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A study of well-posedness in inverse optimal control

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Aufgabenstellung für die Masterarbeit

von Herrn César Aldo Canelo Solórzano

Thema: A study of well-posedness in inverse optimal control

Nowadays, the optimal control problem is widely studied by researchers. In contrast, inverse optimal control (IOC) addresses the problem of inferring the cost function, of which a system behavior is assumed to be optimal. The IOC is useful to reflect the functional objectives of the system, such as the weighting balance between states and control effort. However, it is difficult to design a cost function that accurately reflects desired system behaviors. Since IOC can be regarded as an inverse problem, it is nature to address the well-posedness as a critical property in the problem formulation. Specifically, we concern the following properties for a well-posed IOC:

- The existence (or optimality) problem where it is answered whether there is any cost for which the given trajectories are minimizing.
- The cost function is unique with respect to desired system specifications.
- The inverse application is stable with respect to data perturbations.

The present master thesis aims to estimate weighting matrices for a quadratic cost function of a linear quadratic regulator (LQR) with:

- Objective function: $\int_0^\infty (x'Qx + u'Ru)dt$.
- Linear system: $\dot{x} = Ax + Bu$.
- Specifications: Closed-loop poles for optimal feedback gain.

The well-posedness of IOC is addressed to finding the necessary and sufficient conditions for the existence and uniqueness of weighting parameters in the weighting matrices Q and R.

To validate the theoretical work, we consider the application of path following control for a skid-steering mobile robot.

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I confirm that this Master thesis is my own work and I have documented all sources and material used.

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Resumen

En esta tesis de maestría, se aborda el problema de identificar los parámetros de ponderación en las funciones de coste definidas por un problema de control óptimo. Debido a la naturaleza del problema, abordado como un problema inverso, el enfoque de este trabajo es asegurar el buen planteamiento de los problemas de control óptimo inverso, un aspecto crucial que garantiza la viabilidad, unicidad y estabilidad de las funciones de coste estimadas. El estudio emplea la metodología del regulador cuadrático lineal (LQR) dentro de un sistema lineal.

Un aspecto central de esta investigación es la determinación de los parámetros Q y R en el enfoque LQR, que desempeñan un papel fundamental en la definición de la eficiencia y la eficacia del sistema de control. La tesis examina cómo pueden elegirse óptimamente estos parámetros y el impacto que tienen en el rendimiento del sistema. Además, el estudio explora el uso de restricciones para mejorar la respuesta transitoria del sistema, un factor importante en el diseño de sistemas de control, garantizando que el sistema alcance rápida y eficazmente los requisitos de diseño deseados.

En este trabajo se propone un enfoque de dos niveles para resolver el problema de control óptimo inverso. Se trata de utilizar programación semidefinida para recuperar los parámetros de la función de coste y evaluar la optimalidad de la solución. Además, se aborda el problema para encontrar las condiciones para minimizar la función de coste, estimando los parámetros Q y R a partir de las leyes de control observadas, y aplicando restricciones para la optimización. Se concluye con resultados que demuestran la mejora de la respuesta del sistema y un método alternativo que reduce la dependencia de la matriz de ganancia K.

Abstract

In this master thesis, we address the problem to identify the weighting parameters in the cost functions defined by an optimal control problem. Due to the nature of addressed problem as an inverse problem, the focus of this work is to ensure the well-posedness of inverse optimal control problems, a crucial aspect that guarantees the feasibility, uniqueness, and stability of the estimated cost functions. The study employs the Linear Quadratic Regulator (LQR) methodology within a linear system.

Central to this research is the determination of the parameters Q and R in the LQR approach, which play a pivotal role in defining the efficiency and effectiveness of the control system. The thesis examines how these parameters can be optimally chosen and the impact they have on system performance. Additionally, the study explores the use of constraints to enhance the transient response of the system, a significant factor in control system design, ensuring that the system quickly and effectively reaches its desired design requirements.

A two-level approach to solving the inverse optimal control problem is proposed in this work. It involves using semidefinite programming to recover cost function parameters and evaluating the optimality of the solution. Also, we address the problem to finding conditions for minimizing the cost function, estimating parameters Q and R from observed control laws, and applying constraints for well-posedness. It concludes with results demonstrating improved system response and an alternative method that reduces dependence on the K-gain matrix.

Zusammenfassung

In dieser Masterarbeit befassen wir uns mit dem Problem der Identifizierung der Gewichtungsparameter in den Kostenfunktionen, die durch ein optimales Kontrollproblem definiert werden. Da es sich bei dem behandelten Problem um ein inverses Problem handelt, liegt der Schwerpunkt dieser Arbeit auf der Sicherstellung der Wohlgeformtheit von inversen Optimalsteuerungsproblemen, einem entscheidenden Aspekt, der die Machbarkeit, Eindeutigkeit und Stabilität der geschätzten Kostenfunktionen garantiert. In der Studie wird die Methode des linearen quadratischen Reglers (LQR) in einem linearen System angewandt.

Im Mittelpunkt dieser Untersuchung steht die Bestimmung der Parameter Q und R im LQR-Ansatz, die eine zentrale Rolle bei der Definition der Effizienz und Effektivität des Steuerungssystems spielen. In der Arbeit wird untersucht, wie diese Parameter optimal gewählt werden können und welchen Einfluss sie auf die Systemleistung haben. Darüber hinaus untersucht die Studie die Verwendung von Nebenbedingungen zur Verbesserung des Einschwingverhaltens des Systems, einem wichtigen Faktor beim Entwurf von Regelsystemen, um sicherzustellen, dass das System die gewünschten Entwurfsanforderungen schnell und effektiv erreicht.

In dieser Arbeit wird ein zweistufiger Ansatz zur Lösung des inversen optimalen Steuerungsproblems vorgeschlagen. Dazu gehört die Verwendung der semidefiniten Programmierung, um die Parameter der Kostenfunktion zu ermitteln und die Optimalität der Lösung zu bewerten. Außerdem befassen wir uns mit dem Problem, Bedingungen für die Minimierung der Kostenfunktion zu finden, die Parameter Q und R aus den beobachteten Kontrollgesetzen zu schätzen und Einschränkungen für die Wohlgeformtheit anzuwenden. Abschließend werden Ergebnisse vorgestellt, die eine verbesserte Systemantwort und eine alternative Methode zeigen, die die Abhängigkeit von der K-Gain-Matrix verringert.

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List of Acronyms

LTI	Linear Time-Invariant
IOC	Inverse Optimal Control
OCP	Optimal Control Problem
LQR	Linear Quadratic Regulator
ARE	Algebraic Riccati Equation
DARE	Derivative of the Algebraic Riccati Equation
ODE	Ordinary Differential Equations
LMI	Linear Matrix Inequality
SDP	Semidefinite Programming

List of Symbols

$\mathbb{R}^n$	Space of real column vectors of dimension n
$\mathbb{R}^{n\times m}$	Space of real matrices of dimension $n \times m$
$\mathbb{R}_+$	R+ Set of positive real numbers
$\mathbb{S}^n$	Space of Hermitian matrices with dimension $n \times n$
$\mathbb{S}_+^n$	Space of $n \times n$ positive semi-definite matrices
$\mathbb{S}_{++}^n$	Space of $n \times n$ positive definite matrices
$\bar{x}(t)$	State vector $n \times 1$
$\bar{u}(t)$	Control vector $m \times 1$
$Q \geq 0$	Symmetric $n \times n$ positive semidefinite matrix
$P > 0$	Symmetric $m \times m$ positive definite matrix
∂S	Boundary of set S
$Int(S)$	Interior of set S
$\emptyset$	Empty set.
I_n	Identity matrix of dimension $n \times n$
$X \geq Y$	$X - Y$ is a positive semi-definite matrix
$\|x\|$	l_2 norm of vector x
A^T	Transpose of matrix A
$tr(A)$	Trace of matrix A
$vec(A)$	Vectorization of matrix A
$Im(A)$	Image space of matrix A
$\dim(S)$	Dimension of subspace S
$col\{x_1, \dots, x_n\}$	$[x_1^T, \dots, x_n^T]^T$
$diag(X_1, \dots, X_n)$	Block matrix with $X_1, \dots, X_n$ on its diagonal
$\mathbb{1}_n$	Column vector in Rn with all elements equal to 1
$\mathbb{E}[X]$	Expectation of random variable X
$cov[X]$	Covariance matrix of random variable X
$Re(z)$	Real part of complex number z

Chapter 1
Introduction

At present, a large number of studies have been carried out on the theory of optimal control whose objective is to obtain the control law of a known dynamic system in such a way as to optimize the cost function related to the system. It is worth mentioning that this cost function is intended to regulate the behavior of the system, weighing either the accuracy of the response on the effort that the controller imparts to obtain that response or vice versa. In this sense, since the parameters that integrate the cost function modify the behavior of the system, it is required to establish a balance between these parameters that is in accordance with the purpose of the system. Therefore, in many cases it is often difficult to know in advance the configuration of a cost function that regulates the behavior of the system, so it is useful to have a mathematical approach to estimate this cost function to have a knowledge or, at least, an initial idea of the behavior of the system and this is achieved with the Inverse Optimal Control.

Inverse optimal control (IOC) is an area of mathematics in constant research whose growing interest in recent years is due to its applications especially in the field of robotics. Thus, contrary to forward optimal control, inverse optimal control is based on knowledge of the control law or at least part of the response of the system to subsequently estimate the cost function that could have produced that response. In this sense, one of the main approaches used for the inverse optimal control is the one applied to the Linear Quadratic Regulator (LQR), since its structure, which reflects convexity characteristics, makes it ideal for the study and application in many of the practical systems today [14].

However, the inverse optimal control requires certain necessary and sufficient conditions to be satisfied to produce a reliable result, these conditions ensure the well-posedness to the Inverse Optimal Control problem (IOCP). In this sense, since this type of problems are solved by means of numerical methods, it is necessary that such necessary and sufficient conditions are represented as Linear Matrix Inequalities (LMIs) constraints, also given the convex nature of the Linear Quadratic Regulator it is required to redefine the approach of such constraints as convex equations and inequalities using the theory of Semidefinite Programming (SDP). In this context, this work aims to develop the well-posedness for an Inverse Optimal Control problem by using the Semidefinite Programming (SDP) with Linear Matrix Inequality (LMI) constraints.

Moreover, for practical applications, a good quality controller must also ensure fast and well-damped temporal responses. This is often achieved by strategically placing closed-loop poles in the complex plane, a technique known as regional pole placement. Unlike pointwise pole placement, which assigns poles to specific locations, regional pole placement confines the poles to an area in the complex plane, which is usually achieved by the intersection of a shifted half-plane, a vertical strip, a sector and a disk which is usually known and implemented as a Linear Matrix Inequality (LMI) region. This approach guarantees desirable characteristics such as fast decay and acceptable damping on the system response [11].

In the present work, the linear quadratic regulator (LQR) has been considered as the cost function on which the conditions for a well-posedness to the Inverse Optimal Control problem and the process of obtaining the parameters that compose it have been studied. The codes have been written in Python programming language and have been used libraries for numerical operations such as NUMPY, libraries for numerical optimization such as SCIPY and libraries for convex optimization such as CVXPY.

1.1 Backgrounds

The well-posedness of the Inverse optimal control problem (IOCP) has been a central concern, and researchers have employed a variety of mathematical and computational tools to develop robust and effective solutions. In that sense, inverse optimal control, where starting from an optimal control policy, the objective is to find the parameters that compose a cost function, dates back to the seminal work of Kalman in 1964 where he first characterized the necessary and sufficient conditions for a control policy to be optimal for the Linear Quadratic Regulator (LQR) problem [22]. Also, the Riccati equations are considered one of the fundamental pillars in control theory. These equations were described by Kalman more than 50 years ago, since then the variety of problems involving these equations has expanded, including those with quadratic cost criteria. [24].

As for linear matrix inequalities (LMIs), whose origin dates back to the end of the 19th century, when the Lyapunov theory arises, which establishes that a system $\dot{x}(t) = Ax(t)$ is stable, that is to say that all its trajectories converge to the origin (or zero) if and only if there exists a positive definite matrix $P > 0$ such that $A^T P + PA < 0$, which represents a special form of a linear matrix inequality (LMI) and is known as the Lyapunov inequality on P [20]. Therefore, according to the above, the Lyapunov inequality was the first Linear Matrix Inequality (LMI) used to analyze the stability of a dynamical system. In that sense around the 1940s the methods established by Lyapunov were used in practical control engineering problems, posing elementary linear matrix inequalities whose analytical solution was feasible by hand, thus limiting its application to small or lower order systems. During the 1960s, graphical methods were developed for the solution of LMIs of larger or higher order systems. Later, in the early 1970s, it became known that the same group of LMIs could be solved by the symmetric solution of the algebraic Riccati equation (ARE) in addition to graphical methods. Also, at the same time, the usefulness of linear matrix inequalities applied to computational algorithms, compared to classical or analytical methods, was observed. Finally, during the 1980s it was recognized that many of the LMIs could be solved computationally and conveniently through the formulation of convex programming problems and several interior point algorithms were developed for an efficient solution of such formulations of linear matrix inequalities (LMIs) arising in control theory [20].

