)\,\beta}_{\mu\gamma} - i\epsilon^\beta_{2\gamma}W^{(2)\,\gamma}_{\mu\alpha} + \frac{1}{g_2}(\partial_\mu \epsilon^\beta_{2\,\alpha}) \qquad \alpha,\beta = 1,\cdots,m \tag{3.426}$$

The case where the second rank tensor is a singlet under a symmetry group gives, using an analogous reasoning as with the $O(n) \times O(m)$ symmetry, the symmetry breaking patterns $SU(n)\times SU(m) \to SU(n-1)\times SU(m)$ if it is a singlet under $SU(m)$ and $SU(n)\times SU(m) \to SU(n) \times SU(m-1)$ if it is a singlet under $SU(n)$.

The lowest dimensional irreducible representation is then the tensor product of fundamental representations $\mathbf{n} \times \mathbf{m}$[19](bifundamental representation) transforming infinitesimally as:

$$\psi'_{i\alpha} = \psi_{ia} + \epsilon^j_{1\,i}\psi_{j\alpha} + \epsilon^\beta_{2\,\alpha}\psi_{i\beta} \qquad i,j = 1,\cdots,n; \quad \alpha,\beta = 1,\cdots,m. \tag{3.427}$$

The complex conjugate $\bar{\mathbf{n}} \times \bar{\mathbf{m}}$ transforms as:

$$\psi^{i\alpha\prime} = \psi^{ia} + \epsilon^i_{1\,j}\psi^{j\alpha} + \epsilon^\alpha_{2\,\beta}\psi^{i\beta} \qquad i,j = 1,\cdots,n; \quad \alpha,\beta = 1,\cdots,m. \tag{3.428}$$

As in the orthogonal case, the action of the covariant derivative on ψ under the bifundamental transforms as:

$$\mathcal{D}'_\mu\psi' = U_1\mathcal{D}_\mu\psi U_2^T \tag{3.429}$$

[18] Again assuming $n \geq m$

[19] if $n \neq m$ and $g_1 \neq g_2$

Then the new kinetic term that is gauge invariant is:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\,\text{Tr}[(\mathcal{D}^\mu\psi)^\dagger\mathcal{D}_\mu\psi] \tag{3.430}$$

Explicitely we have:

$$\begin{aligned}\mathcal{D}_\mu\psi_{i\alpha} =&\partial_\mu\psi - ig_1W^{(1)\,b}_{\mu a}(X^b_a)_{ik}\psi_{k\alpha} - ig_2\psi_{i\beta}(W^{(2)b}_{\mu a}X^b_a)^T_{\beta\alpha}\\ =&\partial_\mu\psi_{i\beta} - ig_1W^{(1)\,k}_{\mu i}\psi_{k\beta} - ig_2\psi_{i\beta}W^{(2)b}_{\mu a}(X^b_a)_{\alpha\beta}\\ =&\partial_\mu\psi_{ij} - ig_1W^{(1)k}_{\mu\,i}\psi_{k\alpha} - ig_2W^{(2)\,\beta}_{\mu\alpha}\psi_{i\beta}\end{aligned} \tag{3.431}$$

Thus the kinetic term for a second rank tensor is then:

$$\begin{aligned}\mathcal{L}_{\text{kin}} =&\frac{1}{2}\,\text{Tr}[(\mathcal{D}^\mu\psi)^\dagger\mathcal{D}_\mu\psi] = \frac{1}{2}(\mathcal{D}^\mu\psi)^*_{i\alpha}(\mathcal{D}_\mu\psi)_{i\alpha} = \frac{1}{2}(\mathcal{D}^\mu\psi^*)_{i\alpha}(\mathcal{D}_\mu\psi)_{i\alpha}\\ =&\frac{1}{2}\partial^\mu\psi^*_{i\alpha}\partial_\mu\psi_{i\alpha} - (g_1W^{(1)\mu*k}_i\psi^*_{k\alpha} + g_2W^{(2)\mu*\,\beta}_\alpha\psi^*_{i\beta})\partial_\mu\psi_{i\alpha} + \frac{1}{2}(g_1^2W^{(1)\mu*k}_i\psi^*_{k\alpha}W^{(1)l}_{\mu i}\psi_{l\alpha}\\ &+ g_2^2W^{(2)\mu*\beta}_\alpha\psi^*_{i\beta}W^{(2)\gamma}_{\mu\alpha}\psi_{i\gamma}) + \frac{g_1g_2}{2}(W^{(1)\,\mu*k}_i\psi^*_{k\alpha}W^{(2)\beta}_{\mu\,\alpha}\psi_{i\beta} + W^{(1)\,\mu k}_i\psi_{k\alpha}W^{(2)*\beta}_{\mu\,\alpha}\psi^*_{i\beta})\end{aligned} \tag{3.432}$$

Note that the gauge boson masses will come from the terms of second order in the couplings:

$$\begin{aligned}\mathcal{L}_M =&\frac{1}{2}(g_1^2W^{(1)\mu*k}_i\psi^*_{k\alpha}W^{(1)l}_{\mu i}\psi_{l\alpha} + g_2^2W^{(2)\mu*\beta}_\alpha\psi^*_{i\beta}W^{(2)\gamma}_{\mu\alpha}\psi_{i\gamma}) + \frac{g_1g_2}{2}(W^{(1)\,\mu*k}_i\psi^*_{k\alpha}W^{(2)\beta}_{\mu\,\alpha}\psi_{i\beta}\\ &+ W^{(1)\,\mu k}_i\psi_{k\alpha}W^{(2)*\beta}_{\mu\,\alpha}\psi^*_{i\beta})\end{aligned} \tag{3.433}$$

Case $SU(n) = SU(m)$

In the case we have $n = m$ with $g_1 = g_2$, as we saw in the $O(n)$ section, then second rank representations can be decomposed into the irreducible complete antisymmetric and symmetric representation. In this case the mass therm is:

$$\mathcal{L}_M = g^2(W^{k*}_{\mu i}\psi^*_{kj}W^{\mu\,k'}_i\psi_{k'j}) + g^2(W^{*k}_{\mu\,i}\psi^*_{kj}W^{\mu k'}_j\psi_{i\,k'}) \tag{3.434}$$

Symmetry Breaking patterns

The invariant potential is:

$$V = -\frac{1}{2}\mu^2(\psi_{i\alpha}\psi^{i\alpha})^2 + \frac{1}{4}\lambda_1(\psi_{i\alpha}\psi^{i\alpha})^2 + \frac{1}{4}\lambda_2(\psi^{i\alpha}\psi_{i\beta})(\psi_{j\alpha}\psi^{j\beta}) \tag{3.435}$$

The minimum as always is retrieved from:

$$\left.\frac{\partial V}{\psi_{i\alpha}}\right|_{\psi=\langle\psi\rangle} = \frac{1}{2}\mu^2\psi^{i\alpha} + \frac{1}{2}\lambda_1(\psi_{i\beta}\psi^{i\beta})\psi^{i\alpha} + \frac{1}{2}\lambda_2(\psi_{j\alpha}\psi^{j\beta})\psi^{i\beta} \tag{3.436}$$

This is the same structure equation as Eq.(3.402), thus we can use the results from there. The symmetry breaking is:

$$\lambda_2 > 0 \quad SU(n) \times SU(m) \to SU(m) \times SU(n-m) \tag{3.437}$$
$$\lambda_2 < 0 \quad SU(n) \times SU(m) \to SU(n-1) \times SU(m-1) \tag{3.438}$$

3.6 Summary of results

In this chapter we developed the theory of spontaneous breaking of symmetries for different representations of the $O(n)$ and $SU(n)$ groups. The properties of the representations and the different symmetry breaking patters can be seen in Tables (3.3), (3.4) and (3.5). We started with the vector representation of $O(n)$ where the symmetry breaking pattern was derived in a very standard way.

Next, we started the treatment of the second rank antisymmetric representation of $O(n)$ that was very important since the results are used for the $SU(n)$ case. To retrieve the vev in the second rank antisymmetric representation of $O(n)$ we first noted that all the values a_i of Eq.(3.64) are equal and that the form of the vev, i.e. number K of antisymmetric blocks, depended on the parameter λ_2 of the potential. To explicitly calculate the number K we inserted the generic vev in the potential and analized for which value of K it was minimum. We arrived at the conclusion that for $\lambda_2 > 0$ we got $K = L$ and for $\lambda_2 < 0$ we got $K = 1$. Correspondingly there were two different symmetry breaking patterns, each one corresponding to λ_2 positive Eq. (3.82) or negative Eq. (3.80). Using the fact that all non zero entries of the vev were equal, another important equation Eq.(3.74) was derived. This is the stability condition for the vev.

In the second rank symmetric representation of $O(n)$ the structure equation of the vev is Eq.(3.113). We found directly from the structure equation that the vev is a diagonal matrix composed of three different values Eq.(3.123). Then, analyzing the possible values in the n_1, n_2, n_3 space with conditions given in Eq.(3.124) that made potential at the vev, $V(\langle\phi\rangle) = V_m$, minimum we understood that the vev depended just on two different values. Thus, the set of n's and correspondingly the symmetry breaking patterns depend on the value of the parameter λ_2. After defining the function f and doing the coordinate change $n_1, n_2, n_3 \to x, y$ we found using monotonicity and boundary arguments the values that minimized $V(\phi)$ that was equivalent to the value that maximized f when $\lambda > 0$ and minimized f when $\lambda < 0$ as shown in Table 3.5.

For the symmetric and antisymmetric second rank irreducible representations of $SU(n)$, we, as said above, used the results of the antysimmetric $O(n)$ case. The defining equation of the vev for the second rank symmetric representation Eq. (3.261) and for the second rank antisymmetric representation Eq.(3.300) has the same structure as the vev equation for the second rank antisymmetric tensor of $O(n)$ Eq.(3.71). The difference is that in the $O(n)$ case the equation is defined directly by the tensor ϕ_{ij} where in the other case we are dealing with the tensor $\Sigma_i^j = \psi^{jk}\psi_{ki}$. In the symmetric case, we have a first factorization $\psi \to \tilde{\psi}$ (at the level of the vev) using the operators that diagonalize Σ. Then, after expressing as $\tilde{\psi} = A + iB$ we use the orthogonal matrix O that diagonalizes simultaneously the matrices A and B to diagonalize

$\tilde{\psi} \to \psi'$. Lastly we use the a diagonal phase matrix U to get the vev $\psi' = U\psi'' U^T$of the physical field $\langle \psi'' \rangle$.

For the antisymmetric case, the procedure is similar. After retrieving $\tilde{\psi}$ as before we use U, the matrix that diagonalizes the hermitian matrices iA and iB, and K of Eq.(3.313) to express $\tilde{\psi}$ in canonical form block as seen in Equation (3.319). Lastly we use the matrix V from Eq.(3.321) to get rid of the phases.

For the adjoint representation of $SU(n)$ we use the results of the symmetric representation of $O(n)$ to retrieve the vev since both have the same structure equation. In this case, there is no need for ulterior factorization, we arrive at the vev of Eq.(3.361). The last section included the SSB of product groups $O(n) \times O(m)$ and $SU(n) \times SU(m)$ and some derivations needed to calculate the breaking patterns for the other sections.

Representation	Dimension in $O(n)$	Dimension in $SU(n)$
Vector	n	$2n$
2^{nd} rank Symmetric Tensor	$\frac{1}{2}n(n+1)-1$	$n(2n+1)-1$
2^{nd} rank Antisymmetric Tensor	$\frac{1}{2}n(n-1)$	$n(2n-1)$
Spinor	$2 \times 2^{l-1}$	
	$(n = 2l$ or $n = 2l+1)$	
Adjoint	$\frac{1}{2}n(n-1)$	n^2-1

Table 3.2 Real dimensions of the representations used for $O(n)$ and $SU(n)$

Representation	Transformation Law	Covariant Derivative
Vector	$\phi_i' = \phi_i + \epsilon_{ij}\phi_j$	$\partial_\mu - gW_{\mu ij}\phi_j$
2^{nd} rank Symmetric Tensor	$\phi_{ij}' = \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik}$	$\mathcal{D}_\mu\phi_{ij} = \partial_\mu\phi_{ij} - gW_{\mu ik}\phi_{kj} - gW_{\mu jk}\phi_{ik}$
2^{nd} rank Antisymmetric Tensor	$\phi_{ij}' = \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik}$	$\mathcal{D}_\mu\phi_{ij} = \partial_\mu\phi_{ij} - gW_{\mu ik}\phi_{kj} - gW_{\mu jk}\phi_{ik}$
spinor	$\chi_i' = \chi_i - \frac{1}{4}i\epsilon_{jk}(\sigma^{jk}\chi)_i$	

Table 3.3 Properties of the various representations in $O(n)$

Representation	Transformation Law	Covariant Derivative
Vector	$\psi_i' = \psi\epsilon_i^j\phi_j$	$\partial_\mu - igW_{\mu i}^j\phi_j$
2^{nd} rank Symmetric Tensor	$\psi_{ij}' = \psi_{ij} + i\epsilon_i^k\psi_{kj} + i\epsilon_j^k\psi_{ik}$	$\mathcal{D}_\mu\psi_{ij} = \psi_{ij} - igW_{\mu i}^l\psi_{lj} - igW_{\mu j}^l\psi_{il}$
2^{nd} rank Antisymmetric Tensor	$\psi_{ij}' = \psi_{ij} + i\epsilon_i^k\psi_{kj} + i\epsilon_j^k\psi_{ik}$	$\mathcal{D}_\mu\psi_{ij} = \psi_{ij} - igW_{\mu i}^l\psi_{lj} - igW_{\mu j}^l\psi_{il}$
Adjoint	$\psi_i'^j = \psi_i^j + i\epsilon_i^k\psi_k^j - i\epsilon_k^j\psi_i^k$	$\mathcal{D}_\mu\psi_{ij} = \partial_\mu\psi_{ij} - igW_{\mu i}^l\psi_l^j + igW_{\mu l}^j\psi_i^l$

Table 3.4 Properties of the various representations in $SU(n)$

Representation		$O(n)$	$SU(n)$
Vector		$O(n-1)$	$SU(n-1)$
k Vectors		$O(n-k)$	$SU(n-k)$
2^{nd} rank Symmetric	$\lambda_2 > 0$	$O(l) \times O(n-l), \quad l = [\frac{1}{2}n]$	$O(n)$
Tensor	$\lambda_2 < 0$	$O(n-1)$	$SU(n-1)$
2^{nd} rank Antisymmetric	$\lambda_2 > 0$	$U(l), \quad l = [\frac{1}{2}n]$	$Sp(n)$
Tensor	$\lambda_2 < 0$	$U(1) \times O(n-2)$	$SU(n-2) \times SU(2)$
Adjoint	$\lambda_2 > 0$		$SU(l) \times SU(n-l) \times U(1), \quad l = [\frac{1}{2}n]$
	$\lambda_2 < 0$		$SU(n-1)$

Table 3.5 Summary of the Patterns of Symmetry Breaking for the $O(n)$ and $SU(n)$ groups

Chapter 4

The Standard Model

For this chapter we have used mainly the books [34], [35] and [36]. In addition for Section (4.5) we have used [37]. For Section (4.6) we have used [38] and [39].