On the other hand, semidefinite programming (SDP) as a mathematical programming tool was developed in the 1990s and so far, applications in areas such as convex optimization, combinatorics and control theory are diverse. The popularity of the application of semidefinite programming is largely due to the fact that they offer an efficient solution because they use algorithms based on interior point methods for their solution [23].

1.2 Motivations

In the context of Inverse Optimal Control (IOC), the concept of well-posedness guarantees the existence of a solution to the inverse problem. In the absence of a well-posed solution, there may be multiple solutions or none at all, making it difficult to determine the true underlying control strategy. In that sense, a well-posed problem usually leads to solutions that make sense in the context of the physical system being modeled. This is crucial for applications in robotics and control, where the identified control strategies must be practical and aligned with the expected behavior of the system. Also, well-posed problems often result in numerical methods that are well behaved and computationally tractable. This is crucial in the application of algorithms that efficiently solve Inverse Optimal Control problems [13].

On the other hand, starting from the optimal response of a system is a desirable feature in inverse optimal control. This is evident for example in the movement of biological organisms since the fundamental premise behind biologically inspired engineering, both animal and human, is the assumption that natural processes exhibit optimality [12].

Over the years it has been observed that organisms present in nature exhibit in many of their behaviors a certain degree of intelligence, reinforced by the fact that such behaviors tend to perform optimally according to several studies. In this sense, the optimality principle is investigated as a key tool for modeling mechanisms inspired by such intelligent behaviors. Thus, inverse optimal control has been studied and applied in different fields, mainly in robotics. Since, in inverse optimal control, knowing in advance the dynamics of the system and the behavioral policy considered optimal for a given task, it is sought to recover the optimization criterion or cost function that has generated such behavior. [14].

In Inverse Optimal Control (IOC), the usefulness of estimating objective functions is justified by the fact that processes, whether natural or artificial, require a certain degree of optimization. For example, the predator in nature must weigh the need to feed against the distance traveled, the size of the prey and the energy required. Likewise, a space station must weigh the need for power versus the speed to deploy the solar panels and the amount of fuel used in the positioning jets for that purpose. However, for each component of a process it is difficult to know in advance the weights required in the cost functions that could replicate each behavior. Therefore, an objective function recovery method could help to have a better understanding of the physical system proposed. Another motivation is the imitation of intelligent entities performing complicated tasks such as pouring a glass of water or driving a car in a dynamic environment with obstacles. However, it may be difficult to obtain expressions that characterize the performance of such tasks since in many cases only a rough idea of the desired behavior is available. Therefore, obtaining a cost function that reproduces the desired task is difficult, while expressing the desired behavior may be easier. Thus, inverse optimal control starts from the desired behavior and recovers the objective function that has produced the desired behavior [14][15].

With this in mind, inverse optimal control problems present a different challenge, as it involves situations in which the cost function is not explicitly defined. Instead, what is known is the solution or outcome, or at least the observable aspects of the solution, typically derived from empirical measurements. This distinguishes it from classical forward optimization problems, in which the cost function is fully defined and the task is to determine its solution. Consequently, the objective of Inverse Optimal Control problems is to determine, in the context of a given dynamic process and an observed outcome, the specific optimization criterion or cost function that has generated that particular outcome [13].

1.3 Objectives

Inverse Optimal Control allows obtaining the underlying functional cost associated with a known trajectory that is assumed to be optimal. In this context, the following are the objectives that were planned to be achieved from the development of this work.

General Objective:

The study of the well-posedness of the inverse optimal control problem (IOCP), in order to subsequently determine the weighting parameters that characterize the cost function.

Specific objectives:

- Formulate the necessary and sufficient conditions for a well-posed Inverse Optimal Control problem.
- Estimate the weighting parameters of the cost function of a linear quadratic regulator (LQR) using LMI constraints, convex formulations and from input data considered optimal.
- Elaborate the first solution of an inverse problem using the conditions of well-posedness.
- Elaborate the second solution by including constraints to improve the transient response of the system.
- Elaborate the third solution to avoid dependence on the use of the gain matrix K.
- Evaluate the results by taking two examples. First, a mass-spring-damper system with 4 degrees of freedom (DOF) for the solution of the inverse problem using as input data the gain matrix K. Secondly, it is used as input data for the solution of the inverse problem and, replacing the gain matrix K, the record of the trajectories in two dimensions, i.e., the position at each instant of time of a mobile robot. and thus reconstruct the cost function that could have produced such trajectory.

1.4 Overview

The content of the sections of this work is described as follows:

- **Chapter 1** provides an introduction to the field of Inverse Optimal Control, outlining the motivations driving this research and the specific objectives pursued.
- **Chapter 2** offers a comprehensive review of relevant literature, establishing the context for the present work.
- **Chapter 3** presents the fundamental concepts such as the necessary and sufficient conditions that ensure the well-posedness of the Inverse Optimal Control problem, Semidefinite Programming (SDP), Linear Matrix Inequality (LMI) constraints and LMI regions relevant to the regional pole placement. Subsequently, these theoretical concepts are applied in the formulation of an Inverse Optimal Control problem taking as an example a mass-spring-damper system with four degrees of freedom (DOF).
- **Chapter 4** represents the discussion section of the present work, in which the results derived from solving the proposed system are presented and the performance of the proposed approach is analyzed.
- **Chapter 5** presents the conclusions of this work and possible future research.

Chapter 2
State of the art

In general, the problem of cost function recovery has been intensively investigated. Thus, numerous specialists have examined the necessary and sufficient criteria to establish the well-posed nature of inverse optimal control problems. Various methodologies, including machine learning techniques and optimization algorithms, have been employed to determine the unknown parameters within the underlying model. However, a common feature of these investigations is the initiation of the analytical process from an existing solution, leading to the subsequent derivation of the underlying cost function or objective function responsible for generating that solution.

To tackle the Inverse Optimal Control problem, it's crucial to examine the solution methodology of forward optimal control. Thus, according to [5] the solution to the optimal linear regulator problem involves the determination of a control input, u, that minimizes a performance index or cost function; therefore, the solution is a unique optimal control that is established by a fixed feedback control law $u = -K^T x$ whose gain controller K is determined by the Algebraic Riccati equation (ARE).

On the other hand, with respect to the inverse problem, in [6] a fully controllable scalar linear system $\dot{x} = Fx + gu$ with state feedback control law $u = -K^T x$ and a performance index, J, that accurately reflects the design requirements is considered. Therefore, the inverse problem attempts to determine the performance index or cost function from K such that u minimizes J. However, it is necessary to establish an optimality criterion for K, with $Q \geq 0$, that allows one to determine which system designs can be interpreted as the solution to an optimization problem. In the end, the solution to the inverse optimal control problem is established by $Q \geq 0$, i.e., $Q = diag(a_1, a_2, \dots, a_n)$.

As per reference [4], in forward optimal control, it's preferable to select a criterion or cost function that precisely mirrors the functional goals of the system. Simultaneously, it should produce an optimal control law that imposes minimal weight, cost, and complexity demands on the system. Conversely, the inverse problem in optimal control theory seeks to identify a set of criterion or cost functions for which a provided feedback control law is optimal. In this sense, the standard linear regulator problem is posed with system dynamics $h(c,v) = Ac + Bv$, given or known feedback optimal control law $v = -Kc$ and cost function g having the parameterized form $g(c,v) = \frac{1}{2}[(c,Qc) + (v,v)]$, where Q is an unknown positive semidefinite, $Q \geq 0$, matrix and K is a given or known gain controller, $K = B'L$. The objective is to characterize all Q satisfying the Hamilton-Jacobi-Bellman equation. For this is needed to solve $K = B'L$ for L and, to substitute in the Algebraic Riccati Equation (ARE), $Q - K'K + LA + A'L = 0$, to finally determine the equivalent Q.

Moreover, according to [2] in an inverse optimal control problem, the dynamics are assumed to be known and the data are a set of registered trajectories. The objective is to recover a cost function that is minimized by those given trajectories that are considered optimal. In this sense, for the reconstruction of the cost, it is necessary to ensure the well-posedness of the inverse problem by considering the existence problem where it is answered if there is any cost for which the given trajectories are minimizing, the injectivity problem where it is answered if this cost is unique and the stability problem where it is answered if the inverse application is stable with respect to perturbations of the data. In this approach the standard finite horizon Linear-Quadratic (LQ) problems, $\int_0^T (x^T Qx + 2x^T Su + u^T Ru)dt$ subject to $\dot{x} = Ax + Bu$, are considered. Then, the quadratic cost class is reduced to the canonical cost class $\min\limits_{u} \int_0^T (u + Kx)^T R(u + Kx)$, where the proposed cost reconstruction method takes as data the recorded trajectories, not the control law and outputs the matrices R and K such that the given optimal synthesis is Γ, i.e., the set of all minimizing solutions.

In that sense, in [8] the Inverse Optimal Control problem given the system matrices A, B and a gain matrix K aims to find the necessary and sufficient conditions for K to be the optimum of an infinite-time LQ problem and to determine all the weight matrices Q, R and S that produce the given gain matrix K. However, this method starts from the state trajectories of a Linear Time Invariant (LTI) system and then identifies the Q, R and S matrices that have generated these trajectories, i.e., the problem of estimating a cost function that best approximates a given set of state trajectories is considered.

In [10], the concept of Inverse Optimal Control (IOC) is elucidated as the process of reconstructing a cost function based on observed input and state trajectories, which are either optimal or approximate. Implicit in this process is the assumption that the input observations faithfully represent the system's evolution and that the expert's behavior is approximately optimal. By evaluating the congruence between the demonstration trajectory and the conditions necessary for control optimality, the agent's optimality is assessed. This evaluation relies on deriving a set of functions from the necessary conditions for optimality, typically based on first-order conditions. Subsequently, the Inverse Optimal Control problem (IOCP) is tackled by minimizing these functions while considering the unknown parameters.

According to [3] Inverse Optimal Control is the process of producing the weights Q and R of a time invariant Linear Quadratic Regulator (LQR) problem having beforehand the estimated gain matrix K which is considered the optimal solution. In this regard, the Linear Quadratic Regulator framework is used where the continuous-time Linear Time Invariant (LTI) system is $\dot{x} = Ax + Bu$ and the cost function is $J =$

$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$, with S=0. The inverse problem pursues the task of given a stabilizing control law $u = -Kx$ determining what necessary and sufficient conditions exist in (K, A, B) such that the gain matrix K is optimal for the cost J with $Q \geq 0$ and $R = I$. Second determine all Q for some (K, A, B) that satisfy the conditions previously found.

To solve the inverse problem, a formulation based on Linear Matrix Inequalities (LMIs) is presented to determine whether for a given stabilizing feedback law K, there exists some set (Q, R) for which K is the optimal feedback gain, as shown below.

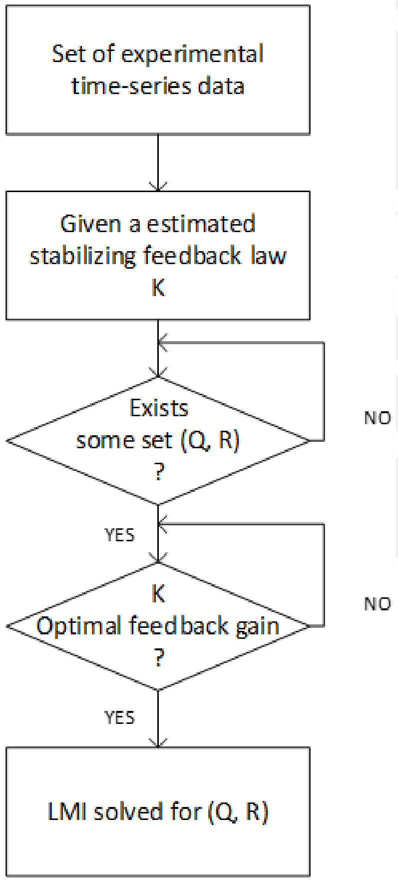


Figure 1 - Formulation of the IOCP given a feedback law K

In cases where the precise solution cannot be obtained due to Linear Matrix Inequalities' (LMI) infeasibility, the development of a gradient descent algorithm becomes imperative. This algorithm relies on the derivative of the Algebraic Riccati Equation (DARE) to minimize the disparity between the optimal solution and the experimental feedback gains.

A contrast between forward and inverse optimal control can be made in [7]. In that sense, being known the quadratic performance index or cost function $\int_0^\infty x'Qx + u'udt$ and the stationary linear dynamic system $\dot{x} = Fx + Gu$. The Forward Optimal Control problem performs the process of finding a feedback matrix K given a certain value of Q. On the other hand, the Inverse Optimal Control problem performs the process of obtaining some symmetric matrix Q for which the given feedback matrix K ($u = -Kx$) must ensure that it is optimal. Therefore, assuming that (F, G) is controllable. The inverse problem is characterized by ensuring that K is optimal for Q if and only if

- Re λ(F-GK) < 0
- $K = G'L$, where L is a real symmetric solution of the Algebraic Riccati Equation (ARE).

With the above it is ensured that there is a unique solution $u = -G'Lx$, i.e., K is optimal for Q. Now, finally, a suitable Q is determined by means of the formula: $Q = K'K - F'L - LF$.

Forward Optimal control problem

$$\min \int_0^\infty x^T Q x + u^T R u dt$$

Subject to $\dot{x} = Ax + Bu$

Solution: $u(t) = -Kx(t)$

where $K = R^{-1}B^T S$ and S is solution to ARE:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0$$

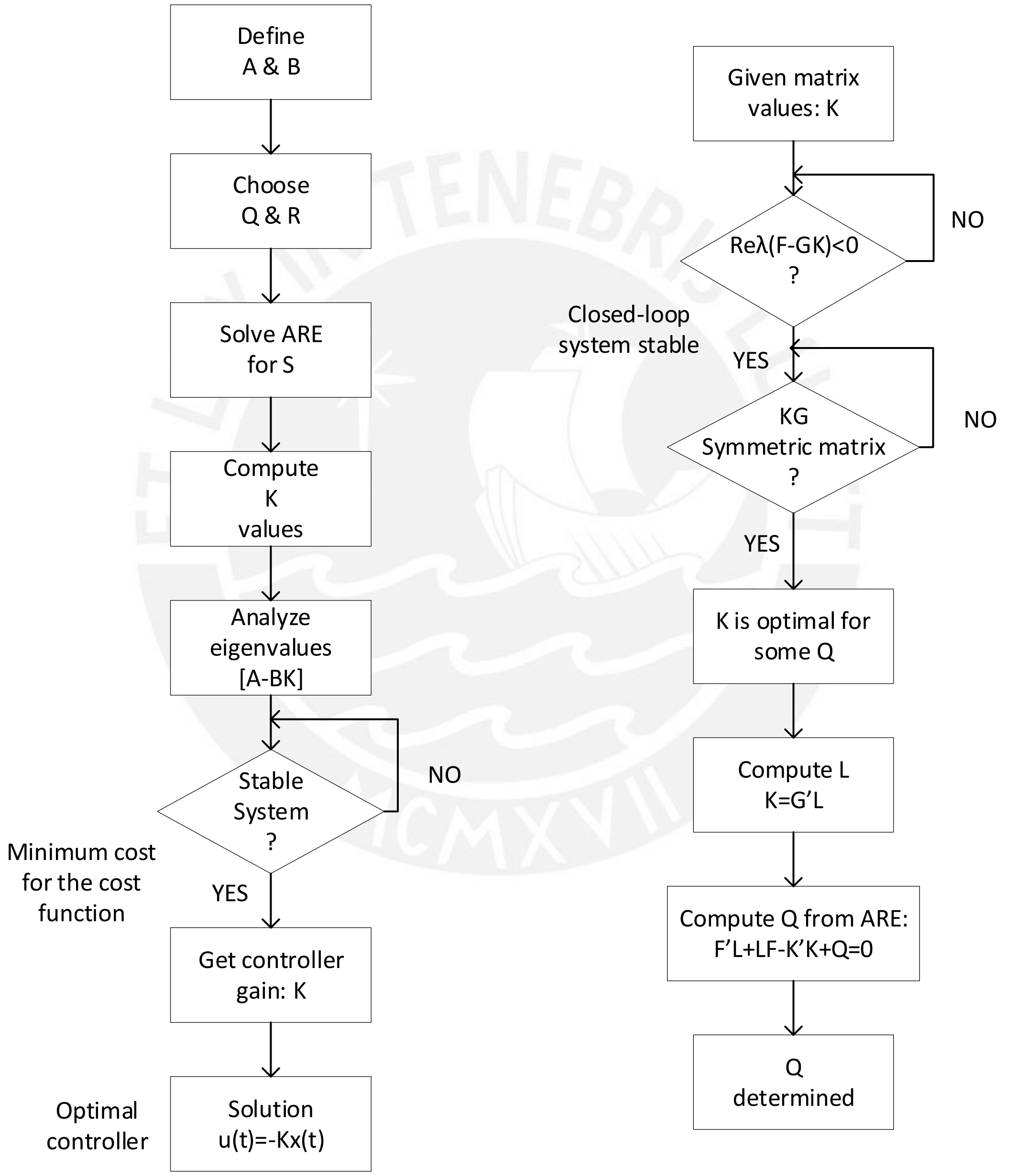
Inverse Optimal Control Problem [7]

Optimal control law: $u = -G'Lx$ with a given $K = G'L$

Subject to $\dot{x} = Fx + Gu$

Unknown: Q for

$$\int_0^\infty x'Qx + u'u dt$$



a) b)

Figure 2 - Solution comparison between OCP and IOCP

The work cited in [9] adopts convex formulations as a methodological approach to tackle the Inverse Optimal Control problem. This problem revolves around the inference of cost function matrices from a given observed control law $u(k) = Kx(k)$, for a linear system and quadratic cost functions of the form $x(k)^T P x(k) = \min \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i$ s. t. $x_{i+1} = Ax_i + Bu_i$, $x_0 = x(k)$, where the system is characterized by given system matrix A and input matrix B, wherein both the state variable x and input u are measurable. It is also assumed a deviation Λ=0 and that the measured K is an optimal gain. In that sense, the Algebraic Riccati Equation (ARE) offers a solution by linking the cost function with the control law, which can then be utilized for inverse optimal control solutions after undergoing vectorization and parameterization. This method stands apart from other learning techniques like reinforcement learning, inverse reinforcement learning, or apprenticeship learning due to its distinct approach involving the ARE.

In that sense the task of inferring Q and R from a given controller gain K can be formulated as a Convex Optimization problem. The gain K can be obtained from the input and output measurements with standard approaches, such as least-squares techniques. The gain K is optimal if and only if the deviation Λ = 0. Neither positive definiteness nor symmetry of P is required, instead if the deviation Λ = 0 then P will be positive definite, and the objective encourages symmetry. Then P is symmetric, if the measure K is the optimal solution of the problem as shown below.

In that sense inferring Q and R from a given controller gain K can be framed as a Convex Optimization problem. The optimal K can be derived from input and output measurements using methods like least-squares techniques. K is considered optimal when the deviation Λ equals zero. Positive definiteness or symmetry of P is not essential; if Λ is zero, P becomes positive definite, and the objective promotes symmetry. Symmetry of P indicates K as the optimal solution as shown below.

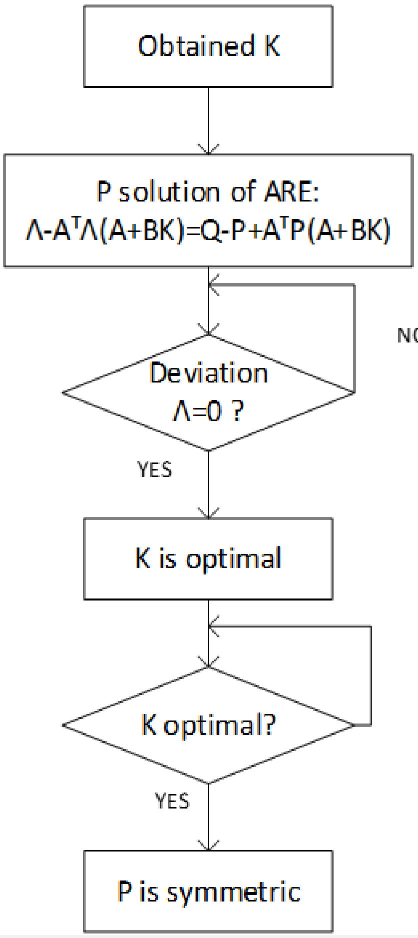


Figure 3 - Optimality condition for K

It is worth mentioning that the classification of convex formulations, shown in Figure 4, includes the Semidefinite Programming (SDP) approach for inferring general objective function matrices from known general gains K, aiming to find Q, R, and Y that minimize deviation to optimality and promote symmetry. Additionally, Linear Programming (LP) is utilized for specific scenarios like diagonal objective function matrices or block diagonal matrices, common in optimal control problems, enabling efficient solutions. Moreover, it's demonstrated that the Inverse Optimal Control problem can be expressed as a convex optimization problem for both closed-loop optimal and non-optimal gains, prevalent in practice, especially with noisy input data. Likewise, an explicit algebraic expression for the inverse problem is derived given an optimal controller gain matrix, establishing sufficient conditions for uniquely inferring the corresponding cost function, in this case particularly for diagonal cost matrices. Consequently, in inverse optimal control problems (IOCP), if Q and R are diagonal and the measured K is optimal, a unique solution exists if the generating cost function matrices are also diagonal.

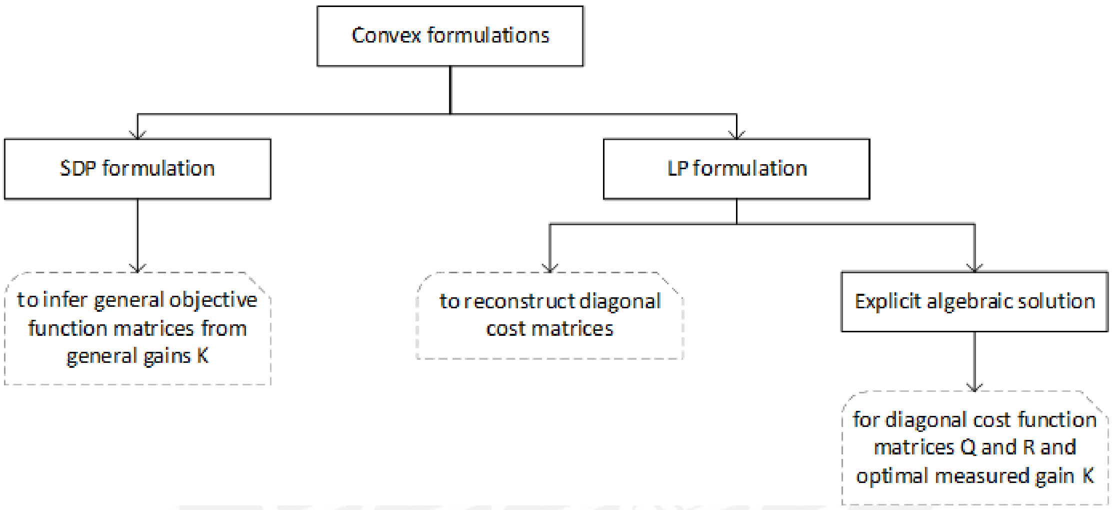


Figure 4 - Classifications for convex functions

In accordance with [1] the linear system $\dot{x} = Ax + Bu$, with control law $u = Dx$ and performance index or cos function $I = \frac{1}{2}x^T(t_1)Fx(t_1) + \frac{1}{2}\int_{t0}^{t1}(x^TQx + u^TRu)dt$ is posed, where the linear optimal inverse control problem consists of finding necessary and sufficient conditions on the matrices A, B, and D of the system such that some performance index is minimized and hence determining all R, Q, and F.

The procedure consists in determining the necessary conditions for the existence of real symmetric matrices $R > 0$ and P such that the feedback matrix $D = -R^{-1}B^TP$ is satisfied. Then, by producing general formulas for R and P, starting from D, the sufficiency of these conditions is demonstrated. These necessary and sufficient conditions are required for the minimum value of the performance index to be nonnegative and positive and for the construction of $Q \geq 0$ from the Algebraic Riccati Equation (ARE). Finally, the class of matrices $\{R, Q, F, P\}$ satisfying the feedback matrix $D = -R^{-1}B^TP$ and the Algebraic Riccati Equation (ARE) is obtained, thus the provided control law minimizes each member of this class of performance indices, effectively solving the inverse problem.

The general outline of the procedure for obtaining the class matrix is shown in Figure 5.