4.1 Content of the Standard Model

The gauge group of the Standard Model $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ is a rank 4 group that is not semisimple. [1] $SU(3)_c$ is the gauge group of the strong interaction and correspondingly has eight gauge boson vectors , the gluons G^a_μ where the internal space coordinate a labels the color indexes. $SU(2)_L \times U(1)_Y$ is the gauge group of the electroweak interactions and has three gauge vector bosons W^a_μ for $SU(2)_L$ and one gauge vector bosons B_μ for $U(1)_Y$ where the subscript L of $SU(2)_L$ denotes the fact that interactions are between fields of left chirality and Y denotes the hypercharge number.

Fermions are divided into of quarks and leptons where the definition is based on the fact that quarks are particles that can interact via strong interactions and leptons are singlets under this gauge group. For one generation, the fermions are composed of 8 left and 7 right Weyl spinors corresponding to a triple set of left handed up and down quarks (6 Weyl spinors), another triple set of right handed up and down quarks, a doublet of left handed leptons and the right handed electron . Since in the SM the number of generations is 3 we have in total 45 Weyl spinors.

For EWSB the Standard Model introduces one complex scalar doublet, the Higgs field. Following the discussion of last chapters, EWSB will give a renormalizable Lagrangian with a mass term for the W^+,W^- and Z gauge vectors. The breaking of symmetry will be done in Section (4.2). Additionally the Higgs field gives masses to the fermions since before EWSB it is not possible to construct a Majorana or Dirac mass terms as these terms are not gauge invariant. All the fields of the Standard model are shown in Table (4.1).

The Lagrangian of the Standard Model is:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_f + \mathcal{L}_H + \mathcal{L}_Y + \mathcal{L}_{CPv} \tag{4.1}$$

[1]it contains the abelian invariant ideal that is the generator of $U(1)_Y$. In other words the lone generator of $U(1)_Y$ commutes with all the rest of generators of the Standard model

Fields of the Standard Model	Representation
Vector Fields	
$G_\mu := G^a_\mu \frac{\hat{\lambda}^a}{2} \quad a = 1, \cdots, 8$	**(8,1,0)**
$W_\mu := W^a_\mu \hat{I}^a \quad a = 1, \cdots, 3$	**(1,3,0)**
$B_\mu := \frac{\hat{Y}}{2} B_\mu$	$(\mathbf{1},\mathbf{1}, 0)$
Spinor Fields	
$Q_{L1} := \begin{pmatrix} u_L \\ d_L \end{pmatrix}$; $Q_{L2} := \begin{pmatrix} c_L \\ s_L \end{pmatrix}$; $Q_{L3} := \begin{pmatrix} t_L \\ b_L \end{pmatrix}$;	$(\mathbf{3}, \mathbf{2}, \frac{1}{3})$
$L_{L1} := \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$; $L_{L2} := \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$; $L_{L3} := \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$;	$(\mathbf{1}, \mathbf{2}, -1)$
$u_{R1} := u_R;\ u_{R2} := c_R;\ u_{R3} := t_R;$	$(\mathbf{3}, \mathbf{1}, \frac{4}{3})$
$d_{R1} := d_R;\ d_{R2} := s_R;\ d_{R3} := b_R;$	$(\mathbf{3}, \mathbf{1}, -\frac{2}{3})$
$\ell_{R1} := e_R;\ \ell_{R2} := \mu_R;\ \ell_{R3} := \tau_R;$	$(\mathbf{1}, \mathbf{1}, -2)$
Scalar Fields	
$\phi := \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, 1)$

Table 4.1 The SM fields in gauge basis classified with respect to the symmetry group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$

Here $\mathcal{L}_B$ contains the kinetic terms of the gauge boson vectors :

$$\mathcal{L}_B = -\frac{1}{4} W^a_{\mu\nu} W^{a\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G^m_{\mu\nu} G^{m\,\mu\nu} \qquad a = 1, 2, 3; m = 1, \cdots, 8 \tag{4.2}$$

The kinetic terms involve the field strengths also known as field tensor since are Lorentz tensors. $W_{\mu\nu}$ corresponds to the field strength of the triplets W^a_μ of $SU(2)_L \times U(1)_Y$:

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc} W^b_\nu W^c_\nu \qquad a, b, c = 1, 2, 3 \tag{4.3}$$

The Yang-Mills field of $U(1)_Y$ is B_μ, with field tensor :

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{4.4}$$

and the Strong Interaction field tensor is:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_3 f^a_{bc} G^b_\mu G^c_\nu \qquad a, b, c = 1, \cdots, 8 \tag{4.5}$$

For a generic fermion f of the Standard Model it's kinetic term is:

$$\mathcal{L}_f = \bar{f} i \mathcal{D}_\mu \gamma^\mu f = \bar{f} i \gamma^\mu \left(\partial_\mu - i g W^a_\mu \hat{I}_a - i g_3 G^m_\mu \frac{\hat{\lambda}^m}{2} - i g' \frac{\hat{Y}}{2} B_\mu \right) f$$
$$\text{with} \qquad a = 1, 2, 3; \quad m = 1, 2, \cdots, 8; \tag{4.6}$$

Where a the index of the generators of $SU(2)$ and i the generators of $SU(3)$. $\hat{I}$ is the representation of the algebra of $SU(2)$and $\frac{\hat{\lambda}^m}{2}$ of $SU(3)$ in the specific representation of the f field (see Table (4.1)).

The kinetic terms of the first generation of fermions are :

$$\begin{aligned}
\mathcal{L}_{f_1} =& i \bar{Q}_L \gamma^\mu \left(\partial_\mu - i g W^a_\mu \frac{\sigma^a}{2} - i g_3 G^m_\mu \frac{\lambda^m}{2} - i \frac{1}{6} g' B_\mu \right) Q_L \\
&+ i \bar{L}_L \gamma^\mu \left(\partial_\mu - i g W^a_\mu \frac{\sigma^a}{2} + \frac{i}{2} g' B_\mu \right) L_L \\
&+ i \bar{u}_R \gamma^\mu \left(\partial_\mu - i g_3 G^m_\mu \frac{\lambda^m}{2} - i \frac{2}{3} g' B_\mu \right) u_R \\
&+ i \bar{d}_R \gamma^\mu \left(\partial_\mu - i g_3 G^m_\mu \frac{\lambda^m}{2} + \frac{i}{3} g' B_\mu \right) d_R \\
&+ i \bar{\ell}_R \gamma^\mu \left(\partial_\mu + i g' B_\mu \right) \ell_R
\end{aligned} \tag{4.7}$$

with λ_i the Gell-Mann matrices. Notice these terms appear for every generation.

The Lagrangian of the Higgs sector is:

$$\mathcal{L}_H = \frac{1}{2} \|\mathcal{D}^\mu \phi\|^2 - V(\phi) = \frac{1}{2} \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - V(\phi) \tag{4.8}$$

Where the potential is defined as:

$$V(\phi) = \frac{\lambda}{4} \left(\|\phi\|^2 - v^2 \right)^2 ; \qquad \text{with} \quad v^2 = \frac{\mu^2}{\lambda} \tag{4.9}$$

and the covariant derivative is:

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - i g W^a_\mu \frac{\sigma^a}{2} \phi - i g' \frac{1}{2} B_\mu \phi \tag{4.10}$$

Eq.(4.8) is the most general renormalizable potential for the vector representation of the group $SU(2)_L \times U(1)_Y$.

The Yukawa sector is:

$$\mathcal{L}_Y = -\frac{1}{\sqrt{2}} \bar{u}_{R\alpha} Y^u_{\alpha\beta} Q_{L\beta} (i \sigma_2 \phi^*) - \frac{1}{\sqrt{2}} \bar{d}_{R\alpha} Y^d_{\alpha\beta} Q_{L\beta} \phi - \frac{1}{\sqrt{2}} \bar{\ell}_{R\alpha} Y^\ell_{\alpha\beta} L_{L\beta} \phi + H.c.$$
$$\alpha = 1, 2, 3 \tag{4.11}$$

Where Y^u, Y^d, Y^ℓ are 3×3 matrices in generation space. From this term we will get the masses for the fermions without spoiling gauge invariance.

The last term of the Standard Model is a strong CP violation term of the strong interaction:

$$\mathcal{L}_{CPv} = \frac{g_3^2 \Theta_{CP}}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^{i\mu\nu} G^{i\rho\sigma} \quad i = 1, \cdots 8; \tag{4.12}$$

where θ_{CP} is an arbitrary parameter.

4.2 Spontaneous Symmetry Breaking of the Standard Model

As we have a scalar in the fundamental representation of $SU(2)_L \times U(1)_Y$, to get the symmetry breaking pattern we can use the results of Section (3.4.1). The vev for the potential (4.8) is then:

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.13}$$

with experimental value $v \approx 246$ GeV. The symmetry breaking is:

$$SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_Q. \tag{4.14}$$

Where Q denotes the generator that forms the Little Group of $\langle \phi \rangle$ and is defined as the electric charge. Using the generators of $SU(3)_c \times SU(2)_L \times U(1)_Y$, $\hat{Q}$ is expressed as a combination of generators:

$$\hat{Q} = \frac{\hat{Y}}{2} + \hat{I}_3 \tag{4.15}$$

This formulation is independent of representation thus valid for all fields in the SM. For the Higgs it is:

$$Q = \frac{Y}{2} + \frac{1}{2}\sigma_3 \tag{4.16}$$

The vector boson masses come from Eq.(3.245):

$$\mathcal{L}_M = \frac{g^2 v^2}{2} W_{\mu 1}^2 W_1^{*\mu 2} + \frac{v^2(g^2 + g'^2)}{2} \left(\sin\theta_W \frac{Y}{2} B_\mu + \cos\theta_W W_{\mu 2}^2 \right) \left(\sin\theta_W \frac{Y}{2} B^\mu + \cos\theta_W W_2^{\mu 2} \right) \tag{4.17}$$

where:

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{4.18}$$

defines the Weinberg angle. Another important relationship is the module of the electric charge:

$$e = \sqrt{4\pi\alpha_{em}} = \frac{g g'}{\sqrt{g^2 + g'^2}} = g \sin\theta_W = g' \cos\theta_W \tag{4.19}$$

Using the usual Pauli matrices representations we have the equivalence:

$$\begin{aligned} W^2_{\mu 1} &= \frac{1}{\sqrt{2}} W^-_\mu = \frac{1}{2}(W_{\mu 1} + iW_{\mu 2}) \\ W^{*2}_{\mu 1} &= \frac{1}{\sqrt{2}} W^+_\mu = \frac{1}{2}(W_{\mu 1} - iW_{\mu 2}) \\ W^2_{\mu 2} &= -\frac{1}{2} W_{\mu 3} \end{aligned} \tag{4.20}$$

Defining:

$$\begin{aligned} \cos\theta_W W_{\mu 3} - \sin\theta_W B_\mu &= Z_\mu \\ \sin\theta_W W_{\mu 3} + \cos\theta_W B_\mu &= A_\mu \end{aligned} \tag{4.21}$$

We obtain mass terms:

$$\mathcal{L}_{M_H} = \frac{g^2 v^2}{4} W^+_\mu W^{-\mu} + \frac{1}{2}\frac{v^2(g^2 + g'^2)}{4} Z_\mu Z^\mu \tag{4.22}$$

This is the BEH Mechanism in the Standard Model.

To get the interactions of the Higgs Field we do a perturbation around the vev. We have the perturbed field:

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ v + \phi_3 + i\phi_4 \end{pmatrix} = e^{i\frac{\sigma^a \rho_a}{2v}} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{4.23}$$

Last factorization clearly shows the Goldstone fields in the exponential and the Higgs boson H. We choose the unitarity gauge that is the gauge where the Goldstone fields are taken away from the theory, obtaining so:

$$\phi = \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{4.24}$$

The Higgs Lagrangian is then:

$$\mathcal{L}_H = \frac{1}{2}\mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - \frac{\lambda}{4}(2vH + H^2)^2 = \frac{1}{2}\mathcal{D}_\mu \phi \mathcal{D}^\mu \phi - \mu^2 H^2 - \sqrt{\lambda}\mu H^3 - \frac{\lambda}{4} H^4 \tag{4.25}$$

where naturally the interactions with the gauge fields will come from the covariant derivative:

$$\mathcal{D}_\mu \phi = \begin{pmatrix} -i\frac{g}{\sqrt{2}} W^+_\mu (v + H) \\ \partial_\mu H - \frac{i}{2}\frac{g}{\cos\theta_W} Z_\mu (v + H) \end{pmatrix} \tag{4.26}$$

So:

$$\frac{1}{2}(\mathcal{D}_\mu \phi)^\dagger \mathcal{D}^\mu \phi = \frac{1}{2}(\partial_\mu H)^2 + \frac{g^2}{4}(v + H)^2 W^-_\mu W^{+\,\mu} + \frac{g^2}{8\cos^2\theta_W}(v + H)^2 Z_\mu Z^\mu \tag{4.27}$$

Expanding:

$$\begin{aligned}\mathcal{L}_H =& \frac{1}{2}(\partial_\mu H)^2 - \mu^2 H^2 - \sqrt{\lambda}\mu H^3 - \frac{\lambda}{4}H^4 + \frac{g^2}{4}v^2 W^-_\mu W^{+\,\mu} + \frac{g^2}{8\cos^2\theta_W}v^2 Z_\mu Z^\mu \\ &+ \frac{g^2}{2}vW^-_\mu W^{+\,\mu}H + \frac{g^2}{4\cos^2\theta_W}vZ_\mu Z^\mu H \\ &+ \frac{g^2}{4}W^-_\mu W^{+\,\mu}H^2 + \frac{g^2}{8\cos^2\theta_W}Z_\mu Z^\mu H^2\end{aligned} \quad (4.28)$$

Last equation shows again the mass terms of the boson vectors and the interactions of them with the Higgs Fields.