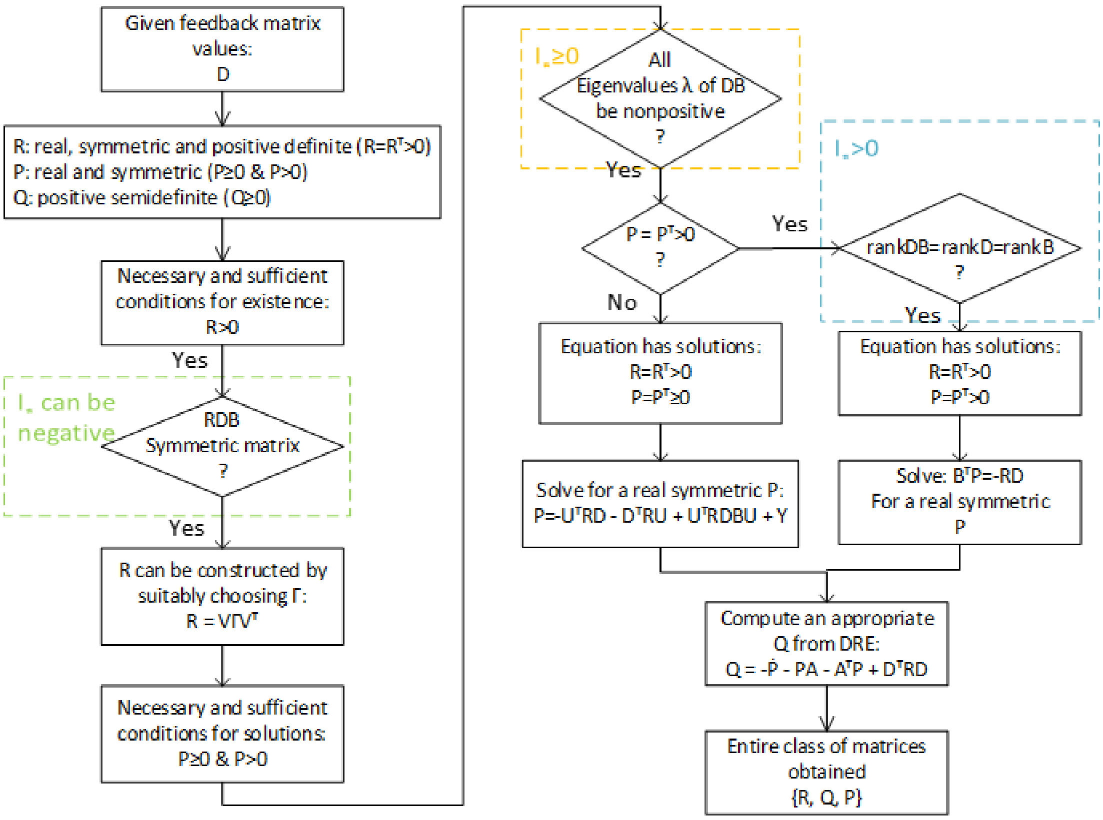


Figure 5 - Procedure for obtaining the performance index parameters

Numerous publications on inverse optimal control for the linear quadratic controller assume that the optimal feedback gain matrix K is known, according to [2], [3], [4], [7], [8], [9]. However, in certain scenarios the feedback gain K cannot be known exactly, so in [32] the feedback gain K is considered to be time-varying for the finite time horizon case. On the other hand, in order to find a stabilizing control law with certain optimality criteria for nonlinear optimal control problems, a lyapunov control function is proposed in [33] to determine the inverse optimal control law. Since its solution usually involves the use of the Hamilton-Jacobi-Bellman equation which does not have an exact analytical solution for the general nonlinear case, but which nevertheless reduces to the Riccati equation in the case of the linear quadratic controller.

The algebraic Riccati equation (ARE) is essential for the solution of the linear inverse quadratic regulator problem. However, in [34] a method is proposed to estimate this equation under the assumption that the system is unknown but that its states and inputs can be observed. In this sense, to solve the inverse problem an equation derived from the algebraic Riccati equation is obtained, but which only requires the trajectory of the system and not the model of the system, representing an alternative to the identification of systems. In [35] two-level evolutionary optimization techniques are used for the solution of inverse optimal control problems. That is, at a higher level it is tried to minimize the error between the calculated and experimental data, while at a lower level it is iteratively searched for a cost function that deviates less from the experimental data.

On the other hand, in [36] with the solution of the Inverse Optimal Control problem LQ necessary and sufficient conditions are established under which the feedback control law places only some of the dominant poles, which affect the response, at specific points that in turn represent optimality for some cost function. In [37] an inverse optimal control design is proposed whose main feature is that the feedback control law is developed after choosing a candidate Lyapunov function with asymptotic properties that converge to equilibrium.

On the other hand, in [38] an optimization approach to the inverse problem is presented for both linear and nonlinear systems. For this, the Inverse Optimal Control matrix approach is used and applied to observed trajectories of systems controlled both in a feedback manner for linear systems and in a feedforward manner for nonlinear systems. In [39] an algorithm is proposed that recovers the control gain K, using a least squares approach, and the cost function parameters from the system trajectories. For this, by certainty equivalence optimality conditions it is guaranteed that stochastic model-free LQR IOC is well posed. Thus, using K by model-free Semidefinite programming, the cost function is obtained.

Finally, in [40] the physical interaction in human-robot collaboration is studied, where the knowledge of the goal of the partner contributes to a natural interaction. For this, the recovery of the cost function that represents such interaction is proposed. Thus, a study of the potentialities and limitations of the Inverse Optimal Control to describe the smooth and natural human-robot interaction is presented.

Chapter 3

Preliminaries and problem formulation

This section presents a review of the main concepts and mathematical tools used throughout this thesis such as control theory tools, optimal control and convex optimization in order to develop the analysis of the inverse problem of the linear quadratic regulator approach. Also, all the theory presented below has been extensively developed in numerous researches and books that can be found in the reference for more details.

3.1 Challenges facing Inverse Optimal Control

Inverse optimal control problems face the challenge of deducing the cost function of an agent from its observed behavior. However, problems arise that make this task difficult and are discussed below.

- **Ill-posedness**

  The ill-posed nature of inverse optimal control problems refers to the inherent difficulty in finding a unique and stable solution. This can be specified as follows.

  - The problem arises because several cost functions can explain the observed behavior equally well.
  - There may be several cost functions leading to the same optimal behavior, making it difficult to unambiguously determine the true underlying cost function.
  - Small variations in the observed trajectories or demonstrations may lead to significantly different inferred cost functions.

Addressing the ill-posedness in inverse optimal control problems often involves introducing additional constraints on the inferred cost functional in order to guide the solution towards more stable and interpretable cost functions.

- **Difficulty in obtaining an explicit gain controller K**

There are situations where having an explicit gain controller K is difficult. Therefore, an alternative approach is to focus on state trajectories, which are sequences of states over time, rather than on the control law. This means trying to understand the underlying cost function by examining the observed states and their transition over time.

By focusing on state trajectories, the idea is to capture the structure and patterns inherent in the observed data and attempt to estimate the underlying cost function based on the observed state trajectories.

This approach is often more practical and applicable in complex, real-world systems where explicit modeling of control gains may be challenging or impractical.

3.2 The well-posedness to Inverse Optimal Control

The well-posedness in inverse optimal control problems addresses the question of whether there is a solution to the inverse problem, such that it is unique and stable with respect to the observed data quality. In that sense, the well-posedness in inverse optimal control involves the following concepts in which necessary and sufficient conditions refer to existence and uniqueness.

3.2.1 Existence

The first aspect of well-posedness is the existence of a solution. This means determining whether there is a cost or objective function that is consistent with the observed behavior of the system. The existence is crucial because, without a solution, the inverse problem cannot be solved. In that sense, the quality of the observed data plays a crucial role. Insufficient or noisy data can lead to ill-posed problems and make it difficult to obtain reliable solutions. Therefore, it is sought to guarantee the optimality of the observed solution, that is, if the solution minimizes the cost function through the following.

- **Necessary conditions:** there must be a feasible solution to the optimization problem associated with the cost function search. This means that there must exist a cost function that, when used in the optimization of the system, produces the observed behavior. In that sense, the existence of a solution to the inverse optimal control problem depends on the controllability of the system, i.e., the system $\dot{x} = Ax + Bu$ must be controllable. A sufficient condition for controllability is that the matrix $[B, AB, A^2B, \ldots, A^{n-1}B]$, where n is the order of the system, has full rank.
- **Sufficient conditions:** in order to address existence, a given gain controller K is optimal for Q if and only if [7]

1) $Re\lambda(A - BK) < 0$
2) $K = B^T L$, where L is the real symmetric solution of the ARE
3) $Q = K^T K - A^T L - LA$

(3.1)

In addition, if the optimization problem is convex, the probability that a solution exists increases. Convexity provides mathematical guarantees of the existence of solutions to optimization problems, so constraints are usually written according to semidefinite programming (SDP).

3.2.2 Uniqueness

Or injectivity, refers to whether the solution of the inverse problem is unique. In other words, whether there is a single cost function that corresponds to the observed behavior of the system. Uniqueness is desirable, as it ensures that the solution is unambiguous and can be identified with certainty [16]. According to the literature one way to approach inverse optimization problems is based on convex optimization [31]. That is, the search for a cost function is treated as the optimization of a convex problem, thus increasing the probability of having a unique and stable solution, according to the following conditions.

- **Necessary conditions:** assuming the controllability of the system (A, B), as in the case of the existence. It is approached in the sense of establishing the relationship between Q_1 and Q_2 such that K is optimal for both. Therefore, K being optimal for Q_1, then it is optimal for Q_2 if and only if there exists a symmetric matrix Y such that $Q_1 - Q_2 = A^T Y + YA$ and $YB = 0$ [7].
- **Sufficient conditions:** in order to guarantee a unique solution, additional constraints are introduced in the cost function Q, such as symmetry i.e., $Q = Q^T$ and positive semidefinite i.e., $Q \geq 0$. Likewise, the symmetry and positive definite of $R > 0$ ensure necessary and sufficient conditions for its existence, and necessary and sufficient conditions for solutions $P \geq 0$.

3.2.3 Stability

Refers to the robustness of the solution with respect to perturbations or uncertainties in the observed data. A well-posed inverse problem should have a solution that is stable, meaning that small changes in the observed behavior should result in small changes in the estimated cost function. This is important for practical applications where there may be noise or uncertainty in the data.

In that sense, stability can be analyzed by examining the eigenvalues of the closed-loop system. The closed-loop system is stable if and only if the eigenvalues of $A - BK$ are in the left half-plane. In this way, the asymptotic stability of the closed-loop system is guaranteed [7]. Additionally, stability can be addressed by including constraints in the optimization problem. Such constraints may come from physical limitations or known properties of the system.

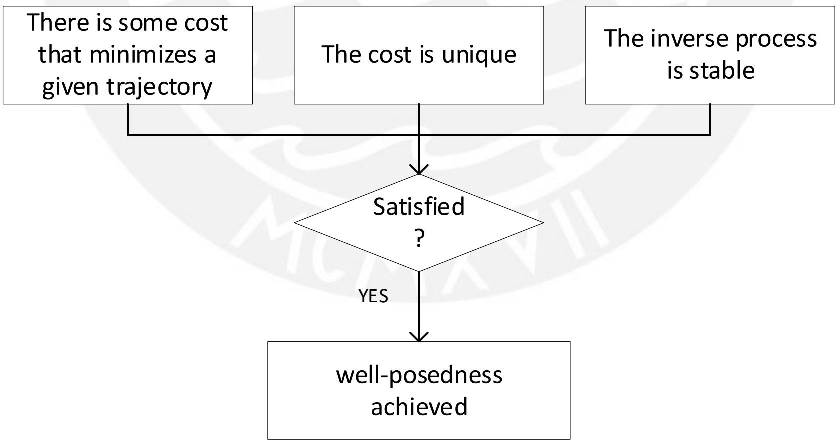


Figure 6 - Scope of well-posedness

These mathematical conditions collectively contribute to a well-posed inverse optimal control problem, according to the diagram shown in Figure 6. However, although these conditions provide a theoretical framework, practical applications often involve a combination of linear inequalities constraints written in semidefinite programming to ensure that the estimated cost function accurately represents the optimization behavior of the underlying system. In this sense, numerical optimization techniques can then be applied to find the solution that satisfies these conditions and estimate the cost function.

3.3 Stability analysis

In general, a continuous-time dynamical system can be represented as a linear time-invariant (LTI) state-space model.

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0 \tag{3.2}$$

$$y(t) = Cx(t) + Du(t) \tag{3.3}$$

Where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output and A, B, C and D are constant matrices with dimensions $n \times n$, $n \times m$, $p \times n$ and $p \times m$ respectively.

Among the main properties of system control are controllability and observability. Controllability refers to the input control's capability to move the system's state from an initial value to the origin within a finite time. Conversely, observability gauges the extent to which the system's states can be deduced from the output information. The following theorem serves as a tool to ascertain these attributes. [21].

Theorem 3.1 *The LTI system, equations 3.2 and 3.3, is controllable if and only if the controllability matrix has full row rank.*

$$Q_c = [B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B] \in \mathbb{R}^{n \times n.m} \tag{3.4}$$

The LTI system is observable if and only if the observability matrix has full column rank.

$$Q_o = [C^T \;\; A^T C^T \;\; (A^T)^2 C^T \;\; \dots \;\; (A^T)^{n-1} C^T] \in \mathbb{R}^{np \times n} \qquad (3.5)$$

And a matrix $A \in \mathbb{R}^{n \times n}$ is $Hurwitz$ or asymptotically stable if all its eigenvalues have strictly negative real part [21].

In view of the above, the stability theory is developed in the Lyapunov domain. The system is shown below

$$\dot{x} = f(x),$$

Where $x = 0$ represents the equilibrium, that is, $f(0) = 0$.