4.3 Gauge Boson sector of the Standard Model

Having obtained the physical configuration of the gauge vector fields we re express $\mathcal{L}_B$ in terms of them . $G_{\mu\nu}$ remains the same but the strength tensors of W and Z change. Using Eq.(4.21) we have:

$$W^3_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu + g(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu) = \cos\theta_W Z_{\mu\nu} + \sin\theta_W A_{\mu\nu} - ig(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu) \quad (4.29)$$

In the same way:

$$B_{\mu\nu} = \cos\theta_W A_{\mu\nu} - \sin\theta_W Z_{\mu\nu} \quad (4.30)$$

with:

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (4.31)$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \quad (4.32)$$

Defining:

$$F_{W\,\mu\nu} = \frac{1}{\sqrt{2}}(W^1_{\mu\nu} - iW^2_{\mu\nu}) \quad (4.33)$$

we see that:

$$-\frac{1}{2}F^\dagger_{W\,\mu\nu}F^{\mu\nu}_W = -\frac{1}{4}W_{1\mu\nu}W^{\mu\nu}_1 - \frac{1}{4}W_{2\mu\nu}W^{\mu\nu}_2 \quad (4.34)$$

then, expanding:

$$F_{W\mu\nu} = (\partial_\mu - igW_{3\mu})W^+_\nu - (\partial_\nu - igW_{3\nu})W^+_\mu \equiv d_\mu W^+_\nu - d_\nu W^+_\mu \quad (4.35)$$

with:

$$d_\mu = \partial_\mu - ieA_\mu - ig\cos\theta_W Z_\mu \quad (4.36)$$

We have then:

$$\begin{aligned}\mathcal{L}_B =& -\frac{1}{4}G^b_{\mu\nu}G^{b\,\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ =& -\frac{1}{4}G^b_{\mu\nu}G^{b\,\mu\nu} - \frac{1}{2}F^\dagger_{W\,\mu\nu}F^{\mu\nu}_W - \frac{1}{4}W^3_{\mu\nu}W^{3\,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ =& -\frac{1}{4}G^b_{\mu\nu}G^{b\,\mu\nu} - \frac{1}{2}F^\dagger_{W\,\mu\nu}F^{\mu\nu}_W - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu}\\ &+ iW^{+\,\mu}W^{-\,\nu}(g\cos\theta_W Z_{\mu\nu} + ieA_{\mu\nu}) + \frac{g^2}{2}[(W^+)^2(W^-)^2 - (W^{+\,\mu}W^-_\mu)^2]\end{aligned} \tag{4.37}$$

From the results we see the interactions of the gauge bosons between themselves.

4.4 Fermion-Gauge interaction in the Standard Model

The fermion-gauge interactions come from the covariant derivative Eq.(4.6). Using:

$$\begin{aligned}\hat{I}_1 + i\hat{I}_2 = \hat{I}_+\\ \hat{I}_1 - i\hat{I}_2 = \hat{I}_-\end{aligned} \tag{4.38}$$

We obtain:

$$gW^1_\mu\hat{I}_1 + gW^2_\mu\hat{I}_2 = \frac{g}{\sqrt{2}}\left(W^+_\mu\hat{I}_+ + W^-_\mu\hat{I}_-\right) \tag{4.39}$$

Also:

$$gW^3_\mu\hat{I}_3 + g'B_\mu\frac{\hat{Y}}{2} = \left(g\cos\theta_W\hat{I}_3 - g'\frac{\hat{Y}}{2}\right)Z_\mu + \left(g\sin\theta_W\hat{I}_3 + g'\cos\theta_W\frac{\hat{Y}}{2}\right)A_\mu \tag{4.40}$$

From last equation, using $\hat{Q} = \frac{\hat{Y}}{2} + \hat{I}_3$ and Eq.(4.19) we have

$$\left((g^2+g'^2)^{\frac{1}{2}}\hat{I}_3 - \frac{g'^2}{\sqrt{g^2+g'^2}}\hat{Q}\right)Z_\mu + e\hat{Q}A_\mu = \frac{g}{\cos\theta_W}\left(\hat{I}_3 - \sin^2\theta_W\hat{Q}\right)Z_\mu + e\hat{Q}A_\mu \tag{4.41}$$

So from Eq.(**??**) we can see the current for each type of interaction:

$$\begin{aligned}\mathcal{L}_I =& \sum_f \bar{f}\gamma^\mu\left(g_3G^i_\mu\frac{\hat{\lambda}_i}{2} + \frac{g}{\sqrt{2}}\left(W^+_\mu\hat{I}_+ + gW^-_\mu\hat{I}_-\right)\right.\\ &\left.+\frac{g}{\cos\theta_W}\left(\hat{I}_3 - \sin^2\theta_W\hat{Q}\right)Z_\mu + e\hat{Q}A_\mu\right)f\\ =& G^i_\mu j^{i\mu}_G + \frac{g}{2\sqrt{2}}\left(W^+_\mu j^\mu_W + W^-_\mu j^{\mu\dagger}_W\right) + \frac{g}{2\cos\theta_W}j^\mu_Z Z_\mu + ej^\mu_\gamma A_\mu\end{aligned} \tag{4.42}$$

In total we have 12 currents corresponding to each generator of G_{SM}. $G^i_\mu j^{i\mu}_G$ denotes the current-gauge interactions with the $SU(3)$ gauge bosons, j^μ_W and $j^{\mu\dagger}_W$ the charged negative and positive weak currents, j^μ_Z the neutral weak current and j^μ_γ the electromagnetic current (that is neutral too). Explicitely, in the so called gauge basis (or generation,family basis), the negative charged current is:

$$j^\mu_W = \sum_f 2\bar{f}\gamma^\mu \hat{I}_+ f = 2\bar{u}_{L\alpha}\gamma^\mu d_{L\alpha} + 2\bar{\nu}_{L\alpha}\gamma^\mu e_{L\alpha}, \quad \alpha = 1, 2, 3. \tag{4.43}$$

and the neutral weak:

$$\begin{aligned} j^\mu_Z =& 2\sum_f \bar{f}\gamma^\mu \left(\hat{I}_3 - \sin^2\theta_W \hat{Q}\right) f \\ =& 2g^\nu_L \bar{\nu}_{L\alpha}\gamma^\mu \nu_{L\alpha} + 2g^e_L \bar{e}_{L\alpha}\gamma^\mu e_{L\alpha} + 2g^e_R \bar{e}_{R\alpha}\gamma^\mu e_{R\alpha} + 2g^U_L \bar{u}_{L\alpha}\gamma^\mu u_{L\alpha} \\ &+ 2g^D_L \bar{d}_{L\alpha}\gamma^\mu d_{L\alpha} + 2g^U_R \bar{u}_{R\alpha}\gamma^\mu u_{R\alpha} + 2g^D_R \bar{d}_{R\alpha}\gamma^\mu d_{R\alpha}. \end{aligned} \tag{4.44}$$

where the constants $g^e_R, \cdots$ are shown in Table (4.2).

Fermion	g_L	g_R
u, c, t	$g^U_L = \frac{1}{2} - \frac{2}{3}s^2_W$	$g^U_R = s^2_W$
d, s, b	$g^D_L = -\frac{1}{2} + \frac{1}{3}s^2_W$	$g^D_R = \frac{1}{3}s^2_W$
e, μ, τ	$g^e_L = -\frac{1}{2} + s^2_W$	$g^e_R = s^2_W$
ν_e, ν_μ, ν_τ	$g^\nu_L = \frac{1}{2}$	$g^\nu_R = 0$

Table 4.2 The result of the operator $\hat{I}_3 - \sin^2\theta_W \hat{Q}$ in each representation of the fermions where $s_W = \sin\theta_W$

4.5 Yukawa interactions

The Standard Model Lagrangian without the Yukawa term has a $[U(3)]^5 = [SU(3)]^5 \times [U(1)]^5$ global symmetry. This is seen from the fact that each different representation of fermions, that are five, is multiplied by three since three are the generations thus is a three dimensional vector in generation space. Equivalently each spinor is seen as a tensor product of the gauge symmetry times the corresponding family symmetry $U(3)$, so $[U(3)]^5 = U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_e$. Adding the Yukawa Lagrangian some of the global symmetries are explicitly broken. Denoting the fermions we are using until now, i.e. represented in the gauge basis , with an additional prime $'$ we see that the Yukawa Lagrangian from Eq. (4.11) is:

$$\mathcal{L}_Y = -\frac{1}{\sqrt{2}}\bar{u}'_{R\alpha} Y^u_{\alpha\beta} Q'_{L\beta}(i\sigma_2\phi^*) - \frac{1}{\sqrt{2}}\bar{d}'_{R\alpha} Y^d_{\alpha\beta} Q'_{L\beta}\phi - \frac{1}{\sqrt{2}}\bar{e}'_{R\alpha} Y^\ell_{\alpha\beta} L'_{L\beta} + H.c. \tag{4.45}$$

Independent rotations in the different $U(3)$ spaces do not leave last equation invariant since we have trilinears couplings between the Higgs boson with two fermions in different representations. Since are complex matrices, the 3×3 matrices Y^u, Y^d, Y^e can be diagonalized via a biunitary transformation [36]:

$$\begin{aligned} V_R^u Y^u V_L^{u\,\dagger} = Y_D^u; \qquad & V_R^d Y^d V_L^{d\,\dagger} = Y_D^d; \\ & V_R^e Y^e V_L^{e\,\dagger} = Y_D^e; \end{aligned} \tag{4.46}$$

Where Y_D^u, Y_D^d, Y_D^e are diagonal 3×3 real matrices, the Yukawa mass matrices. The fermions are redefined:

$$\begin{aligned} V_L^u Q_L' = Q_L; \qquad & V_R^u u_R' = u_R \\ & V_R^d d_R' = d_R \\ V_L^e L_L' = e_L; \qquad & V_R^e e_R' = e_R \end{aligned} \tag{4.47}$$

Following $[U(3)]^5$ symmetry in the Lagrangian minus $\mathcal{L}_Y$ we have for example for the three dimensional vector Q_L':

$$\bar{Q}_L' \mathcal{D}_\mu Q_L' = \bar{Q}_L' \mathcal{D}_\mu V_L^{u\dagger} V_L^u Q_L' = \bar{Q}_L \mathcal{D}_\mu Q_L \tag{4.48}$$

and so for each of the five different representations of fermions. This new basis is called the flavor basis for all fermions and the mass basis of all fermions minus the d_L. In fact:

$$V_L^u Q_L' = \begin{pmatrix} V_L^u u_L' \\ V_L^u d_L' \end{pmatrix} := \begin{pmatrix} u_L \\ d_L^u \end{pmatrix} = Q_L \tag{4.49}$$

where $V_L^u d_L' = d_L^u$ is the flavor basis of the down left quarks. After the factorization of the Yukawas, we obtain:

$$\mathcal{L}_Y = -\frac{1}{\sqrt{2}} \bar{u}_{Ri} Y_{Dii}^u Q_{Li} (i\sigma_2 \phi^*) - \frac{1}{\sqrt{2}} \bar{d}_{Ri} Y_{Dii}^d (V_L^d V_L^{u\,\dagger})_{ij} Q_{Lj} \phi - \frac{1}{\sqrt{2}} \bar{e}_{Ri} Y_{Di}^e L_{Li} \phi + h.c. \qquad i = 1, 2, 3 \tag{4.50}$$

In this way we see that d_L^U is not the mass eigenstate of down quarks. We also see the first appearance of the unitary Cabibbo-Kobayashi-Masawaka matrix (CKM) defined as:

$$U_{\text{CKM}} = V_L^u V_L^{d\,\dagger} \tag{4.51}$$

Note that the $[U(3)]^5 = [SU(3)^5 \times [U(1)]^5$ is broken at the full Lagrangian level with the the only remaining global symmetries being $U(1)_B \times [U(1)_\ell]^3$. In fact the CKM matrix restricts the possible $[U(3)]^3$ symmetry of the baryons into only a symmetry given by the diagonal phase matrix $e^{i\alpha_B} \mathbf{1}_{3\times 3}$ that is interpreted as the Baryon number conservation. Then, the transformations under $U(1)_B$ are:

$$Q_L' = e^{iB\alpha_B} \mathbf{1}_{3\times 3} Q_L; \quad u_R' = e^{iB\alpha_B} \mathbf{1}_{3\times 3} u_R; \quad d_R' = e^{iB\alpha_B} \mathbf{1}_{3\times 3} d_R; \tag{4.52}$$

with B the baryonic number of the representation where the transformation acts. [2] B is $+1/3$ for quarks and $-1/3$ for antiquarks. The rest of the fields are invariant under this transformation.
Since in the Standard Model there is not an analogous to the CKM matrix (the PMNS matrix) for the leptons, there is a $U(1)$ symmetry in each lepton mas basis. The transformations:

$$\begin{aligned} L'_{L1} &= e^{iL_e\alpha_e}L_{L1}; \quad e'_{R1} = e^{iL_e\alpha_e}e'_{R1}; \\ L'_{L2} &= e^{iL_\mu\alpha_\mu}L_{L2}; \quad e'_{R2} = e^{iL_\mu\alpha_\mu}e'_{R2}; \\ L'_{L3} &= e^{iL_\tau\alpha_\tau}L_{L3}; \quad e'_{R3} = e^{iL_\tau\alpha_\tau}e'_{R3}; \end{aligned} \tag{4.53}$$

leave the Yukawa term invariant and correspond the $U(1)_e, U(1)_\mu$ and $U(1)_\tau$ symmetry that conserve leptonic L_e, L_μ, L_τ number, respectively. Note again that since there is no PMNS matrix the SM the mass basis is the same as the flavor basis for the leptons thus we can make the correspondence of indices $e = 1, \mu = 2, \tau = 3$. Each L_e, L_μ, L_τ is $+1$ for the corresponding lepton and -1 for the corresponding antilepton.

After spontaneous symmetry breaking from Eq.(4.50) we get:

$$\mathcal{L}_Y = (v+H)\left(\frac{1}{\sqrt{2}}\bar{u}_{Ri}Y^u_{ii}u_{Li} + \frac{1}{\sqrt{2}}\bar{d}_{Ri}Y^d_{Dii}(U^\dagger_{\text{CKM}})_{i\alpha}d^u_{L\alpha} + \frac{1}{\sqrt{2}}\bar{e}_{Ri}Y^\ell_{ii}e_{Li} + h.c.\right) \tag{4.54}$$

Redefining the down left quarks into a mass diagonal base:

$$[U^\dagger_{\text{CKM}}]_{i\alpha}d^u_{L\alpha} \equiv d_{Li} \tag{4.55}$$

Thus last equation is:

$$d^u_{L\alpha} = [U_{\text{CKM}}]_{\alpha i}d_{Li} \tag{4.56}$$

It is equal to:

$$(U_{\text{CKM}})_{ij}(V^u_L)_{j\alpha}d'_{L\alpha} = (V^d_L V^{u\,\dagger}_L)_{ij}(V^u_L)_{j\alpha}d'_{L\alpha} = V^d_{Li\gamma}d'_{L\gamma} = d_{Li} \tag{4.57}$$

The CKM matrix depends on four parameters, three angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a CP-violating phase δ_{13}. It can be shown that one parametrization of the CKM, the standard parametrization is:

$$U_{\text{CKM}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.58}$$

$$= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{bmatrix} \tag{4.59}$$

[2] B is given by the representation since baryonic number conservation is a $U(1)$ symmetry. See the discussion at Chapter 1.

or using the Wolfenstein parametrization, an approximation of the standard parameterization:

$$U_{\text{CKM}} \approx \begin{bmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix} \tag{4.60}$$

The four Wolfenstein parameters have the property that all are of order 1 and are related to the standard parameterization:

$$\lambda = s_{12} \tag{4.61}$$

$$A\lambda^2 = s_{23} \tag{4.62}$$

$$A\lambda^3(\rho - i\eta) = s_{13}e^{-i\delta} \tag{4.63}$$

with $\lambda \approx 0.22$. It can be shown that the CP violation can be determined by measuring $\rho - i\eta$.