Definition 3.1 Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar function. V is a Lyapunov function candidate if it is a locally positive-definite function, i.e.,

$$V(0) = 0, \text{ and } V(x) > 0 \;\forall x \in \mathfrak{B} \backslash \{0\},$$

With $\mathfrak{B}$ being some neighborhood around $x = 0$.

Theorem 3.2 *The equilibrium $x = 0$ of the system is*

- *Stable if there is a locally positive definite Lyapunov function candidate $V(x)$ such that $\dot{V}(x) \leq 0$ for all $x \neq 0$, or*
- *Asymptotically stable if there is a locally positive definite Lyapunov function candidate $V(x)$ such that $\dot{V}(x) < 0$ for all $x \neq 0$.*

3.4 Optimal control of the Linear Quadratic Regulator

Optimal control theory has allowed the analysis and solution of a variety of control problems. In this sense, the study of the optimal control or forward optimal control problems is important for the solution of the inverse optimal control problems mentioned in the preceding chapters. The general formulation of a continuous time-optimal control problem is shown below.

In general, optimal control seeks to determine a control law that satisfies the physical constraints of a dynamic system and also minimizes the cost function or performance index that regulates the behavior of the system. In that sense, the Linear Quadratic Regulator (LQR) approach seeks to regulate such behavior by penalizing the state deviations on the control effort applied for that purpose or vice versa.

Therefore, the task in the Linear Quadratic Regulator (LQR) problem is to find a Full State Feedback (FSFB) control law for a continuous-time Linear Time Invariant (LTI) [19], whose representation in the state space of the dynamical system is determined by equations 3.2 and 3.3.

In such a way that a cost function with the following structure is minimized

$$J = \int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{3.6}$$

$$J = \int_0^{\infty} x^T(t) Q x(t) + u^T(t) R u(t) dt \tag{3.7}$$

Where the meaning of the variables is:

$x(t)$:	$n \times 1$ state vector
$u(t)$:	$m \times 1$ control vector
$y(t)$:	$p \times 1$ output vector
A:	$n \times n$ state matrix

B: $n \times m$ control matrix

C: $p \times n$ output matrix

Q: $n \times n$ symmetric positive semidefinite matrix ($Q \geq 0$). Matrix that penalizes the state deviation

R: $m \times m$ symmetric positive definite matrix ($R > 0$). Matrix that penalizes the control effort.

S: $m \times m$ symmetric positive semidefinite matrix ($S \geq 0$). Matrix that weights the terminal cost.

Assuming that

$x^T Qx \geq 0$ is positive semidefinite $\forall x$.

$u^T Ru > 0$ is positive definite $\forall u$.

$S = 0$

(A, B) is controllable

and (A, M) is observable with $Q = M^T M \geq 0$

The optimization problem is posed for the Linear Quadratic Regulator (LQR)

$$\underset{u \in \mathbb{R}^m}{\text{minimize}} \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt \tag{3.8}$$

$$\begin{aligned} subject\ to\quad & \dot{x}(t) = Ax(t) + Bu(t), \\ & x(t_0) = x_0 \end{aligned}$$

Whose solution involves finding some control law or policy, u, such that it minimizes the cost function, J, according to the following scenarios

If Q "is bigger" than R: in this case a fast regulation is obtained, i.e., the state returns to the origin ($x \to 0$) quickly, since u is large. Aggressive controller.

If R "is bigger" than Q: in this case, a slow regulation is obtained, since u is small. Conservative controller.

Therefore, Q and R act as knobs that allow adjusting the behavior of the controller depending on how important the state is on the control and vice versa.

In this sense, the optimal solution is the full state feedback controller (FSFB)

$$u(t) = -Kx(t) \tag{3.9}$$

where the gain matrix is

$$K = R^{-1}B^{T}P \tag{3.10}$$

Where P is the nxn symmetric unique positive semidefinite solution to the Algebraic Riccati Equation (ARE)

$$A^{T}P + PA - PBR^{-1}B^{T}P + Q = 0 \tag{3.11}$$

The procedural scheme for the solution of the linear quadratic controller is as follows:

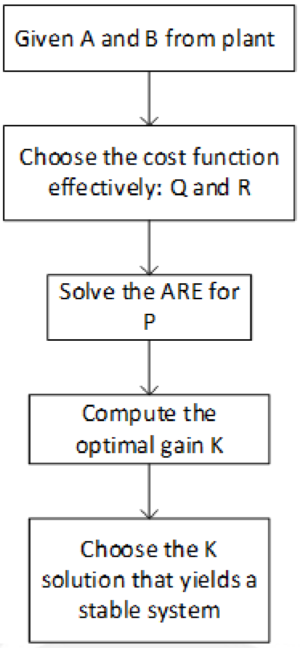


Figure 7 - LQR problem solution scheme

An example for the solution of the optimal problem by determining the feedback gain K is shown below.

Example. Mass-damper system

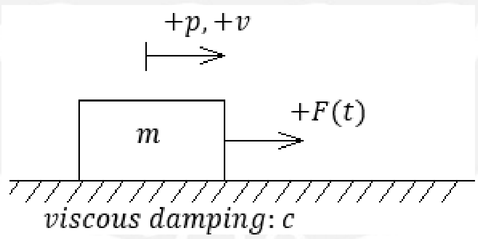


Figure 8 - Diagram mass-damper system

$$\sum \bar{F} = 0$$

$$m\ddot{p}(t) + c\dot{p}(t) = F(t)$$

$$x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}, u(t) = [F(t)]$$

State space representation of the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -{}^{c}/_{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ {}^{1}/_{m}\end{bmatrix}u(t)$$

Step 1: consider m=1 and c=0.2

$$A = \begin{bmatrix}0 & 1\\ 0 & -{}^{1}/_{5}\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}$$

Step 2: choose Q and R

$$Q = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, R = [0.01]$$

Step 3: solve the ARE for symmetric matrix P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Solutions for P:

$$P_1 = \begin{bmatrix}0.89465 & -0.1\\ -0.1 & -0.091465\end{bmatrix} \quad P_2 = \begin{bmatrix}-0.89465 & -0.1\\ -0.1 & 0.087465\end{bmatrix}$$

$$P_3 = \begin{bmatrix}-1.09563 & 0.1\\ 0.1 & -0.111563\end{bmatrix} \quad P_4 = \begin{bmatrix}1.09563 & 0.1\\ 0.1 & 0.107563\end{bmatrix}$$

Step 4: compute K

$$K_1 = [-10 \quad -9.14651] \qquad K_2 = [-10 \quad 8.74651]$$

$$K_3 = [10 \quad -11.1563] \qquad K_4 = [10 \quad 10.7563]$$

Only K that have stable closed-loop eigenvalues will allow the system to return to the origin and therefore have a minimum cost for the cost function.

Step 5: in order to find a solution that gives rise to a stable system it is necessary to analyze the eigenvalues of $A_{CL} = A - BK$.

$$\lambda_1 = Eigenvalues[A - BK_1] = \{9.95139, -1.00488\}$$
$$\lambda_2 = Eigenvalues[A - BK_2] = \{-9.95139, 1.00488\}$$
$$\lambda_3 = Eigenvalues[A - BK_3] = \{9.95139, -1.00488\}$$
$$\lambda_4 = Eigenvalues[A - BK_4] = \{-9.95139, -1.00488\}$$

Finally, the set of values that lie on the negative part of the real axis determine the solution (λ_4)

$$K = K_4 = [10 \quad 10.7563]$$

3.5 Semidefinite Programming

Semidefinite programming (SDP) finds in convex optimization one of its most important applications. In this sense, the versatility in the type of constraints that can be formulated in semidefinite programming is varied, such as linear inequalities, convex quadratic inequalities, lower bounds in matrix norms, lower bounds for symmetric positive semidefinite matrices, etc. Therefore, by means of semidefinite programming (SDP) linear programming problems can be modeled as well as the optimization of convex quadratic structures subject to constraints of the convex quadratic inequalities type [23].

The following definitions, according to [19], are based on $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, therefore

a) Positive semidefinite matrix
   Matrix A, with dimensions $n \times n$, is positive semidefinite, i.e., $A \geq 0$ if for all $x \neq 0$ the following is true

$$x^T Ax \geq 0 \; for\; any\; x \in \mathbb{R}^n$$

b) Positive definite matrix
   Matrix A, with dimensions $n \times n$, is positive definite, i.e., $A > 0$ if for all $x \neq 0$ the following is true

$$x^T Ax > 0 \; for\; any\; x \in \mathbb{R}^n, x \neq 0$$

Therefore, the set of symmetric and positive semidefinite matrices is denoted as follows

$$S_+^n = \{A \in \mathbb{R}^{n\times n} | A = A^T; A \geq 0\}$$

Consequently, having symmetric matrices X and Y. The matrix X is said to be symmetric and positive semidefinite if $X \geq 0$, likewise $X - Y \geq 0$ if $X \geq Y$. On the other hand, X is said to be symmetric and positive definite if $X > 0$.

With the above it is good to mention that semidefinite programming is a part of convex optimization in which a linear function is minimized such that certain constraints written as a composition of affine, symmetric, positive definite matrices are satisfied. In addition, semidefinite programming provides a standard way to derive algorithms and study the properties of various convex problems, including but not limited to the linear and quadratic cases [19]. A positive semidefinite formulation of an optimization problem is shown below [23][20].

$$
\begin{aligned}
&\textit{minimize} && c^T x \\
&\textit{subject to} && A_i x = b_i, i = 1, \dots, m \\
& && x \geq 0
\end{aligned}
$$

Where: $c^T x = c_{ij}^T x_{ij}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$

The following is an example for the formulation of a semidefinite programming [23]. For n=3 and m=2, the following matrices are defined:

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix}, b_1 = 11 \textit{ and } b_2 = 19$$

Consequently, the symmetric variable matrix x with dimensions 3×3 is denoted as:

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

hence, the objective function to be minimized will be:

$$
\begin{aligned}
c^T x &= x_{11} + 2x_{12} + 3x_{13} + 2x_{21} + 9x_{22} + 0x_{23} + 3x_{31} + 0x_{32} + 7x_{33} \\
&= x_{11} + 4x_{12} + 6x_{13} + 9x_{22} + 0x_{23} + 7x_{33}
\end{aligned}
$$

Finally, the semidefinite programming (SDP) can be formulated as follows:

$$
\begin{aligned}
&\textit{minimize} && x_{11} + 4x_{12} + 6x_{13} + 9x_{22} + 0x_{23} + 7x_{33} \\
&\textit{subject to} && x_{11} + 0x_{12} + 2x_{13} + 3x_{22} + 14x_{23} + 5x_{33} = 11
\end{aligned}
$$

$$0x_{11} + 4x_{12} + 16x_{13} + 6x_{22} + 0x_{23} + 4x_{33} = 19$$

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \geq 0$$

Therefore, the semidefinite programming fulfills the restriction that the variable X must belong to the cone of positive semidefinite matrices.

3.6 Linear Matrix Inequalities

Linear matrix inequalities (LMIs) are a mathematical tool to reduce a wide variety of control theory problems to a few convex or quasiconvex optimization problems. The advantage lies in the fact that these resulting optimization problems can later be solved numerically in an efficient way by means of software that implement interior point methods. It is worth mentioning that the use of linear matrix inequalities (LMIs) represents the solution to the original system itself but differs from other conventional methods such as analytical or frequency domain solution. Likewise, from the formulation of LMIs as convex optimization problems, a reliable solution is obtained, which represents an important advantage in cases where an analytical solution cannot be found [20].

A linear matrix inequality (LMI) is a constraint written in a convex form, i.e., according to convex programming, as follows

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0$$

where $x \in \mathbb{R}^m$ is the variable and the matrices $F_0, \ldots, F_m$ are symmetric, i.e., $F_i = F_i^T$ with dimensions nxn. The symbol of inequality indicates that $F(x)$

is positive definite, i.e., $u^T F(x) u > 0$. However, non-strict linear matrix inequalities (LMIs) are also found whose form is as follows

$$F(x) \geq 0$$

As can be seen the IML is in essence a convex constraint for x, i.e.

$$the\ set\ \{x|F(x) > 0\}\ is\ convex$$

Also, the general form of the linear matric inequality (LMI) effectively represents a wide variety of convex constraints for x, such as linear inequalities, quadratic inequalities and constraints present in control theory such as convex quadratic matrix inequalities and Lyapunov inequalities [20]. On the other hand, multiple LMIs can be represented as a single LMI, this is achieved with the Schur complement.

The Schur complement is an important tool for reformulating nonlinearities in linear matrix inequalities (LMI). Because in convex optimization problems the formulated inequalities are linear. In Schur complement the nonlinear (convex) inequalities can be converted to an individual LMI by this method. That is, the following LMI formulation [20]

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$$

Where matrices $Q(x)$ and $R(x)$ are symmetric according to $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$ respectively, and $S(x)$ depend affinely on x is equivalent to

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^T(x) > 0$$

That is, the set of nonlinear inequalities shown above can be represented as an individual linear matrix inequality (LMI) whose components are linear.