From Eq.(4.54) we arrive, using $\psi = \psi_L + \psi_R$ for the fermions, at:

$$\mathcal{L}_Y = \frac{1}{\sqrt{2}}(v+H)\left(\bar{u}Y_D^u u + \bar{d}Y_D^d d + \bar{e}Y_d^e e\right) \tag{4.64}$$

We clearly see the masses and interactions with the Higgs boson of all the fermions less the neutrino. Explicitly the mass matrices are:

$$M^I = \frac{1}{\sqrt{2}}vY_D^I; \qquad I = u, d, \ell \tag{4.65}$$

Note that the interactions of the fermions with the Higgs are proportional to the Yukawa Couplings and thus to the fermion mass $\sim \frac{m_f}{v}$, with m_f mass of the fermion. This fact has important phenomenological consequences for Higgs physics.

Expressing the charged current term using Eq.(4.43) in the mass basis we have:

$$\begin{aligned} j_W^\mu =& 2\bar{u'}_L\gamma^\mu d'_L + 2\bar{\nu'}_L\gamma^\mu e'_L = 2\bar{u}_L\gamma^\mu(V_L^u V_L^{d\dagger})d_L + 2\bar{\nu}_L\gamma^\mu e_L \\ =& 2\bar{u}_{L\,k}\gamma^\mu(U_{\text{CKM}})d_L + 2\bar{\nu}_L\gamma^\mu e_L \end{aligned} \tag{4.66}$$

Last equation implies charged that current interactions between quarks mix the mass eigenstates of quarks (equivalently mix flavor eigentstates). On the other hand, since in the SM there are not right handed neutrinos we cannot construct a second Yukawa matrix Y^ν for neutrinos thus they are massless. In this way swe see that doing the transformation (4.47) we already defined also the flavor basis for neutrinos. So Eq.(4.66) implies that there are not flavor changing charged currents for leptons.

For the weak neutral current we have:

$$\begin{aligned} j_Z^\mu =& 2g_L^\nu \bar{\nu'}_L \gamma^\mu \nu'_L + 2g_L^e \bar{e'}_L \gamma^\mu e'_L + 2g_R^e \bar{e'}_R \gamma^\mu e'_R + 2g_L^u \bar{u'}_L \gamma^\mu u'_L \\ &+ 2g_L^d \bar{d'}_L \gamma^\mu d'_L + 2g_R^u \bar{u'}_R \gamma^\mu u'_R + 2g_R^d \bar{d'}_R \gamma^\mu d'_R \\ =& 2g_L^\nu \bar{\nu}_L \gamma^\mu \nu_L + 2g_L^e \bar{e}_L \gamma^\mu e_L + 2g_R^e \bar{e}_R \gamma^\mu e_R + 2g_L^u \bar{u}_L \gamma^\mu u_L + 2g_L^d \bar{d}_L^u \gamma^\mu d_L^u \\ &+ 2g_R^u \bar{u}_R \gamma^\mu u_R + 2g_R^d \bar{d}_R \gamma^\mu d_R \end{aligned} \tag{4.67}$$

Where we have used Eq.(4.47) to go into the flavor basis. Note that at tree level there are not flavor changing neutral currents (FCNC). This is the GIM mechanism.

Table 4.3 Analytic form of Tree level Masses of the different fields and experimental value.[1]

Field	Analytic Form	Experimental Value
H	$\sqrt{2}\mu$	125.03+0.26 0.27 (stat) +0.13 0.15 (sys) GeV (CMS)
$W^\pm$	$\frac{g}{2}v$	80.385±0.015 GeV
Z	$\frac{\sqrt{g^2+g'^2}}{2}v$	91.1876±0.0021 GeV
γ	0	$\leq 1^{-18}$ eV
G	0	
e	$\frac{1}{\sqrt{2}}vY^e_{D1}$	0.510998928 ± 0.000000011 MeV
μ	$\frac{1}{\sqrt{2}}vY^e_{D2}$	105.6583715 ± 0.0000035 MeV
τ	$\frac{1}{\sqrt{2}}vY^e_{D3}$	1776.82 ± 0.16 MeV
ν_e	0	$\leq$ 2.05 eV (95 % C.L.)
ν_τ	0	"
ν_τ	0	"
u	$\frac{1}{\sqrt{2}}vY^u_{D1}$	$2.3^{+0.7}_{-0.5}$ MeV
d	$\frac{1}{\sqrt{2}}vY^d_{D1}$	$3.5^{+0.7}_{-0.2}$ MeV
s	$\frac{1}{\sqrt{2}}vY^u_{D2}$	95 ± 5 MeV
c	$\frac{1}{\sqrt{2}}vY^d_{D2}$	1.275 ± 0.025 GeV
t	$\frac{1}{\sqrt{2}}vY^u_{D3}$	173.21 ± 0.51 ± 0.71 GeV [a]
b	$\frac{1}{\sqrt{2}}vY^d_{D3}$	4.66 ± 0.03 GeV [b]

[a] direct measurement
[b] in bounded state

4.6 Cancellation of Anomalies in the Standard Model

Following the discussion at the beginning of the Chapter 2, a strong motivation for the use of gauge theories was the fact that they are renormalizable. It can be shown that when trying to prove the renormalizability of the Electroweak Sector of the classical Standard Model Lagrangian, $\mathcal{L}_{\text{SM}}$, one arrives at diagrams of the type of diagram (4.1). The amplitude of the

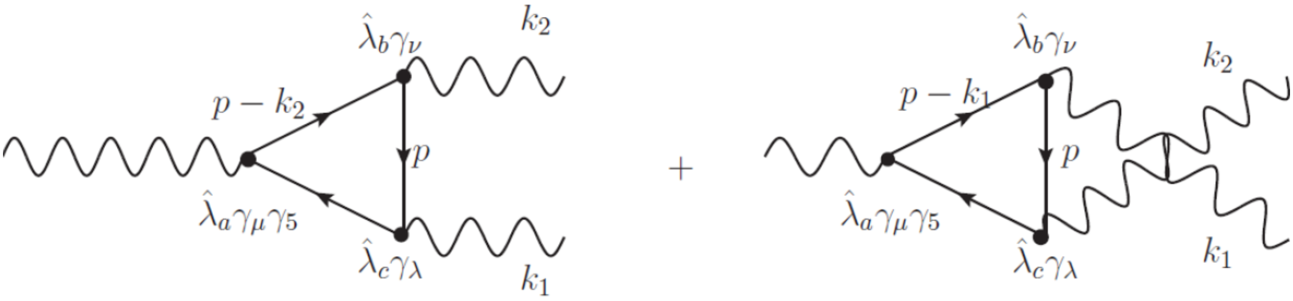


Figure 4.1 Feynman Diagram

3-point function using Feynman rules on the sum of both diagrams in (4.1) is then:

$$T^{\mu\nu k}_{abc}(k_1,k_2) = -i\int\frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{i}{\not{p}-\not{k}_2-m}\frac{\hat{\lambda}_b}{2}\gamma^\nu\frac{i}{\not{p}-m}\frac{\hat{\lambda}_c}{2}\gamma^\lambda\frac{i}{\not{p}+\not{k}_1-m}\frac{\hat{\lambda}_a}{2}\gamma^\lambda\gamma^5\right] + \begin{pmatrix} k_1\leftrightarrow k_2\\ \nu\leftrightarrow\mu\\ b\leftrightarrow c\end{pmatrix}\right\} \tag{4.68}$$

Where $\hat{\lambda}$ are proportional to the four generators in the fundamental representation of $SU(2)_L\times U(1)_Y$. Explicitly they take the values:

$$\begin{aligned}\hat{\lambda}_i &= \sigma_i, \quad i=1,2,3\\ \hat{\lambda}_4 &= Y\end{aligned} \tag{4.69}$$

Using the fact that in the Standard Model the fermions are massless before EWSB we have $m=0$. Then using symmetry properties we arrive at the formula:

$$T^{\mu\nu k}_{abc}(k_1,k_2) = \frac{1}{8}D_{abc}\left(-i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not{p}-\not{k}_2}\gamma^\nu\frac{i}{\not{p}}\gamma^\lambda\frac{i}{\not{p}-\not{k}_1}\gamma^\mu\gamma^5\right]\right), \tag{4.70}$$

where :

$$D_{abc} = \sum_f \mathrm{Tr}[\hat{\lambda}_a\{\hat{\lambda}_b,\hat{\lambda}_c\}]. \tag{4.71}$$

The sum is over the chiral fermions of the Standard Model. When evaluating Eq.(4.70), one finds that these kind of diagrams break the $SU(2)_L\times U(1)_Y$ symmetry. Such an effect is called an "anomaly ". In order to keep the symmetry of $SU(2)_L\times U(1)_Y$ intact, one requires that the term D_{abc} vanishes for all combinations of $\hat{\lambda}$.

Taking all $\hat{\lambda}$ as the Pauli matrices we see that D_{abc} is null since the σ are traceless, in fact:

$$\mathrm{Tr}[\{\sigma_a\sigma_b\},\sigma_c] = 2\delta_{ab}\,\mathrm{Tr}[\sigma_c] = 0 \tag{4.72}$$

When we have one $\hat{\lambda}$ as the hypercharge $\hat{\lambda} = Y$ we have:

$$\sum_f \mathrm{Tr}[\{\sigma_a,\sigma_b\},Y] = 2\delta_{ab}\sum_f \mathrm{Tr}[Y] = 2\delta_{ab}\left(\sum_f Y_f\right) = 0 \tag{4.73}$$

using the results of Table (4.1). When we have two $\hat{\lambda}$ as the hypercharge we have:

$$\sum_f \mathrm{Tr}[\{\sigma_a,Y\},Y] = \sum_f \mathrm{Tr}[\sigma_a]Y_f^2 = 0 \tag{4.74}$$

using again the fact that Pauli matrices are traceless.
When we have all $\hat{\lambda}$ as the hypercharge we have:

$$\sum_f \mathrm{Tr}[\{Y,Y\},Y] = \sum_f Y^3 = 3\left(\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^3\right) + 2(-1)^3 - 3\left(\frac{4}{3}\right)^3 - 3\left(-\frac{2}{3}\right)^3 - (-2)^3 = 0 \tag{4.75}$$

So we see that the Standard Model is anomaly free.

4.7 Problems of the Standard Model

At present time the Standard Model has been experimentally verified with incredible accuracy. The last ingredient was the discovery of the Higgs boson. Despite the success, the SM has some important problems. Let's enumerate some of them:

4.7.1 Observational Problems

(a) **Dark Matter and Dark Energy**. The Planck mission team measured using the Standard model of Cosmology, the total mass–energy of the known universe. It contains 4.9 % ordinary matter, 26.8 % dark matter and 68.3 % dark energy. Dark matter corresponds to matter with at most weak interactions. Dark energy can be interpreted as a cosmological constant.

(b) **Matter-Antimatter Asymmetry.** Currently there is no consistent explanation for the perceived excess of matter over antimatter . The Standard Model gives a possible explanation, the asymmetry was created after the electroweak phase transition (100 GeV). The SM satisfies Shakarov's conditions for baryogenesis: it has processes that violate B number (thought not at tree level), has plenty of C and CP violation (in the phases of the CKM) and if it happened at the electroweak phase transition (EWPT) it is also out of equilibrium. Nevertheless quantitatively, the results are not consistent with current measurements; there is not enough production of asymmetry. Mainly after EWPT the asymmetry diminishes.

(c) **Neutrino masses and mixing.** One of the last remarkable experimental results in HEP was the recent (1998-2002) measurement of oscillations between the different flavors of neutrinos. This implies a non-zero neutrino mass, completely in disagreement with the massless neutrino of the Standard Model.

(d) **Anomalous Magnetic moment of the muon.** The prediction of the factor $\frac{1}{2}(g-2)$ related to the Magnetic moment of the muon is in disagreement with the SM at the level of $\sim 3\sigma$.

4.7.2 Theoretical Problems

(a) **The gauge hierarchy problem and Naturalness.** Renormalizable Quantum Field Theories connect physical observables not to fundamental parameters i.e. parameters at tree level, but to the renormalized version of them. Since the physical mass of the Higgs is in the GeV scale, a problem happens when calculating the quantum corrections to the Higgs mass. Using only the fields of the Standard Model we have a set of quadratically divergent terms $\sim \Lambda^2$(after imposing a cutoff a lot larger than the EW scale) whose sum must cancel in a very delicate way, such that the renormalized mass is in the order of the fundamental mass. Without knowing any other mechanism except the ones of the Standard Model the cancellation happens using a very strong fine tuning thus needing explanation. In fact if assuming naturalness[3] one would expect both Higgs mass and radiative corrections to be at the same scale.This is related to the question of why there exists this difference in scale between the Higgs mass and the scale of Λ (for example, the Planck Scale, $M_{\text{Planck}} \approx 10^{19}$ GeV). This is referred as the gauge hierarchy problem.

(b) **Hypercharge quantization** Since the quantization of the hypercharge implies quantization of Electric Charge we focus in the first one. In the SM we don't have a clear explanation in the choice of the hypercharge representations (i.e. hypercharge numbers see Section(1.4)) for the matter fields since in principle the spectrum of the generator of the hypercharge is continuous because it is the generator of the Abelian $U(1)_Y$ group of the G_{SM}. On the other hand,if somehow the gauge group $U(1)_Y$ is embedded into a larger semi-simple group we could obtain discrete eigenvalues [4] for the generator of $U(1)_Y$ thus we would "quantize" the Hypercharge. From the formula $\hat{Q} = \hat{I}_3 + \frac{\hat{Y}}{2}$ this would imply the quantization of the Electric charge.

(c) **Arbitrariness in the Fermion Representations and the "miracle" of anomaly cancellation.** The fermion representations of the SM and their hypercharge numbers miraculously canceled the anomalies of the theory. Since the G_{SM} is not a simple group there

[3] given a theory with a cutoff we expect all parameters of order $\mathcal{O}(1)$

[4] Eigenvalues here means the entries of the matrix in the fundamental representation of the generator of $U(1)_Y$ when embedded in the larger Group. See Chapter 5 for an example.

it is not a clear recipe on how to select the representations of the fermions such that anomalies cancel, contrary to the case of gauge theories using simple groups where from the structure of the group we can find representations of combinations of them where anomalies cancel.