So far, in the formulation of the linear matrix inequality (LMI) shown above, the variable is a scalar value of the form $x \in \mathbb{R}^m$. However, it is very often in practice that the variables encountered are of the matrix type. For example, in the Lyapunov inequality $A^T P + PA < 0$ where $A \in \mathbb{R}^{n \times n}$ is known, the variable is symmetric matrix, $P = P^T$.

Taking into account the above and considering the Schur complement, a formulation for linear matrix inequality can be established in the following example [20]. Therefore, considering quadratic matrix inequality

$$A^T P + PA + PBR^{-1}B^T P + Q < 0 \tag{3.12}$$

where the matrices $A, B, Q = Q^T, R = R^T > 0$ are known and of appropriate dimensions, and also the symmetric matrix $P = P^T$ is the variable. It can be seen that it is a quadratic matrix inequality in the variable P, which can be reformulated as a linear matrix inequality in the following way

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0 \tag{3.13}$$

With this reformulation it can be seen that the quadratic matrix inequality is convex in P.

On the other hand, according to the above, below are defined the Linear Matrix Inequality (LMI) regions that are presented by D-stability which is a generalization of Hurwitz stability and Schur stability [25]. The D-stability of a matrix is a region that is established by the linear matrix inequalities (LMI) conditions which are known as linear matrix inequalities regions.

Definition 3.2 Let $\mathbb{D}$ be a region on the complex plane. If there exist matrices $L \in \mathbb{S}^m$, and $M \in \mathbb{R}^{m\times m}$ such that

$$\mathbb{D} = \{s \mid s \in \mathbb{C}, L + sM + \bar{s}M^T < 0\} \tag{3.14}$$

Then $\mathbb{D}$ is called an LMI region and is usually denoted by $\mathbb{D}_{(L,M)}$

$$F_{\mathbb{D}}(s) = L + sM + \bar{s}M^T \tag{3.15}$$

Is called the characteristic function of the LMI region $\mathbb{D}_{(L,M)}$. Where it is an intersection of the following regions [25].

Strip region:

$$\begin{gathered} P > 0 \\ ATP + PA + 2\alpha P < 0 \\ ATP + PA + 2\beta P > 0 \end{gathered} \tag{3.16}$$

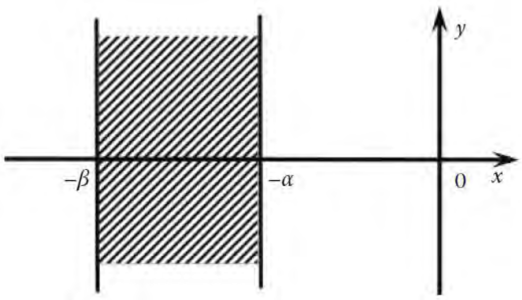


Figure 9 - Strip region

Disk region:

$$\begin{bmatrix} -rP & qP + AP \\ qP + PA^T & -rP \end{bmatrix} < 0 \tag{3.17}$$

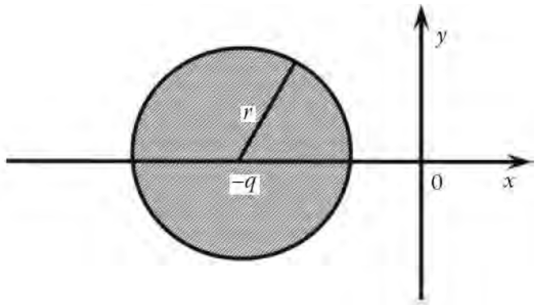


Figure 10 - Disk region

Sector:

$$\begin{bmatrix} (s+\bar{s})sin\theta & (s+\bar{s})cos\theta \\ (-s+\bar{s})cos\theta & (s+\bar{s})sin\theta \end{bmatrix} < 0 \tag{3.18}$$

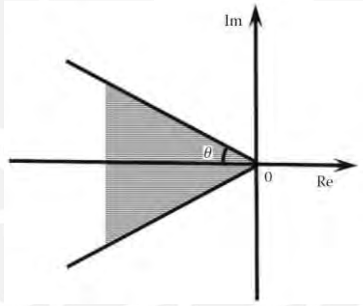


Figure 11 - Sector D_θ

3.7 Inverse Linear Quadratic Regulator (LQR) problem

The inverse optimal control problem for the linear quadratic regulator (LQR) starts with a given dynamic system and a feedback gain matrix K defined by equations 3.2, 3.3 and 3.10.

The task is to estimate the parameters, i.e., the matrices Q and R of the cost function J. This requires determining whether the matrices $Q \geq 0$ and R> 0 exist, such that (Q, A) is detectable and whether $u = Kx$ minimizes the corresponding linear quadratic regulator (LQR) cost function J according to the formula defined by equation 3.7 [20].

In that sense, the search of values for Q and R as well as their positivity conditions $Q \geq 0$ and $R > 0$ can be formulated as linear matrix inequalities (LMIs) constraints [22]. Where the variables $P \geq 0$ and $P_1 > 0$ satisfy the following

$$(A + BK)^T P + P(A + BK) + K^T RK + Q = 0 \qquad (3.19)$$

$$B^T P + RK = 0 \qquad (3.20)$$

Likewise, the condition of detectability for (Q, A) can be formulated as

$$A^T P_1 + P_1 A < Q \qquad (3.21)$$

Thus, through the formulation of linear matrix inequalities the values of Q and R are determined. The solution scheme can be presented in the following way

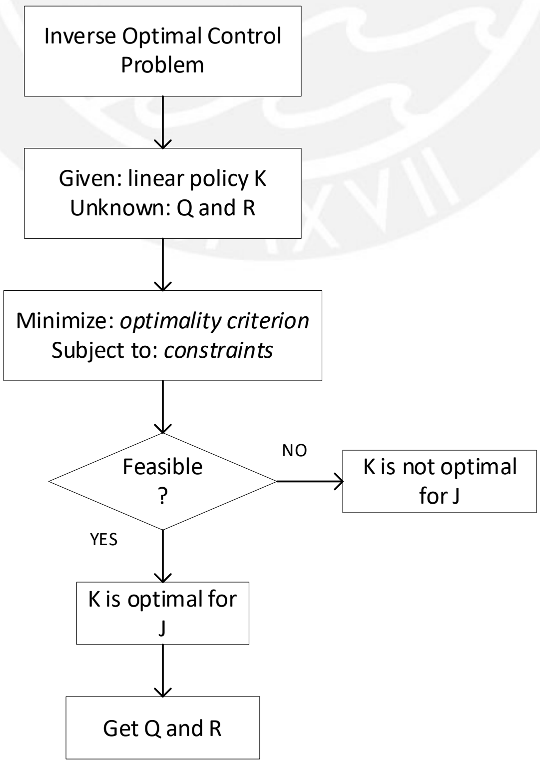


Figure 12 - Inverse optimal control problem solution scheme

According to the above, a semidefinite convex feasibility problem, defined by equation 3.1, can be formulated in whose solution the values of Q and R are obtained with the help of the following constraints [22].

$$\begin{gathered}(A+BK)^T P + P(A+BK) + K^T R K + Q = 0, \\ B^T P + RK = 0, \\ A^T P_1 + P_1 A < Q, \\ Q \geq 0, R > 0, P \geq 0, P_1 > 0\end{gathered} \tag{3.22}$$

The semidefinite programming formulation shown contains constraints of linear matrix inequalities (LMIs) and Q, R, P and P_1 variables where if the problem is feasible then K is optimal for the Q and R matrix components of the cost function.

Chapter 4
Solution approach and Results

In this chapter, due to the fact that inverse control problems suffer from an ill-posedness, the well-posedness of an inverse optimal control problem is analyzed, breaking it down into such important concepts as existence, injectivity and stability. Likewise, a dynamical system of study is proposed in which the concept of well-posedness is applied for its subsequent solution. Finally, the results obtained after the analysis of the well-posedness and the application of the solution algorithms for this inverse optimal control are presented.

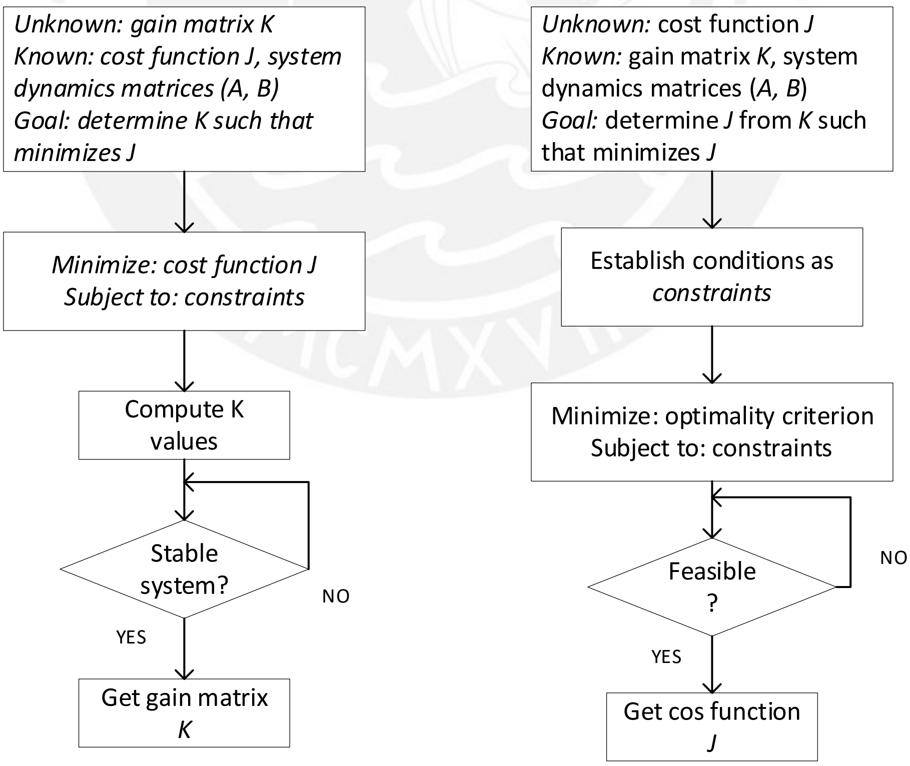
4.1 Inverse optimal control approach

In forward optimal control, the aim is to determine a solution or control law from the optimization of a cost function, according to diagram 13a. On the other hand, in inverse optimal control, starting from an existing solution that is assumed to be optimal, the aim is to estimate the parameters of a cost function in such a way that this solution minimizes it.

The procedure is performed by first analyzing the necessary and sufficient conditions that ensure the well-posedness to the inverse optimal control problem with respect to existence, uniqueness and stability, in order to subsequently estimate the parameters that compose the cost function for the linear quadratic controller (LQR) approach.

In that sense, since these types of problems are generally solved by numerical methods, such conditions for a well-posed inverse problem are represented as Linear Matrix Inequalities (LMI) constraints, which is a mathematical tool to reduce a variety of control problems to a few convex optimization problems. The advantage lies in the fact that such problems can be solved numerically in an efficient way through algorithms involving interior point methods, which represents an advantage in cases where an analytical solution cannot be found.

On the other hand, given the convex nature of the linear quadratic regulator, such constraints are redefined as convex forms using the theory of Semidefinite Programming, in which a function is minimized according to certain constraints that are formulated as a composition of affine, symmetric, positive definite matrices. Finally, the algorithm verifies if the set of constraints for the well-posedness admits a feasible solution which results in the solution of the inverse problem, otherwise the feasibility set is determined to be empty. The summary of the process can be seen in diagram 13b.



a) b)

Figure 13 - Optimization problem formulation for an OCP and the IOCP

4.2 Proposal of a dynamic system for study

This section presents the formulation of the problem to which the theoretical concepts reviewed in Chapter 3 are applied. The example considered is a mass-spring-damper system with four degrees of freedom (DOF) represented in a rolling cart system. It is considered that there is no friction between the wheels and the floor [29]. The representation of the state-space model of the mass-spring-damper system is derived below, as a first step the dynamic model of the following system is formulated.

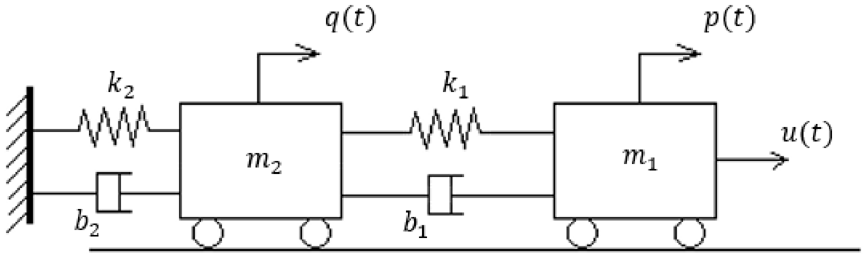


Figure 14 - Diagram of the mass-spring-damper system

Where:

q(t): displacement of mass m_2.

p(t): displacement of mass m_1, and is selected as output p(t).

u(t): applied force, input.