(d) **Gauge Unification.** Reiterating, the gauge group of the Standard Model $SU(3)_c \times SU(2)_L \times U(1)_Y$ is not simple. A particular consequence is that each coupling corresponding to each gauge subgroup is not clearly related to the other. This is what we see at EW scale where we have three different values for the coupling but it is desirable that they unify at a higher energy such that there is only one coupling related to a simple Lie group. This will restrict the number of parameters and also solve the problem of the gauge coupling divergence. Besides this, gauge unification will explain the quantization of charge and provide a guide in the choosing of fermion representations such that anomalies cancel.

(e) **Flavor theory.** Currently there is not a theory of flavour even when 13 of the 19 parameters correspond to flavor physics in the SM (Yukawa couplings, CKM mixing, CP phase).

(f) **Violation of strong CP.** In principle QCD would violate CP but experimentally this has never been measured. This implies that the value of Θ_{CP} in $\frac{g_3^2\Theta_{CP}}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}G^{i\mu\nu}G^{i\rho\sigma}$ is very small and not of order one. This is very unnatural and in the need of a more exhaustive explanation.

(g) **Quantum Gravity.** The Standard Model does not include Gravitation for technical reasons. In particular gravitation is not renormalizable

(h) **Vacuum Stability.** In the Higgs potential of the SM we had $\lambda > 0$ for stability but it can be shown that a running of this parameter to higher energies can make it zero or negative.

Chapter 5

Minimal $SU(5)$ Grand Unification Model

For this chapter we have mainly used [40],[41] , [38] and [34]. For the Young diagram treatment we used [42] and [4]. In this Section we use the following convention. For a Left handed spinor ψ_L the spinor that results after applying the operator C of charge conjugation is $\hat{C}\psi_L = C\gamma^0 P_L \psi^* = P_R C\gamma^0\psi^* = (\psi^c)_R \equiv \psi^c_R$ and viceversa.

5.1 Extensions of the Standard Model

As show in Section (4.7), the Standard Model is weak in many fronts. Correspondingly, there have been a lot of extensions of it since the dawn of the model, even before it became "Standard" . One possible solution to solve some of the problems of the SM is Grand Unification.

The main idea of Grand Unification is to construct a Lagrangian with a gauge symmetry given by a bigger compact[1] group G that contains the Standard model group $SU(3)_c \times SU(2)_L \times U(1)_Y \subset G$ such that after at least one spontaneous symmetry breaking mechanism (usually two or more) we arrive at the interactions of the Standard Model after Electroweak Symmetry Breaking (EWSB) plus other possible new interactions. In this view, the specification of the group is the first important choice. Since G_{SM} is rank 4 group, in fact it is a simple product of rank 2 $SU(3)$, rank 1 $SU_L(2)$ and rank 1 $U_Y(1)$, we require G is of at least rank 4.

We also need a group G to have complex representations and not only real nor pseudoreal representations. Remembering, a representation $D(g)$ with $g \in G$ is real if there exists a transformation S such that:

$$SD(g)S^{-1} = D^*(g) \tag{5.1}$$

with $D^*(g)$ the conjugate representation. In principle, using real representations we can construct a Majorana mass term for the Weyl spinor ψ_L [2]:

$$\mathcal{L}_M = -\frac{1}{2}M(\overline{\psi^c}_R S\psi_L + h.c.) = -\frac{1}{2}M(\psi_L^T CS\psi_L + h.c.) \tag{5.2}$$

that is gauge invariant as can be seen using the transformation $\psi_L \to D(g)\psi_L$, Eq.(5.1) and the

[1] We again reiterate that we need compact groups in order to have finite unitary representations

[2] technically it is a left handed Dirac Spinor but a field of this type is equivalente to a $(1/2, 0)$ Weyl Spinor

Table 5.1 Classical Groups

Group	Rank	Dimension	Complex Rep.
$SU(n)$	$n-1$	n^2-1	$n \geq 3$
$SO(2n)$	n	$n(2n-1)$	$n = 2l+3, l \geq 1$
$SO(2n+1)$	n	$n(2n+1)$	no
$Sp(2n)$	n	$n(2n+1)$	no
G_2	2	14	no
F_4	4	52	no
E_6	6	78	YES
R_7	7	133	no
G_2	8	248	no

fact that $D(g)$ is unitary. Expanding $D(g)$ in the generators we can show that S is symmetric or antisymmetric. Since the fermions of the SM do not acquire mass until EWSB we need them to be embedded in a chiral representations at the GUT scale. Even more, suppose we have a chiral theory with ψ_L and ψ_R chiral components of a fermion. All left handed fields ψ_L and ψ_L^c live in the representation (usually reducible) D_L and the right handed in the representation $D_R = D_L^*$. So in the the case the representation is real we have that both left and right handed components transform in the same way thus the representation is not chiral, it is "vectorlike" implying that we necessarily need complex representations. For example in QCD we have the reducible representation D_L labeled as $(\mathbf{3} \oplus \bar{\mathbf{3}})$ and it is real, in fact $D_R = D_L^* = D_L$ consequently the theory is not chiral. Last fact limits the space of possible groups as can be seen in the Table (5.1).

Another important requirement, though not restrictive, is to use a simple group. This is done in order to have a single gauge coupling at the scale of grand unification, thus explaining the origin of the coupling parameters of the Standard model. Summarizing, after requiring rank ≥ 4, complex representations and simplicity of the group, from Table (5.1) we see that the possible choices for a Grand Unification group are $SU(n)$ for $n \geq 5$, $SO(4l+6)$ for $l \geq 1$ and E_6[3].

Historically the first model was grand unification in $SU(5)$ [43] in 1974. This is the probably the most simple GUT theory and will be the main topic of this chapter. All the other simple GUTs contain this group as a subgroup. Contemporaneously, in 1974 Pati and Abdus Salam [44] developed the eponymous model with gauge symmetry $SU_L(2)\times SU_R(2)\times SU(4)$. It is very important since it is a main subgroup of a larger GUT based on the $SO(10)$ group and would explain the $V-A$ (chiral) structure of the interaction of the Standard model since is left-right symmetric. This last model [45] was created in 1975.

[3] Table (5.1) should really be a Table of groups generated by semisimple Lie algebras since $SO(k)$ does not have complex representations. Instead is the group $Spin(k)$ that has them though both have the same algebra.

5.2 Fields of the Minimal $SU(5)$

As outlined in Section (3.2), after the choice of the Gauge group, we naturally need to add $dim(G)$ gauge vector bosons in order to construct covariant derivatives. In addition we need to find adequate representations for the fermionic fields and for the scalar fields. Following chapter 3, we can express the Yang Mills fields using Eq.(3.205) as : $W_\mu = W^j_{\mu i} X^j_i$ with $i, j = 1, \cdots, 5$ with scalar product $\mathrm{Tr}[X^j_i, X^l_k] = \delta_{ik}\delta_{jl}$ and with each gauge fields $W^j_{\mu i}$ satisfying conditions (3.213) and (3.214). In this chapter we will use a different basis to express the YM field. We will use the generalized Gell-Mann matrices λ_a (hermitian and traceless) as generators such that the YM is expressed as $W_\mu = A^a_\mu \lambda^a$ with $a = 1, \cdots, 24$. The scalar product now is:

$$\mathrm{Tr}[\lambda_a, \lambda_b] = 2\delta_{ab}, \qquad a, b = 1, \cdots, 24. \tag{5.3}$$

and the gauge fields A^a_μ are now real. This new choice representation of generators is chosen since in this way the embedding of the generators of the SM $SU(3)_c \times SU(2)_L \times U(1)_Y$ is explicit. Since $SU(5)$ is a rank 4 group the Cartan subalgebra (maximal abelian subalgebra) is composed of $\lambda_3, \lambda_8, \lambda_{21}$ and λ_{24}. Explicitly.

$$\lambda_3 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad \lambda_{23} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \tag{5.4}$$

$$\lambda_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} \tag{5.5}$$

We can see the correspondence between λ_3 and λ_8 with the two generators that generate the Cartan subalgebra of $SU(3)_c$ and λ_{21} corresponding to the lone generator of the Cartan subalgebra of $SU(2)_L$. Thus λ_{24} generates the hypercharge (in the fundamental representation):

$$Y = c\lambda_{24} \tag{5.6}$$

with a constant that will be found later. This last result is important since it solves the problem of quantization of the Electric charge since the spectrum of the hypercharge and consequently the charge will take only discrete values. The complete set of generators are, using λ_G as the

3×3 Gell-Mann matrices and using block matrices:

$$\lambda_i = \begin{pmatrix} \lambda_{Gi} & \\ & \end{pmatrix}, \quad i = 1, \cdots 8 \tag{5.7}$$

with the rest being

$$\lambda_9 = \begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda_{10} = \begin{pmatrix} & & & i & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda_{11} = \begin{pmatrix} & & & 0 & 1 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \end{pmatrix} \tag{5.8}$$

$$\lambda_{12} = \begin{pmatrix} & & & 0 & i \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ i & 0 & 0 & & \end{pmatrix} \quad \lambda_{13} = \begin{pmatrix} & & & 0 & 0 \\ & & & 1 & 0 \\ & & & 0 & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda_{14} = \begin{pmatrix} & & & 0 & 0 \\ & & & i & 0 \\ & & & 0 & 0 \\ 0 & i & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix} \tag{5.9}$$

$$\lambda_{15} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 1 \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \end{pmatrix} \quad \lambda_{16} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & i \\ & & & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & i & 0 & & \end{pmatrix} \quad \lambda_{17} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \end{pmatrix} \tag{5.10}$$

$$\lambda_{18} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & i & 0 \\ 0 & 0 & i & & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \lambda_{19} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix} \quad \lambda_{20} = \begin{pmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & i \\ 0 & 0 & 0 & & \\ 0 & 0 & i & & \end{pmatrix} \tag{5.11}$$

$$\lambda_{2a} = \begin{pmatrix} & \\ & \sigma_a \end{pmatrix}, \quad a = 1, 2, 3 \tag{5.12}$$

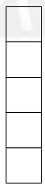
and λ_{24} as above. The λ_i's with $i = 1, c \ldots, 8$ form the algebra of $SU(3)_c$. The λ_i's with $i = 21, 22, 23$ form the algebra of $SU(2)_L$.

5.2.1 Construction of irreducible representations of $SU(n)$ using the fundamental representation

The procedure of getting all possible tensorial irreducible representations for the $SU(n)$ group can be easily done using the Young tableaux. A Young tableaux is a tableaux made of D boxes arranged in $R \leq D$ rows such that the number of boxes in each row is non increasing. Explicitly, let $\{r_1, \cdots, r_R\}$ denote the number of boxes in each row of the tableux or equivalently a non increasing partition of D. A Young tableaux have to satisfy:

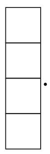
$$\sum_{i=1}^{R} r_i = D \qquad r_1 \geq r_2 \geq \cdots \geq r_R \tag{5.13}$$

It can be shown [42] that each Young tableaux with D boxes in at most n rows corresponds to an irreducible D-tensorial representation of $SU(n)$. Using the general formalism for tensors in $SU(n)$ a general tensor $K + K'$ tensor is written as $\psi = \psi_{i_1 \cdots i_K}^{j_1 \cdots j'_K}$ where lower indices denote indices that transform under the fundamental and higher ones that transform under the antifundamental. We can use Levi Civita's to lower indices. Using Young tableau each box correspond to an index of the tensor and we have that for two vertically aligned boxes we have correspondingly two indices antisymmetric for them and for horizontal boxes we have indices that are symmetric. The conjugate or not nature of the index will depend on how the tableux is constructed.
The fundamental representation corresponds symbolically to a box, $\square$. Since the fundamental $SU(n)$ representation for $n > 2$ is complex we have also an antifundamental representation. It's Young tableaux is defined as the complementary tableaux (complement) of the fundamental. In general for a given Young tableaux to retrieve it's complement we first fill the original Young tableaux with boxes such that we get a rectangular Young tableaux of n rows and an equal number of columns as the original Young tableaux. Then to the tableaux made of the new boxes we rotate by 180 degrees and that is the conjugate tableaux of the original Young tableaux For example fixing $n = 5$ for the complement of $\square$, defined symbolically as $\bar{\square}$, we use the Young tableaux:



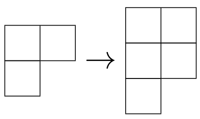
(5.14)

and so the Young Tableaux of $\bar{\square}$ is:



(5.15)

Another example, for $n = 4$:



(5.16)

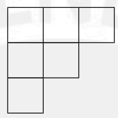
Since the n column Young diagram correspond to a singlet in $SU(n)$ it is sufficient to use $n-1$ indices to label unambiguously any representation of $SU(n)$. One parametrization is as:

$$[r_1 - r_n, r_2 - r_n, \cdots, r_{n-1} - r_n] \tag{5.17}$$

in this way if taking a full n-row row we see that it is not a representation of $SU(n)$ since it's labeling is made of zero entries thus a singlet. The number of labels $n-1$ corresponds to the number of Casimirs of $SU(n)$ or equivalently the rank of the Group.

Dimension of the the Young Tableaux

To know the irrep that each diagram corresponds to we need to calculate its dimension. To calculate the dimension of an arbitrary representation of $SU(n)$ we do as follows. Suppose we have the tableaux:



(5.18)

and we want to calculate it's dimension. First fill the tableaux inserting n on the leftmost element of the first row and then inserting $n+1$ to the right to it and so on. Then, in the second row, start with $n-1$ in the leftmost elements and so on. For example, for the empty tableaux above.

$$\begin{array}{|c|c|c|} \hline n & n{+}1 & n{+}2 \\ \hline n{-}1 & n & \\ \cline{1-2} n{-}2 & & \\ \cline{1-1} \end{array} \tag{5.19}$$

Now multiply all the entries to get a number A. Next construct a similar tableau and fill each box with the number of boxes directly below and directly to the right of the box plus one. This is called the Hook Lenght of the box. We then have the following Young's Diagram:

$$\begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 3 & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array} \tag{5.20}$$

Multiplying all entries we get the number B. Then the dimension is shown to be:

$$d = \frac{A}{B} \tag{5.21}$$

We give a more explicit formula. The dimension of the representation $[\lambda_1, \cdots, \lambda_{n-1}]$ for $SU(n)$ is:

$$dim[\lambda] = \prod_i \frac{m_i}{g_i} \prod_{i<j} \left(\frac{m_i - m_j}{g_i - g_j} \right) \qquad i, j = 1, \cdots n-1 \tag{5.22}$$

where

$$g_i = n - i; \qquad m_i = g_i + r_i \tag{5.23}$$

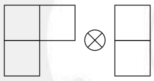
Usually we use the dimension of the irrep as the symbol to denote the irrep. For example for the fundamental rep. of $SU(5)$ we have $\mathbf{5}$. To the denote conjugate representations we use a bar as $\bar{\mathbf{5}}$. A Young tableaux made of only vertical boxes is antisymmetric irreps, with each box indicating A totally antisymmetric representation of $SU(n)$ is equivalently labelled using $[m]$ with m denoting the number of rows of the Young Diagram. When dealing with full antisymmetric representations the formula (5.22) can easily be seen to reduce into:

$$dim[m] = \frac{n!}{m!(n-m)!} \tag{5.24}$$

Multiplication of tableauxs

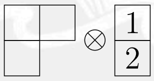
Let n be the dimension of $SU(n)$. We give some examples of irreps and the corresponding labeling in $SU(5)$ in Table(5.2). Note that in general the conjugate representation has the same dimensionality as the original one. Ths can be deduced from the general formula.