The procedure to obtain the state-space model of the system is as follows

a) Equations of motion

For mass m_1:

$$\sum F = m_1\ddot{p}$$

$$u(t) - k_1(p-q) - b_1(\dot{p}-\dot{q}) = m_1\ddot{p} \tag{4.1}$$

$$m_1\ddot{p} + k_1 p + b_1\dot{p} = u(t) + k_1 q + b_1\dot{q}$$

For mass m_2:

$$\sum F = m_2\ddot{q}$$

$$k_1(p-q) + b_1(\dot{p}-\dot{q}) - k_2 q - b_2\dot{q} = m_2\ddot{q} \tag{4.2}$$

$$m_2\ddot{q} + \dot{q}(b_1+b_2) + q(k_1+k_2) = k_1 p + b_1\dot{p}$$

b) Definition of state variables

$$x_1 = p(t)$$

$$x_2 = q(t)$$

$$x_3 = \dot{p}(t) = \dot{x}_1$$

$$x_4 = \dot{q}(t) = \dot{x}_2$$

Represented in vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

c) Write the differential equations for each state variable

Equation 1

$$m_1\ddot{p} + b_1\dot{p} + k_1 p = u(t) + k_1 q + b_1\dot{q}$$

$$m_1\dot{x}_3 + b_1 x_3 + k_1 x_1 = u(t) + k_1 x_2 + b_1 x_4 \tag{4.3}$$

$$\dot{x}_3 = \frac{1}{m_1}u(t) + \frac{k_1}{m_1}x_2 + \frac{b_1}{m_1}x_4 - \frac{b_1}{m_1}x_3 - \frac{k_1}{m_1}x_1$$

Equation 2

$$
\begin{gathered}
m_2\ddot{q} + (k_1 + k_2)q + (b_1 + b_2)\dot{q} = k_1 p + b_1 \dot{p} \\
m_2\dot{x}_4 + (k_1 + k_2)x_2 + (b_1 + b_2)x_4 = k_1 x_1 + b_1 x_3 \\
\dot{x}_4 = \frac{k_1}{m_2}x_1 + \frac{b_1}{m_2}x_3 - \frac{(k_1 + k_2)}{m_2}x_2 - \frac{(b_1 + b_2)}{m_2}x_4
\end{gathered}
\tag{4.4}
$$

Now proceed to write it in the matrix form:

$$\dot{x} = Ax + Bu \tag{4.5}$$

Where:

$u(t)$: input control

$x(t)$: state

$\dot{x}(t)$: derivative of the state

$A,\ B$: matrices to determine

4) Write in Matrix form

$$
\begin{gathered}
\dot{x}_1 = x_3 \\
\dot{x}_2 = x_4 \\
\dot{x}_3 = -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 - \frac{b_1}{m_1}x_3 + \frac{b_1}{m_1}x_4 + \frac{1}{m_1}u(t) \\
\dot{x}_4 = \frac{k_1}{m_2}x_1 - \frac{(k_1 + k_2)}{m_2}x_2 + \frac{b_1}{m_2}x_3 - \frac{(b_1 + b_2)}{m_2}x_4
\end{gathered}
\tag{4.6}
$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & \frac{b_1}{m_2} & -\frac{(b_1+b_2)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t) \quad (4.7)$$

It is now necessary to derive an expression for the system output. In this case, the output is the displacement of mass 1, i.e., $p(t) = x_1 = y$.

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \quad (4.8)$$

4.3 Solution

The inverse optimal control problem analyzed as an infinite horizon Linear Quadratic Regulator (LQR) is solved in a scheme of two level, as shown in figure 15, where in the outer loop the inverse optimal control problem (IOCP) is used for the recovery of parameters of the cost function and is posed as a Semidefinite Programming (SDP) with Linear Matrix Inequality (LMI) constraints that express characteristics of well-posedness. On the other hand, in the inner loop, a forward optimal control problem is solved with the objective of updating the parameter (K) and evaluating the optimality of the recovered cost function.

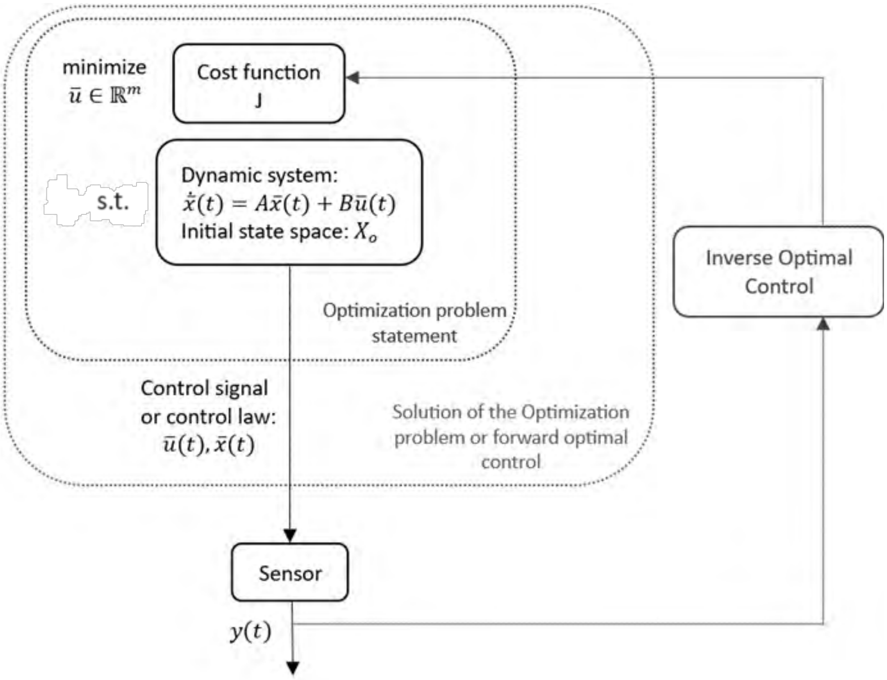


Figure 15 - Inverse optimal control approach

In order to address the well-posedness to the inverse control problem, it is assumed that the registered trajectories are optimal [2], so that the optimality criterion would be satisfied and it would remain to address the problem of uniqueness and stability. In this sense, the solution is approached as follows.

a) Find the necessary and sufficient conditions on the matrices K, A and B such that the control law $u(t) = -Kx(t)$ minimizes the cost function for given values of Q and R.

   In this case, K is optimal for certain values of Q and R if and only if $A + BK$ is asymptotically stable as well as it must fulfill the conditions established in section 3.1.

b) Determine all parameters Q and R in such a way that they produce the same value of K.

In this case, the parameters can be obtained by establishing the following conditions

$$P = P^T > 0, Q = Q^T, R = R^T > 0 \tag{4.9}$$

and by solving the matrix equation

$$B^T P = -RK \tag{4.10}$$

where the variable Q is obtained from the Algebraic Riccati Equation (ARE)

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{4.11}$$

It is worth mentioning that the solution obtained varies if Q is required to be positive semidefinite, i.e., $Q = Q^T \geq 0$ [8], because it is a closed constraint that is used in semidefinite programming [22]. In addition, $P > 0$ describes the infinite-horizon cost, $Q \geq 0$ describes the state-penalty matrix, and R≥ 0 describes the input-penalty matrix [9][18].

Next, the objective is to estimate the parameters Q and R of the cost function from the observed control law $\mathrm{u(t) = Kx(t)}$. Specifically, from the convex formulations optimize the values of Q and R from a known value of K, which is known as convex formulation of the inverse optimal controller problem [17].

As mentioned above, in forward optimal control, the control gain K is related to the cost function by means of the Algebraic Riccati Equation (ARE), which is used as the basis for the solution of inverse optimal control [9]. Therefore, the following is derived.

$$K = -(B^T P B + R)^{-1} B^T P A$$
$$-RK = B^T P(A + BK) \tag{4.12}$$

Therefore, the semidefinite programming (SDP) for the estimation of parameters of the inverse optimal control problem cost function from the gain matrix K, is proposed by imposing symmetry and positive definiteness constraints for the matrices P, Q and R. Thus, given K, the objective is to estimate the values for P, Q and R in such a way that the imposed restrictions are satisfied to ensure the well-posedness to the problem. In that sense, the solution diagram involving the constraints that impose a well-posedness to the inverse optimal control problem is shown in Figure 16.

Regarding the implementation of the solution algorithm, several investigations have used different numerical methods for the estimation of the cost function based, for example, on numerical optimal control [2]. Likewise, for the estimation of the cost functions of inverse optimal control problems for linear systems, convex formulations are used. In this sense, in the present work, the Python libraries CVXPY and SCIPY, whose problem formulations are written in SDP and with LMI constraints, are used.

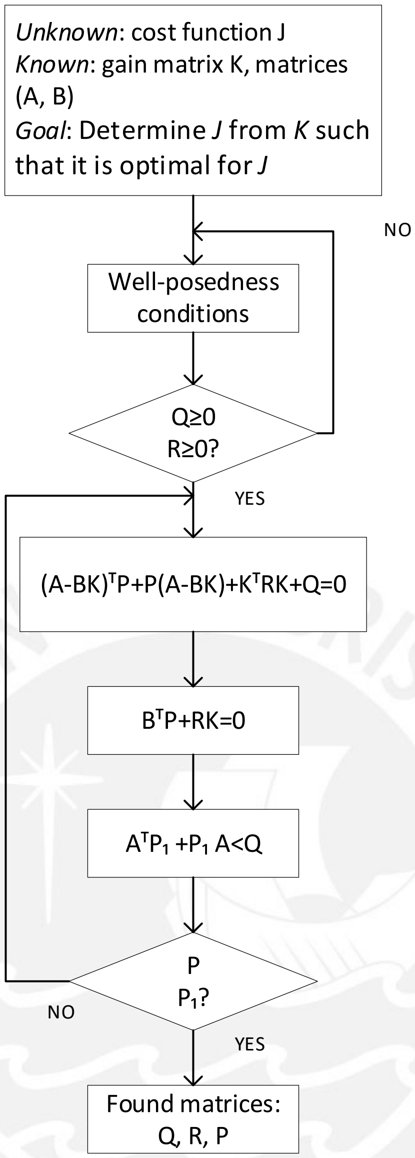


Figure 16 - Proposed solution scheme for IOCP

4.4 Results

This section shows the results regarding the application of the well-posedness constraints for the solution of the inverse problem, then presents the results after including constraints that restrict the poles to a certain region on the left side of the half-plane to improve the transient response and finally presents the results for the solution of the inverse optimal control problem by penalizing the deviation of observed states and proposing an alternative to avoid dependence on the use of the K-gain matrix.

4.4.1 Inverse optimal control with well-posedness constraints

Taking into account the mass-spring-damper system described in section 4.2, the forward optimal control problem, according to equation 4.13, will be formulated in order to be compared with the response obtained after solving the inverse optimal control problem.

$$\begin{aligned} &\underset{u\in\mathbb{R}^m}{\text{minimize}} \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt \\ &subject\ to \quad \dot{x}(t) = Ax(t) + Bu(t) \\ &\qquad\qquad\qquad x(t_0) = x_0 \end{aligned} \tag{4.13}$$

According to the formulation of the system described in section 4.2, the dimension of the state vector $x(t)$ is 4×1 ($n \times 1$) and the dimension of the control vector $u(t)$ is 1×1 ($m \times 1$), whereby, according to the matrix multiplication rules, the dimensions of the weight parameters of the cost function are as follows.

Q: $n \times n$, symmetric positive semidefinite matrix with dimensions 4×1.

R: $m \times m$, symmetric positive definite matrix with dimensions 1×1.

Three scenarios are presented below, each using a different set of values for the matrix penalizing state deviation Q and the matrix penalizing control effort R.

4.4.1.1 Scenario 1

In this case, the system state is prioritized to return to zero quickly, i.e., a fast regulation is expected by penalizing matrix Q. For this purpose, matrix Q is assigned a higher weight than matrix R. Therefore, the components considered for the cost function, i.e., the matrix penalizing the deviation of states Q, and the matrix penalizing the control effort R are as follows.

State weighting matrix:

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Input weighting matrix

$$R = [0.1]$$

The solution to the forward optimal control problem for the linear quadratic regulator (LQR) result in the following optimal controller gain matrix K.

$$K = [0.45613562, 0.2254168, 2.23401424, 0.30041099]$$

Also, the eigenvalues confirm that the system is stable and its oscillatory nature.

Eigen values of the close-loop A matrix:

$$-3.94938826 + 13.36737958j$$
$$-3.94938826 - 13.36737958j$$
$$-0.45486352 + 3.19717636j$$
$$-0.45486352 - 3.19717636j$$

In that sense, the response of the mass-spring-damper system described above after solving the forward optimal control problem is shown below.