We can multiply two irreps to get new irreducible representations. To do this correctly we use a simple algorithm, for example for the product in $SU(n)$:



(5.25)

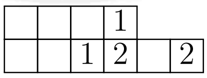
We start labeling each box of the second tableaux with the number of the corresponding row:



(5.26)

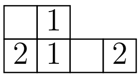
Then multiply each box of the second tableaux, starting from the rightmost box of the first row. The restrictions on the new Young diagram besides being a proper one for $SU(n)$ (satisfy condition (5.13), at most $n-1$ vertical boxes) are such that:

(a) For each row, the number on the boxes originating from the second diagram must not decrease from left to right.



(5.27)

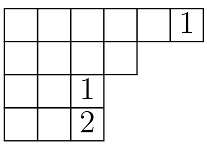
is valid but



(5.28)

it is not since the 2 is before 1 in the same row.

(b) For each column the number on the boxes originating from the second diagram must increase from top to bottom.



(5.29)

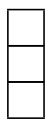
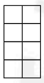
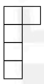
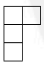
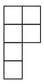
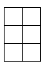
Young Tableaux	Dimension in $SU(n)$	Labeling in $SU(5)$	Tensor rep.
	n	$\mathbf{5}$	ψ_{i_1}
	n	$\bar{\mathbf{5}}$	ψ^{i_1}
	$\frac{n(n-1)}{2}$	$\mathbf{10}$	$\psi_{i_1 i_2}$
	$\frac{n(n-1)}{2}$	$\bar{\mathbf{10}}$	$\psi^{i_1 i_2}$
	$\frac{n(n+1)}{2}$	$\mathbf{15}$	$\psi_{i_1 i_2}$
	$\frac{n(n+1)}{2}$	$\bar{\mathbf{15}}$	$\psi^{i_1 i_2}$
	$n^2 - 1$	$\mathbf{24}$	$\psi^{j_1}_{i_1}$
	$\frac{(n+1)n(n-1)(n-2)}{8}$	$\mathbf{45}$	$\psi^{j_1}_{i_1 i_2}$
	$\frac{(n+1)n(n-1)(n-2)}{8}$	$\bar{\mathbf{45}}$	$\psi^{i_1 i_2}_{j_1}$
	$\frac{(n+1)^2(n-1)n}{12}$	$\mathbf{50}$	$\psi_{i_1 i_2}$
	$\frac{(n+1)n(n-1)(n-2)}{8}$	$\bar{\mathbf{50}}$	$\psi^{i_1 i_2}$
	$\frac{(n+1)n(n-1)}{6}$	$\mathbf{20}$	$\psi_{i_1 i_{2 i 3}}$

Table 5.2 Tableaux of different irreps

is valud but

$$\begin{ytableau} \ & \ & \ & \ & \ & 1 \\ \ & \ & \ & \ \\ \ & \ & 2 \\ \ & \ & 1 \end{ytableau} \tag{5.30}$$

is not

(c) Counting labeled boxes starting from the upper right box of the new diagram, the number of boxes labeled by a must never be more than the ones labeled by the number $a-1$.

$$\begin{ytableau} \ & \ & \ & \ & \ & 1 \\ \ & \ & 1 & 2 \\ \ & \ & 2 \\ \ & \ & 1 \end{ytableau} \tag{5.31}$$

is valid but

$$\begin{ytableau} \ & \ & \ & \ & \ & 1 \\ \ & \ & 2 & 2 \\ \ & \ & 2 \\ \ & \ & 1 \end{ytableau} \tag{5.32}$$

it is not since the second 2 apperars before the second 1.

We give an example using Eq.(5.26)

$$\ydiagram{2,1} \otimes \begin{ytableau} 1 \\ 2 \end{ytableau} = \left(\begin{ytableau} \ & \ & 1 \\ \ \end{ytableau} \oplus \begin{ytableau} \ & \ \\ \ & 1 \end{ytableau} \oplus \begin{ytableau} \ & \ \\ \ \\ 1 \end{ytableau} \right) \otimes \begin{ytableau} 2 \end{ytableau} = \begin{ytableau} \ & \ & 1 \\ \ & 2 \end{ytableau} \oplus \begin{ytableau} \ & \ & 1 \\ \ \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ \\ \ & 1 \\ 2 \end{ytableau} \oplus \begin{ytableau} \ & \ \\ \ \\ 1 \\ 2 \end{ytableau}$$

(5.33)

Thus:

$$\ydiagram{2,1} \otimes \ydiagram{1,1} = \ydiagram{3,2} \oplus \ydiagram{3,1,1} \oplus \ydiagram{2,2,1} \oplus \ydiagram{2,1,1,1} \tag{5.34}$$

where each resultant Young Tableaux can or cannot be a representation of $SU(n)$. This will depend on n. For example for $SU(2)$ we at most can have 2 rows thus some diagrams are not valid so the decomposition is just:

$$\ydiagram{2,1} \otimes \ydiagram{1,1} = \ydiagram{3,2} \tag{5.35}$$

Examples with $SU(5)$

On the other hand for $n = 5$ the decomposition is exactly as in Eq.(5.34).

We retrieve some general decompositions for $SU(n)$ the correspondent representations can be seen from Table(5.2) and the last result in $SU(5)$:

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} = \mathbf{10} \oplus \mathbf{15} \tag{5.36}$$

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1} = \mathbf{20} \oplus \bar{\mathbf{10}} \tag{5.37}$$

$$\ydiagram{1,1,1,1} \otimes \ydiagram{1,1,1,1} = (\ydiagram{1})^c \otimes (\ydiagram{1})^c = \left(\ydiagram{2}\right)^c \oplus \left(\ydiagram{1,1}\right)^c = \bar{\mathbf{10}} \oplus \bar{\mathbf{15}} \tag{5.38}$$

$$(\ydiagram{1})^c \otimes \ydiagram{1,1} = \ydiagram{1,1,1,1} \otimes \ydiagram{1,1} = \begin{ytableau} \ & 1 \\ \ \\ \ \\ \ \end{ytableau} \otimes \begin{ytableau} 2 \end{ytableau} = \begin{ytableau} \ & 1 \\ \ & 2 \\ \ \\ \ \end{ytableau} \oplus \begin{ytableau} \ & 1 \\ \ \\ \ \\ \ \\ 2 \end{ytableau} = \ydiagram{2,2,1,1} \oplus \ydiagram{1} = \left(\ydiagram{2,1,1}\right)^c \oplus \ydiagram{1} = \bar{\mathbf{45}} \oplus \mathbf{5} \tag{5.39}$$

$$\ydiagram{1,1} \times \ydiagram{1,1} = \left(\begin{ytableau} \ & 1 \\ \ \end{ytableau} \oplus \begin{ytableau} \ \\ \ \\ 1 \end{ytableau} \right) \otimes \begin{ytableau} 2 \end{ytableau} = \begin{ytableau} \ & 1 \\ \ & 2 \end{ytableau} \oplus \begin{ytableau} \ & 1 \\ \ \\ 2 \end{ytableau} \oplus \begin{ytableau} \ \\ \ \\ 1 \\ 2 \end{ytableau} = \mathbf{50} \oplus \mathbf{45} \oplus \bar{\mathbf{5}} \tag{5.40}$$

In addition to retrieve the adjoint representation we have:

$$\ydiagram{1} \otimes \left(\ydiagram{1}\right)^c = \mathbf{1} + dim(SU(n)) \tag{5.41}$$

If there are subgroups of $SU(m)$ embedded in a larger $SU(n)$ we can see the branching rules. For the fundamental rep. of $SU(5)$

$$\square_5 = (\square_3, 1_2) \oplus (1_3, \square_2) \tag{5.42}$$

and, since $SU(2)$ is pseudoreal for the antifundamental:

$$\bar{\square}_5 = (\bar{\square}_3, 1_2) \oplus (1_3, \square_2) \tag{5.43}$$

To get the decomposition of $\mathbf{24}$ we use the Young multiplication procedure:

$$\begin{aligned}
\mathbf{5}\otimes\bar{\mathbf{5}} &= \Big((\mathbf{1},\mathbf{2})_{(+1)}\oplus(\mathbf{3},\mathbf{1})_{(-2/3)}\Big)\otimes\Big((\mathbf{1},\mathbf{2})_{(-1)}\oplus(\bar{\mathbf{3}},\mathbf{1})_{(+2/3)}\Big)\\
&= \Big((\mathbf{1},\mathbf{2})_{(+1)}\otimes(\mathbf{1},\mathbf{2})_{(-1)}\Big)\oplus\Big((\mathbf{1},\mathbf{2})_{(+1)}\otimes(\bar{\mathbf{3}},\mathbf{1})_{(+2/3)}\Big)\oplus\Big((\mathbf{3},\mathbf{1})_{(-2/3)}\otimes(\mathbf{1},\mathbf{2})_{(-1)}\Big)\\
&\quad\oplus\Big(\mathbf{3},\mathbf{1})_{(-2/3)}\otimes(\bar{\mathbf{3}},\mathbf{1})_{(+2/3)}\Big)\\
&=(\mathbf{1},\mathbf{2}\otimes\mathbf{2})_{(0)}\oplus(\bar{\mathbf{3}},\mathbf{2})_{(+5/3)}\oplus(\mathbf{3},\mathbf{2})_{(-5/3)}\oplus(\mathbf{3}\otimes\bar{\mathbf{3}},\mathbf{1})_{(0)}\\
&=(\mathbf{1},\mathbf{1})_{(0)}\oplus(\mathbf{1},\mathbf{3})_{(0)}\oplus(\mathbf{8},\mathbf{1})_{(0)}\oplus(\bar{\mathbf{3}},\mathbf{2})_{(+5/3)}\oplus(\mathbf{3},\mathbf{2})_{(-5/3)}
\end{aligned} \tag{5.44}$$

5.2.2 Fermion Representations

Fields in Left chirality	Representation
Spinor Fields	
$Q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2},\frac{1}{3})$
$L_L := \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2},-1)$
u_L^c	$(\bar{\mathbf{3}},\mathbf{1},-\frac{4}{3})$
d_L^c	$(\bar{\mathbf{3}},\mathbf{1},+\frac{2}{3})$
e_L^+	$(\mathbf{1},\mathbf{1},+2)$

Table 5.3 The SM field for one generation classified with respect to the symmetry group $G_{SM} = SU(3)\times SU(2)\times U_Y(1)$ in Left Chirality

As seen before all possible irreducible representation of $SU(n)$ for $n \geq 3$ are constructed from the tensor products of the fundamental representation $\square$ and the antifundamental representation $\bar{\square}$. This is a consequence of the complexity of representations. From the explicit algebra of generators of $SU(5)$ we see that $\mathfrak{su}(3)\oplus\mathfrak{su}(2)\subset\mathfrak{su}(5)$ thus the fundamental irrep. of $SU(5)$ is decomposed as:

$$\square_5 = (\square_3, 1_2)\oplus(1_3,\square_2) \tag{5.45}$$

Using the tensorial product we arrive at the braching rules of Table (5.4).

Now we want to embed the first generation of the SM fermionic fields into a representation(s) of $SU(5)$.

Table 5.4 Branching Rules for $SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y$

$\mathbf{5}$	$(\mathbf{1},\mathbf{2})_{(+1)} \oplus (\mathbf{3},\mathbf{1})_{(-2/3)}$
$\bar{\mathbf{5}}$	$(\mathbf{1},\mathbf{2})_{(-1)} \oplus (\bar{\mathbf{3}},\mathbf{1})_{(+2/3)}$
$\mathbf{10}$	$(\mathbf{1},\mathbf{1})_{(+2)} \oplus (\bar{\mathbf{3}},\mathbf{1})_{(-4/3)} \oplus (\mathbf{3},\mathbf{2})_{(+1/3)}$
$\mathbf{15}$	$(\mathbf{1},\mathbf{3})_{(2)} \oplus (\mathbf{3},\mathbf{2})_{(1/3)} \oplus (\mathbf{6},\mathbf{1})_{(-4/3)}$
$\mathbf{24}$	$(\mathbf{1},\mathbf{1})_{(0)} \oplus (\mathbf{1},\mathbf{3})_{(0)} \oplus (\mathbf{8},\mathbf{1})_{(0)} \oplus (\mathbf{3},\mathbf{2})_{(-5/3)} \oplus (\bar{\mathbf{3}},\mathbf{2})_{(5/3)}$
$\mathbf{45}$	$(\mathbf{1},\mathbf{2})_{(1)} \oplus (\mathbf{3},\mathbf{1})_{(-2/3)} \oplus (\mathbf{3},\mathbf{3})_{(-2/3)} \oplus (\bar{\mathbf{3}},\mathbf{1})_{(8/3)} \oplus (\bar{\mathbf{3}},\mathbf{2})_{(-7/3)}$ $\oplus(\bar{\mathbf{6}},\mathbf{1})_{(-2/3)} \oplus (\mathbf{8},\mathbf{2})_{(1)}$
$\mathbf{50}$	$(\mathbf{1},\mathbf{1})_{(-4)} \oplus (\mathbf{3},\mathbf{1})_{(-2/3)} \oplus (\bar{\mathbf{3}},\mathbf{2})_{(-7/3)} \oplus (\mathbf{6},\mathbf{3})_{(-2/3)}$ $\oplus(\mathbf{6},\mathbf{1})_{(8/3)} \oplus (\mathbf{8},\mathbf{2})_{(1)}$

A first restriction is that they cannot possible be other than totally antisymmetric representations otherwise we include new fermions not belonging to the standard model. For example for the $\mathbf{15}$ representation we have a sextet of quarks $\mathbf{6} = \square\square_3$. Then the only possible representations are the K totally antisymmetric tensors denoted by $[K]$.

Fermions are a collection of 15 Weyl spinors so if we want to embed them into a single representation we need at least one with 15 complex dimension. There are not totally antisymmetric reps of $SU(5)$ with this dimensions so we can't use just one representation, need at least two. Since the combinations of representations of the type $[K] + [n-K]$ are real (thus fermions are vectorlike) we can't use representations of these types.

Nevertheless, the most important restriction is to select fermion representations such that anomalies cancel. [4] For a $[K]$ representation we have that the normalized anomaly contribution is:

$$A([m]) = \frac{(n-2)!(n-2m)}{(n-m-1)!(n-1)!} = -A([n-m]) \tag{5.46}$$

Since we only can have five different totally antisymmetric representations the only combination for $SU(5)$ that cancel anomalies is: $[2] + [4]$ or equivalently, since $[4] = \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} = \bar{\square} = \bar{\mathbf{5}}$, $\bar{\mathbf{5}} + \mathbf{10}$. In order to see which fields compose the $SU(5)$ field in the $\bar{\mathbf{5}}$, denoted as $\mathbf{5}^c_F$, we start from its branching rule, $\bar{\mathbf{5}} = (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{2}})$, and see that is composed of a multiplet that is color antitriplet in $SU(3)_c$ and singlet in the $SU(2)_L$ and another one that is an antidoublet in $SU(2)$ and singlet in $SU(3)$. The color antitriplet can be filled, without knowing the hypercharge of $\mathbf{5}^c_F$, with u^c_L or d^c_L implying that the chiral field $\mathbf{5}^c_F$ has to be left handed. On the other hand, since the fundamental representation of $SU(2)$ is pseudoreal then $\mathbf{2}$ and $\bar{\mathbf{2}}$ are equivalent implying that the antidoublet $SU(2)_L$ can only be filled with the fields of L_L,

[4] In fact we could just imposed anomaly cancellation to retrieve $\mathbf{5}_f$ and $\mathbf{10}_F$ as the fermion fields.

and not its charge conjugated, since have to be left handed. Although we know there are the leptons of L_L we still don't know the order of them in $\mathbf{5}_F^c$.

Thus we need to find which quark antitriplet belongs to $\mathbf{5}_F^c$ and the position of the leptons. First, since $\mathbf{5}_F^c$ transforms as an antifundamental we need to know the generators in the antifundamental representation. Denoting λ as the generator in the fundamental representation and $\bar{\lambda}$ in the antifundamental using the fact that for a field in the antifundamental we have

$$\bar{U}\psi = e^{i\bar{\lambda}}\psi \equiv U^*\psi = e^{-i\lambda^*}\psi \tag{5.47}$$

but λ is hermitian so we have:

$$\bar{\lambda} = -\lambda^T \tag{5.48}$$

Using last equation we see the order of the leptons of $\mathbf{5}_F^c$. Since the charge operator is defined as:

$$\hat{Q} = \hat{I}_3 + \frac{\hat{Y}}{2} \tag{5.49}$$

since $\hat{I}$ in the fundamental is the generator λ_{23} we have for the doublet of $\mathbf{5}_F^c$, using Eq.(5.48) that it has the value of:

$$\hat{I}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & +\frac{1}{2} \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \tag{5.50}$$

then using Eq.(5.49) on $\mathbf{5}_F^c$ knowing that the value of the hypercharge is -1 we obtain:

$$Q\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \tag{5.51}$$

Thus [5]:

$$\mathbf{5}_F^c = \begin{pmatrix} q_{1L}^c \\ q_{2L}^c \\ q_{3L}^c \\ e^- \\ -\nu \end{pmatrix} \tag{5.52}$$

To find the hypercharge of the quarks of $\mathbf{5}_F^c$ we need to the constant c from Eq.(5.6). We have:

$$-c\lambda_{24}^T\mathbf{5}_F^c = -c\lambda_{24}\mathbf{5}_F^c = Y\mathbf{5}_F^c \tag{5.53}$$

[5] there is an ambiguity on the sign of ν

Explicitly:

$$-c\lambda_{24}\mathbf{5}_F^c = \frac{c}{\sqrt{15}}\begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}\begin{pmatrix} q_{1L}^c \\ q_{2L}^c \\ q_{3L} \\ e^- \\ -\nu \end{pmatrix} = \begin{pmatrix} Y_q & & & & \\ & Y_q & & & \\ & & Y_q & & \\ & & & Y_L & \\ & & & & Y_L \end{pmatrix}\begin{pmatrix} q_{1L}^c \\ q_{2L}^c \\ q_{3L} \\ e^- \\ -\nu \end{pmatrix} \tag{5.54}$$

From the value of the hypercharge of the doublet $Y_L = -1$ we have that $c = \sqrt{\frac{5}{3}}$. Thus $Y_q = +\frac{2}{3}$ so $q_L^c = d_L^c$. The result is:

$$\mathbf{5}_F^c = \begin{pmatrix} d_{1L}^c \\ d_{2L}^c \\ d_{3L}^c \\ e^- \\ -\nu \end{pmatrix} \tag{5.55}$$

We will use greek symbols to denote $1, 2, 3$ and latin symbols to index $4, 5$.

The decomposition of the $\mathbf{10}$ representation under $SU(3)_c \times SU(2)_L$ is:

$$\mathbf{10} = (\mathbf{1}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \tag{5.56}$$

Then the rest of the fermions are contained in the $\mathbf{10}$ representation :

$$\mathbf{10}_F = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & u_{3L}^c & -u_{2L}^c & u_{1L} & d_{1L} \\ -u_{3L}^c & 0 & u_{1L}^C & u_{2L} & d_{2L} \\ u_{2L}^c & -u_{3L}^C & 0 & u_{3L} & d_{3L} \\ -u_{1L} & -u_{2L} & -u_{3L} & 0 & e_L^+ \\ -d_{1L} & -d_{2L} & -d_{3L} & -e_L^+ & 0 \end{pmatrix} \tag{5.57}$$

$\hat{X}$ is an arbitrary operator. For an antisymmetric tensor we have:

$$[\hat{X}\psi]_{ij} = X_i + X_j \tag{5.58}$$

and for an adjoint:

$$[\hat{Q}\psi]_i^j = X_i - X_j \tag{5.59}$$

In this way we complete the branching rules Table using $\hat{Y}$. Using $\hat{Q}$ we have:

$$\hat{Q}\mathbf{10}_F = \frac{1}{\sqrt{2}}\begin{pmatrix} -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ -2/3 & -2/3 & -2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & 2 & 1 \\ -1/3 & -1/3 & -1/3 & 1 & 0 \end{pmatrix} \tag{5.60}$$

5.2.3 Gauge Bosons

The $SU(5)$ Adjoint representation ($\mathbf{5}\otimes\bar{\mathbf{5}}=\mathbf{24}\oplus\mathbf{1}$) decomposes under $SU(3)_c\times SU(2)_L$ as:

$$\mathbf{24}=(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{2})\oplus(\bar{\mathbf{3}},\bar{\mathbf{2}}) \tag{5.61}$$

where $(\mathbf{8},\mathbf{1})$ is the adjoint representation of $SU(3)$, $(\mathbf{1},\mathbf{3})$ the adjoint representation of $SU(2)$ and the $(\mathbf{1},\mathbf{1})$ as a singlet under $SU(3)_c\times SU(2)_L$. The new gauge fields will live in the $(\mathbf{3},\mathbf{2})$ and $(\bar{\mathbf{3}},\bar{\mathbf{2}})$. The Yang-Mills field $A_\mu=\frac{1}{2}\lambda^a A^a_\mu$ after the correspondent redefintion of fields as in the SM is:

$$A_\mu=\frac{1}{\sqrt{2}}\begin{pmatrix} & & & X_{1\mu} & Y_{1\mu}\\ & \frac{1}{\sqrt{2}}\sum_{a=1}^{8}G^a_\mu\lambda^a & & X_{2\mu} & Y_{2\mu}\\ & & & X_{3\mu} & Y_{3\mu}\\ X^c_{1\mu} & X^c_{2\mu} & X^c_{3\mu} & \frac{W^3_\mu}{\sqrt{2}} & W^+_\mu\\ Y^c_{1\mu} & Y^c_{2\mu} & Y^c_{3\mu} & W^-_\mu & \frac{W^3_\mu}{\sqrt{2}}\end{pmatrix}+\frac{B_\mu}{2\sqrt{15}}\begin{pmatrix}-2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3\end{pmatrix} \tag{5.62}$$

Or compactly:

$$A_\mu=\frac{1}{\sqrt{2}}\begin{pmatrix}[G_{\mu 8}]^\beta_\alpha & [X_\mu]^a_\alpha\\ [X^\dagger_\mu]^\alpha_a & [W_{\mu 3}]^b_a\end{pmatrix}+\frac{1}{2}B_\mu\lambda_{24} \tag{5.63}$$

We also have:

$$\hat{Q}A_\mu=\begin{pmatrix}0 & 0 & 0 & -4/3 & -1/3\\ 0 & 0 & 0 & -4/3 & -1/3\\ 0 & 0 & 0 & -4/3 & -1/3\\ 4/3 & 4/3 & 4/3 & 0 & 1\\ -1/3 & -1/3 & -1/3 & -1 & 0\end{pmatrix} \tag{5.64}$$

5.2.4 Scalar Representations

Using the results of Section 3 we see that we need at least two scalars to arrive to the low energy symmetry $U(1)_Q$. The first scalar Σ in the adjoint representation will give us the symmetry breaking pattern $SU(5)\to SU(3)_c\times SU(2)_L\times U(1)_Y$.

$$\Sigma=\frac{1}{\sqrt{2}}\begin{pmatrix}[\Sigma_8]^\beta_\alpha-\frac{2}{\sqrt{30}}\Sigma_0 & [\Sigma_X]^a_\alpha\\ [\Sigma^\dagger_X]^\alpha_a & [\Sigma_3]^b_a+\frac{3}{\sqrt{30}}\Sigma_0\end{pmatrix} \tag{5.65}$$

The second scalar has to gives us the the electroweak symmetry breaking. To satisfy this it needs to contain the Standard Model Higgs field. The minimal representation that contains it, is again the vector representation of $SU(5)$, **1**.

$$\phi = \begin{pmatrix} T_1 \\ T_1 \\ T_1 \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} T_\alpha \\ H_a \end{pmatrix}, \qquad \alpha = 1,2,3. \quad a = 4,5. \tag{5.66}$$

5.2.5 Calculation of the Weinberg Angle

Expressing the coupling of the hypercharge as in the SM, that is with the coupling g', we have that :

$$\frac{1}{2} g_5 \lambda_{24} B_\mu = \frac{1}{2} g' Y B_\mu \tag{5.67}$$

Since $c\lambda_{24} = Y$ we arrive at $g' = \sqrt{\frac{3}{5}} g_5$. From the other components we have that the $SU(2)_L$ is equal to $g = g_5$ so the Weinberg angle at grand unification scale ($\mu > M_{\text{GUT}}$) is:

$$\sin^2 \theta_W = \frac{g^2}{g'^2 + g^2} = \frac{3}{8} \tag{5.68}$$

To arrive at testable experimental values at the EW scale we need to use the renormalization group equations.The result is:

$$\sin \theta_W^2 = 0.208 \tag{5.69}$$

in discordance with the result of the SM of $\sin \theta_W^2 = 0.23120 \pm 0.00015$.

5.3 $SU(5)$ Lagrangian

The complete Lagrangian is composed of:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_f + \mathcal{L}_H + \mathcal{L}_Y \tag{5.70}$$

where the kinetic term of the gauge bosons is:

$$\mathcal{L}_B = -\frac{1}{4} A^a_{\mu\nu} A^{a\mu\nu}, \qquad a = 1, \cdots, 24 \tag{5.71}$$

with

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_5 f_{abc} A^b_\mu A^c_\nu, \qquad a,b,c = 1, \cdots, 24 \tag{5.72}$$

where f_{abc} is the structure constant of the Lie Algebra of $SU(5)$.

The fermionic Lagrangian (implying scalar product on generation space) is:

$$\begin{aligned}\mathcal{L}_f =& i\bar{\mathbf{5}}^c_L\gamma^\mu\mathcal{D}_\mu\mathbf{5}^c_F + i\,\mathrm{Tr}[\overline{\mathbf{10}_F}\gamma^\mu\mathcal{D}_\mu\mathbf{10}_F]\\ =& i\bar{\mathbf{5}}^c_L(\gamma^\mu\partial_\mu - iA^a_\mu\frac{1}{2}\gamma^\mu\bar{\lambda}^a)\mathbf{5}^c_F + i\,\mathrm{Tr}[\bar{\mathbf{10}}_F(\gamma^\mu\partial_\mu\mathbf{10}_F - i\gamma^\mu A^a_\mu\frac{\lambda^a}{2}\mathbf{10}_F - i\gamma^\mu A^a_\mu\mathbf{10}_F\frac{(\lambda^a)^T}{2}]\\ =& i\bar{\mathbf{5}}^c_F\partial\!\!\!/\mathbf{5}^c_F + \frac{1}{2}i\,\mathrm{Tr}[\bar{\mathbf{10}}_F\partial\!\!\!/\mathbf{10}_F] - \bar{\mathbf{5}}^c_F\gamma^\mu A^a_\mu\frac{(\lambda^a)^T}{2}\mathbf{5}^c_F + \mathrm{Tr}[\bar{\mathbf{10}}_F\gamma^\mu A^a_\mu\lambda^a\mathbf{10}_F]\end{aligned} \tag{5.73}$$

where we used $\frac{\bar{\lambda}^a}{2}$ as the antifundamental generators of $SU(5)$ and the antisymmetry propriety of $\mathbf{10}$.

We impose parity in the scalar Lagrangian $_h$ for practical reasons:

$$\mathcal{L}_h = \frac{1}{2}\,\mathrm{Tr}[(\mathcal{D}^\mu\Sigma)^\dagger\mathcal{D}_\mu\Sigma] + \frac{1}{2}(\mathcal{D}^\mu\phi)^\dagger\mathcal{D}_\mu\phi - V(\Sigma,\phi) \tag{5.74}$$

with

$$V(\Sigma,\phi) = V(\Sigma) + V(\phi) + \frac{\lambda_4}{4}\,\mathrm{Tr}[\Sigma^2](\phi^\dagger\phi) + \frac{\lambda_5}{4}(\phi^\dagger\Sigma^2\phi) \tag{5.75}$$

$$V(\Sigma) = -\frac{m_1^2}{2}(\mathrm{Tr}[\Sigma^2]) + \frac{\lambda_1}{4}(\mathrm{Tr}[\Sigma^2])^2 + \frac{\lambda_2}{4}\,\mathrm{Tr}[\Sigma^4] \tag{5.76}$$

$$V(\phi) = -\frac{m_2^2}{2}(\phi^\dagger\phi) + \frac{\lambda_3}{4}(\phi^\dagger\phi)^2 \tag{5.77}$$

The Yukawa Lagrangian (as will be retrieve in Section (5.5)) is:

$$\mathcal{L}_Y = \mathbf{5}^c_F Y_5\mathbf{10}_F\phi^* + \frac{1}{8}\epsilon_5\mathbf{10}_F Y_{10}\mathbf{10}_F\phi \tag{5.78}$$

which contain the Yukawa terms that gives masses to the fermions at EW scale.

Fermionic Interactions

Expanding the covariant derivatives from $\mathcal{L}_f$ we have new interactions at GUT scale:

$$\begin{aligned}
\mathcal{L}_{\text{int}f} =& \frac{g_5}{\sqrt{2}}\left(\bar{u}_i^\alpha\gamma^\mu[G_{\mu 8}]_\alpha^\beta u_{i\beta} + \bar{d}_i^\alpha\gamma^\mu[G_{\mu 8}]_\alpha^\beta d_{i\beta}\right)\\
&+ \frac{g_5}{\sqrt{2}}\left(\bar{Q}_{iL}^a\gamma^\mu[W_{\mu 3}]_a^b Q_{iLb} + \bar{L}_{iL}^a\gamma^\mu[W_{\mu 3}]_a^b L_{Lb}\right)\\
&+ \sqrt{\frac{3}{5}}\frac{g_5}{2}\Bigg(- \bar{L}_{iL}\gamma^\mu B_\mu L_{iL} - 2\bar{e}_{iL}^{+}\gamma^\mu B_\mu e_{iL}^{+} + \frac{1}{3}\bar{Q}_{Li}\gamma^\mu B_\mu Q_{iL}\\
&+ \frac{4}{3}\bar{u}_{iR}\gamma^\mu B_\mu U_{iR} - \frac{2}{3}\bar{d}_{iR}\gamma^\mu B_\mu d_{iR}\Bigg)\\
&- \frac{g_5}{\sqrt{2}}\epsilon_{ab}\bar{L}_L^b\gamma^\mu[X_\mu]_\alpha^a[d_{iL}^c]^\alpha + \frac{g_5}{\sqrt{2}}\epsilon^{\alpha\beta\gamma}\bar{u}_{iL\gamma}^c\gamma^\mu[X_\mu]_\alpha^a Q_{iL\beta a} + \frac{g_5}{\sqrt{2}}\bar{Q}_{iL}^{\alpha b}\gamma^\mu[X_\mu]_\alpha^a\epsilon_{ab}e_{iL}^{+} + h.c.
\end{aligned} \tag{5.79}$$

Note that there are new interactions mediated by the gauge boson X that violate B baryon and leptonic numbers but preserve $B-L$ symmetry.

We can retrieve an order of magnitude for the GUT scale M_X. An amplitude for the decay of the proton viable from the last term of the full fermionic interacions (5.79) is:

$$\Gamma(p \to \pi^0 e^+) \approx \frac{m_p^5}{M_X^4} \tag{5.80}$$

Since the actual constraints on proton decay are of the order of $\approx 10^{34}$ years we obtain $M_X \geq 10^{16}$ GeV.

5.4 Spontaneous symmetry breaking of $SU(5)$

To retrieve the vev of the potential (5.75) we can use two approaches. The first one is assuming a priori the existence of two mass scales, the EW scale M_W and the very much large GUT mass scale M_X, introducing the hierarchy $M_W \ll M_X$. One can have an approximate order of magnitude using the Renormalization Group Equation (RGE) on each of the couplings of $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ independently. In this case we can assume a two step breaking mechanism. The first breaking will be of the potential $V(\Sigma)$ from Eq.(5.76) and after that we will retrieve the Standard Model Higgs potential after assuming certain conditions. The other, more direct procedure, is retrieving the vev of the full potential $V(\Sigma, \phi)$ as done in [46]. We will not use this approach though the results, as for example when retrieving the masses of the different vector and scalar bosons, go like $\frac{M_W}{M_X}$, thus essentially irrelevant for calculations but not so much for the conceptual understanding.

The first stage of the breaking corresponds finding the vev $\langle\Sigma\rangle$ of $V(\Sigma, \phi)$ with $\phi = 0$ (note that we just take the field as the null vector we don't minimize the potential in any form for ϕ) that is just the vev of the potential $V(\Sigma)$ of Eq.(5.76). Following Section (3.4.5) if

$(\lambda_1 > 0, \lambda_2 > 0)$ and $30\lambda_1 + 7\lambda_2 > 0$ the pattern of symmetry breaking is:

$$SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y. \tag{5.81}$$

The vev of Eq.(5.76) is retrieved from the general formula Eq.(3.361) with Eq.(3.179) for $n_1 = 3$, $n_2 = 2$ and:

$$\langle\phi_1\rangle^2 = \frac{4m_1^2}{30\lambda_1 + 7\lambda_2} \equiv 4V^2 \tag{5.82}$$

thus:

$$\langle\Sigma\rangle = \langle\phi_1\rangle \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix} = V \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{5.83}$$

To see the bosons that acquire mass we use Eq. (3.366). From the expression of the gauge fields we have the relation $W_\alpha^a = \frac{1}{\sqrt{2}} X_\alpha^a$ then the gauge mass term is:

$$\mathcal{L}_M = g^2(\phi_1 - \phi_2)^2 \frac{1}{2} X_{\mu\alpha}^a X_\alpha^{*\mu a} = \frac{25}{2} V^2 g^2 X_{\mu\alpha}^a X_\alpha^{*\mu a} \tag{5.84}$$

So the mass of the six X (12 real fields) bosons is $M_X = \sqrt{\frac{25}{2}} gV$. From the potential $V(\Sigma)$ we retrieve the masses of the components of Σ:

$$V(\langle\Sigma\rangle + \Sigma) = -\frac{m_1^2}{2} \operatorname{Tr}[(\langle\Sigma\rangle + \Sigma)^2] + \frac{\lambda_1}{4} (\operatorname{Tr}[(\langle\Sigma\rangle + \Sigma)^2])^2 + \frac{\lambda_2}{4} \operatorname{Tr}[(\langle\Sigma\rangle + \Sigma)^4] \tag{5.85}$$

arriving at the results in Table (5.5).

The ϕ scalar also develops a mass after SSB at grand unification scale:

$$V(\phi) = -\frac{m_2^2}{2}(\phi^\dagger\phi) + \frac{\lambda_3}{4}(\phi^\dagger\phi)^2 + \frac{\lambda_4}{4} \operatorname{Tr}[\langle\Sigma\rangle^2](\phi^\dagger\phi) + \frac{\lambda_5}{4}(\phi^\dagger \langle\Sigma\rangle^2 \phi). \tag{5.86}$$

restricting to second order in ϕ we have:

$$\begin{aligned} V_{\text{quadratic}} =& -\frac{m_2^2}{2}(T^\dagger T + H^\dagger H) + \frac{30}{4}\lambda_4 V^2 (T^\dagger T + H^\dagger H) + \frac{\lambda_5}{4} V^2 (4T^\dagger T + 9H^\dagger H) \\ =& \frac{1}{2}\left(-m_2^2 + 15\lambda_4 V^2 + 2\lambda_5 V^2\right) T^\dagger T + \frac{1}{2}\left(-m_2^2 + 15\lambda_4 V^2 + \frac{9}{2}\lambda_5 V^2\right) H^\dagger H \\ =& \frac{1}{2} m_T^2 T^\dagger T + \frac{1}{2} m_H^2 H^\dagger H. \end{aligned} \tag{5.87}$$

Scalar Fields	$SU(3)_c \times SU(2)_L$ quantum numbers	Mass after separate breaking
$[\Sigma_8]^\beta_\alpha$	$(\mathbf{8}, \mathbf{1})$	$5\lambda_2 V^2$
$[\Sigma_3]^b_a$	$(\mathbf{1}, \mathbf{3})$	$20\lambda_2 V^2$
Σ_0	$(\mathbf{1}, \mathbf{1})$	m_1^2
$[\Sigma_X]^a_\alpha$	$(\mathbf{3}, \mathbf{2})$	0
T_α	(3,1)	$-m_2^2 + (15\lambda_4 + 2\lambda_5)V^2$
H_a	(1, 2)	$-m_2^2 + (15\lambda_4 + \frac{9}{2}\lambda_5)V^2$

Table 5.5 Masses of the Scalar Bosons

In principle after SSB at GUT scale all masses in Table (5.5) would be expected to be of the order of the GUT scale if the SSB is done contemporaneously. This is not the case for us, since at the beginning of the Section we have imposed the hierarchy $M_W \ll M_X$ implying that the Goldstone Bosons of the SM are absorbed only by the Higgs similar to the one of the SM that is H. In consequence it will have a mass M_H of the order M_W but from seeing how the analytical form of M_H is we see that this requirement can be satisfied only at the expense of Large Fine Tuning $\mathcal{O}\left(\frac{M_W}{M_X}\right) = 10^{-12}$. For the triplet with mass M_T we don't have this requirement. This discordance is the doublet-triplet splitting. Even more it is not known if the triplet T satisfies the Appelquist-Carazzone theorem [47] since in principle could interact with SM particles. We can get a boundary condition for M_T from the couplings at the Yukawa sector that produce the proton decay. Using the values of the diagonalized Yukawas of the SM, $[Y^u_D]_{11}$ and $[Y^d_D]_{11}$, we obtain:

$$\frac{[Y^u_D]_{11}[Y^d_D]_{11}}{M_T^2} \leq \frac{1}{M_X^2} \tag{5.88}$$

that imply $M_T \geq 10^{12}$ GeV and lessening the fine tuning requirement for the Doublet-Triplet splitting to $\mathcal{O}\left(\frac{M_W}{M_T}\right) = 10^{10}$.

Then, at EW scale we get the effective potential.

$$V_{\text{eff}} = \frac{m_H^2}{2} H^\dagger H + \frac{\lambda_3}{4}(H^\dagger H)^2 \tag{5.89}$$

that is the Standard Model Higgs potential and the vev is retrieved in the usual way. Summarizing the SSB pattern is:

$$SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_Q \tag{5.90}$$

5.5 Yukawa sector and fermion masses

The combination of fermions we can have for a possible Yukawa coupling and the corresponding representation of the product is:

$$\begin{array}{rc} \mathbf{5}_F^{c\,T} C \mathbf{5}_F^c \rightarrow & \bar{\mathbf{5}} \otimes \bar{\mathbf{5}} = \bar{\mathbf{10}} \oplus \bar{\mathbf{15}} \\ \mathbf{5}_F^{c\,T} C \mathbf{10}_F \rightarrow & \bar{\mathbf{5}} \otimes \mathbf{10} = \mathbf{5} \oplus \bar{\mathbf{45}} \\ \mathbf{10}_F^T C \mathbf{10}_F \rightarrow & \mathbf{10} \otimes \mathbf{10} = \bar{\mathbf{5}} \oplus \mathbf{45} \oplus \mathbf{50} \end{array}$$

This shows that the $\mathbf{24}$ scalar boson cannot couple to the fermions fields via a Yukawa coupling thus the fermions remain massless after SSB at grand unification scale. The Yukawa Lagrangian is then as in Eq. (5.5), only with the $\mathbf{5}$ scalar: [6]

$$\mathcal{L}_Y = \frac{1}{\sqrt{2}}(\mathbf{5}_F^c)^x_{ia} C_{xy} (Y_5)_{ij} (\mathbf{10}_F)^{yab}_j (\phi^*)_b + \frac{1}{8\sqrt{2}} \epsilon_{abcde} (\mathbf{10}_F)^{abx}_i C_{xy} (Y_{10})_{ij} (\mathbf{10}_F)^{cdy}_j \phi^e + h.c. \tag{5.91}$$

From the first term we get:

$$\begin{aligned} \frac{1}{\sqrt{2}} \mathbf{5}_F^c Y_5 \mathbf{10}_F \phi^* =& d^c Y_5 Q \phi^* + L Y_5 e^c \phi^* + d^c Y_5 \epsilon_3 u^c T^* + L \epsilon_2 Y_5 Q^T T^* + h.c. \\ =& d^c Y_5 Q \phi^* + e^c Y_5^T L \phi^* + d^c Y_5 \epsilon_3 u^c T^* + L \epsilon_2 Y_5 Q^T T^* + h.c. \end{aligned} \tag{5.92}$$

Thus the traspose of the Yukawa matrix for down quarks is the Yukawa matrix for leptons:

$$Y_D = Y_L^T \tag{5.93}$$

This implies equal masses of leptons and down quarks at GUT scale. Since leptons and quarks evolve differently, we can use the RGE to see the ratio of masses at EW scale. At scale $\mu = 10$ GeV it is:

$$\frac{m_d}{m_e} = 3.5 \tag{5.94}$$

Implying a mass of $m_d = 1.6$ MeV, different from the measured value in (4.22). From the second term we retrieve:

$$\frac{1}{2}\frac{1}{\sqrt{2}} u^c (Y_{10} + Y_{10}^T) Q H + h.c. = \frac{1}{2}\frac{1}{\sqrt{2}} u^c Y_U Q H \tag{5.95}$$

that implies a symmetric up quark mass Yukawa matrix:

$$Y_U = Y_U^T \tag{5.96}$$

Note that Eq.(5.92) implies fermions-T interactions that violate Baryon and leptonic quantum numbers.

[6] using mass proportional to $\psi_L^T C \psi'_L$. The indices i, j denote family indices, the ab $SU(5)$ indices and x, y Lorentz

5.6 Problems of the minimal $SU(5)$ model

(a) **No Unification** It can be shown using RGE starting from EW with corrections induced by the minimal $SU(5)$ that the couplings of the SM do not converge at GUT scale.

(b) **Gauge hierarchy problem. Doublet-Triplet splitting** After the introduction of a new mass scale M_{GUT}, we see that we need to fine tune the values of many observables and explain the existence of a scalar with two different mass scales (Doublet-Triplet splitting).

(c) **No Flavor theory.** Although we have constrained some parameters on the Minimal $SU(5)$, ($Y_L = Y_L, Y_U = Y_U^T$) there is still not an explanation for the existence of three generations. To deal with this in the context of GUT theries we need larger gauge groups or horizontal symmetries.

(d) **No Neutrino mass.** We need to introduce a right handed neutrino in order to construct a Yukawa coupling analogous to the PMNS matrix. In the minimal $SU(5)$ model this cannot be done using only the **5** and **10** irreps thus we need at least to introduce a right handed singlet.

(e) **Wrong Charged Lepton Mass.** The results $Y_D = Y_L^T$ is wrong. A possible new scalar that modifies this relationship can be seen in Eq.(5.5).

(f) **Too fast proton decay.** The initial Georgi-Glashow model, the one we have used in this chapter, predicted a lifetime of the proton of $\tau_P \approx 10^{31}$ [48]years for $M_x \approx 10^{14}$ GeV. This was disproved very soon for example at the Kamiokande experiment (1989) that gave $\tau_P \geq 2.6 \times 10^{32}$ years.

(g) **Magnetic Monopoles** Following hypercharge quantization, the minimal $SU(5)$ predicted heavy (order $\approx M_X$) magnetic monopoles that have been falsified by the MACRO experiment (2011).

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