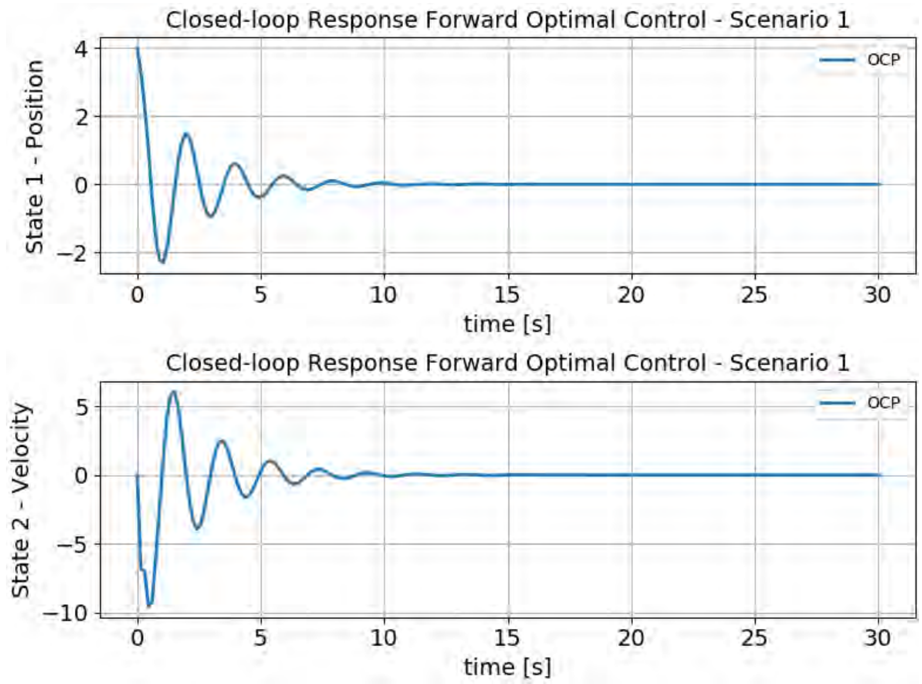


Figure 17 - System response for scenario 1

4.4.1.2 Scenario 2

In this case, priority is given to the least possible effort for the controller so that the state returns to zero, i.e., a slow regulation will be obtained by penalizing the R matrix. Therefore, the components considered for the cost function, i.e., the matrix penalizing the deviation of states Q, and the matrix penalizing the control effort R are as follows.

State weighting matrix:

$$Q = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

Input weighting matrix

$$R = [20]$$

The solution to the forward optimal control problem for the linear quadratic regulator (LQR) result in the following optimal controller gain matrix K.

$$K = [0.00030937, -0.00063223, 0.00102951, 0.00012672]$$

Also, the eigenvalues confirm that the system is stable and its oscillatory nature.

Eigen values of the close-loop A matrix:

$$-3.94814015 + 13.37281284j$$
$$-3.94814015 - 13.37281284j$$
$$-0.17698854 + 3.2024485j$$
$$-0.17698854 - 3.2024485j$$

In this case, the response of the mass-spring-damper system after solving the forward optimal control problem is shown below.

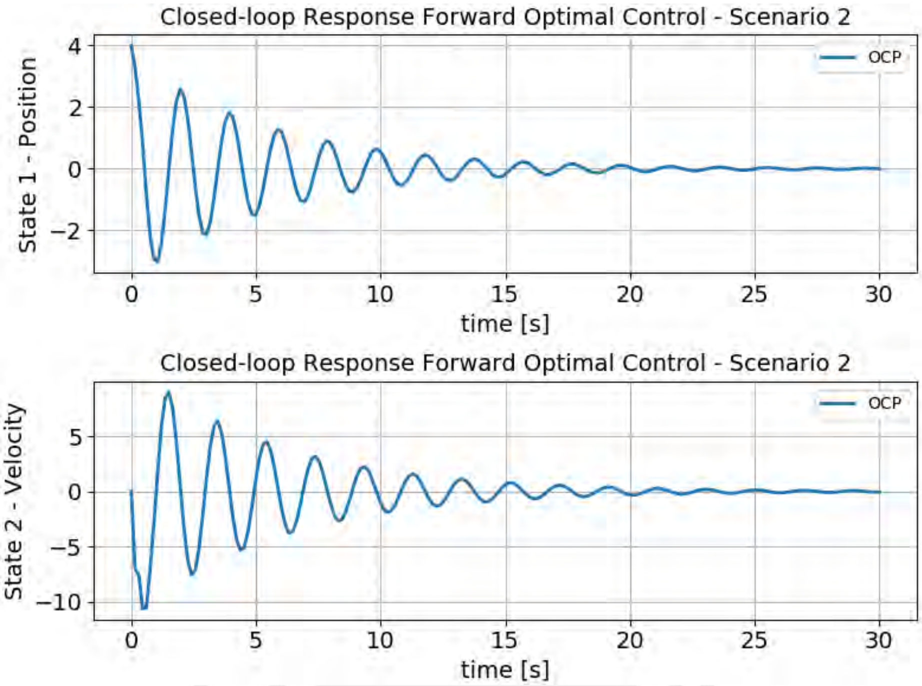


Figure 18 - System response for scenario 2

4.4.1.3 Scenario 3

In this case it is prioritized that the system velocity returns to zero quickly. For this purpose, the velocity dimension of the Q matrix is penalized while maintaining a relatively high weight for the R matrix. Therefore, the components considered for the cost function, i.e., the matrix penalizing the deviation of states Q, and the matrix penalizing the control effort R are as follows.

State weighting matrix:

$$Q = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Input weighting matrix

$$R = [30]$$

The solution to the forward optimal control problem for the linear quadratic regulator (LQR) result in the following optimal controller gain matrix K.

$$K = [-0.38338825, 0.4335758, 0.57580537, 0.08289382]$$

Also, the eigenvalues confirm that the system is stable and its oscillatory nature.

Eigen values of the close-loop A matrix:

$$-3.94776849 + 13.36922043j$$
$$-3.94776849 - 13.36922043j$$
$$-0.24920718 + 3.19329694j$$
$$-0.24920718 - 3.19329694j$$

Finally, the response of the mass-spring-damper system after solving the forward optimal control problem is shown below.

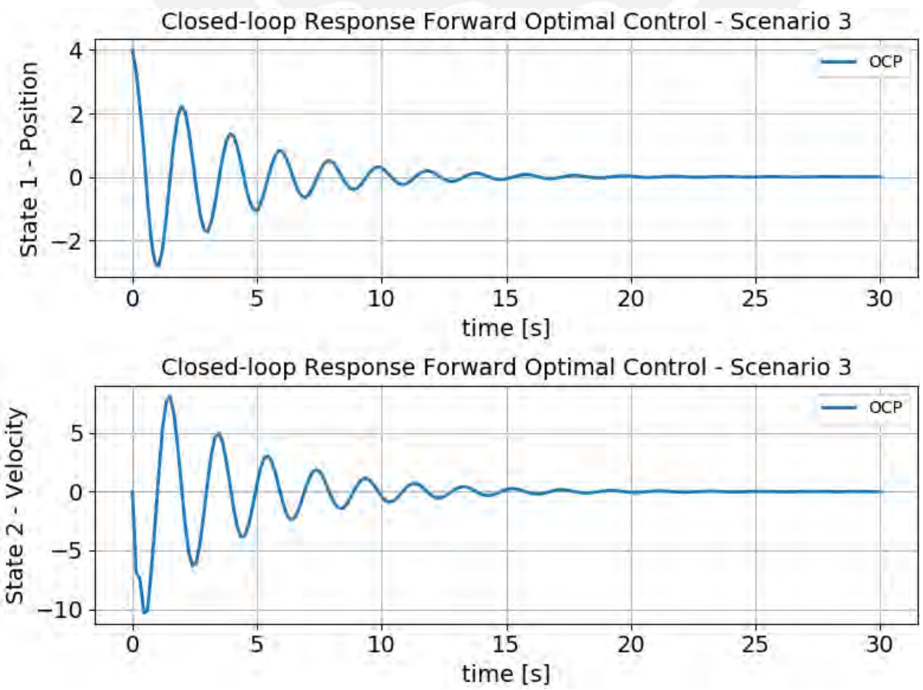


Figure 19 - System response for scenario 3

Now, to solve the inverse optimal control problem of the linear quadratic regulator (LQR), the optimal controller gain matrix K determined earlier is used as input.

4.4.1.4 Inverse optimal control for scenario 1

The restrictions for the well-posedness of the inverse optimal control problem are stated, such as the positive semidefinite conditions on the variables (Q, R) components of the cost function, as well as for the variable P which is the solution of the algebraic Riccati equation (ARE) and the variable P_1 which verifies the controllability of the system. Therefore, the linear matrix inequalities (LMI) constraints written in semidefinite programming (SDP) are shown below.

$$
\begin{gathered}
Q = Q^T, Q \geq 0 \\
R = R^T, R \geq 0 \\
P = P^T, P \geq 0 \\
P_1 = P_1^T, P_1 \geq 0 \\
(A - BK)^T P + P(A - BK) + K^T R\,K + Q = 0 \\
B^T P - RK = 0 \\
A^T P_1 + P_1 A - Q \leq 0
\end{gathered}
\tag{4.14}
$$

For this approach and for the development of the algorithm, it is necessary to verify whether the set of constraints admits a feasible solution, defined by equations 3.1, which satisfies the existence criterion. Indeed, it is required that a feasible solution is returned or else the feasible set is determined to be empty.

Therefore, after solving the inverse optimization problem, the values of the Q and R parameters of the cost function are estimated.

State weighting matrix:

$$Q = \begin{bmatrix} 10.2595604 & 0.2349327 & 0.21997224 & -0.11097164 \\ 0.2349327 & 10.2836208 & 0.37746966 & -0.13114467 \\ 0.21997224 & 0.37746966 & 10.4298761 & -0.15699507 \\ -0.11097164 & -0.13114467 & -0.15699507 & 10.1328847 \end{bmatrix}$$

Input weighting matrix

$$R = [0.0368504]$$

Likewise, the response of the mass-spring-damper system to a unit step input with parameters Q and R obtained from the inverse optimal control problem and with an initial condition of $x_0 = (4, 0, 0, 0)$ is as follows

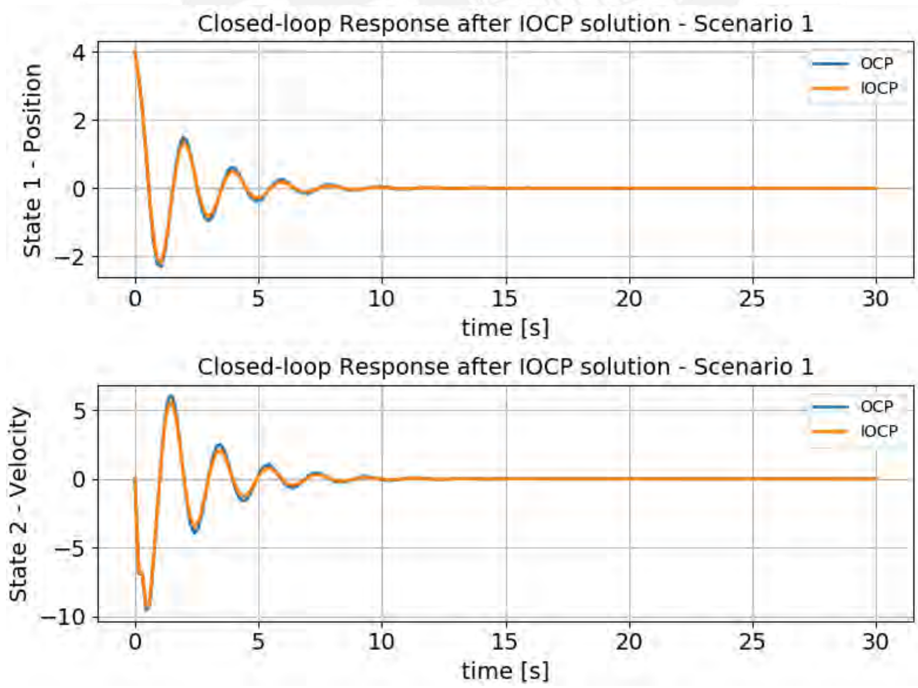


Figure 20 - System response after IOC for scenario 1

As can be seen from the above, the response obtained is similar to the response obtained in the forward optimal control of scenario 1. The configuration of the weight parameters allows a stable response and a settling time of 10 seconds.

4.4.1.5 Inverse optimal control for scenario 2

In this case after solving the inverse optimization problem, the values of the Q and R parameters of the cost function are estimated.

State weighting matrix:

$$Q = \begin{bmatrix} 0.18491162 & -1.52335861e-05 & -4.53560052e-07 & -2.22856261e-08 \\ -1.52335861e-05 & 0.12734808 & 8.14924254e-07 & 3.87914547e-08 \\ -4.53560052e-07 & 8.14924254e-07 & 0.14583880 & -1.37713395e-08 \\ -2.22856261e-08 & 3.87914547e-08 & -1.37713395e-08 & 0.13959121 \end{bmatrix}$$

Input weighting matrix

$$R = [21.66500493]$$

Likewise, the response of the mass-spring-damper system to a unit step input with parameters Q and R obtained from the inverse optimal control problem and with an initial condition of $x_0 = (4, 0, 0, 0)$ is as follows

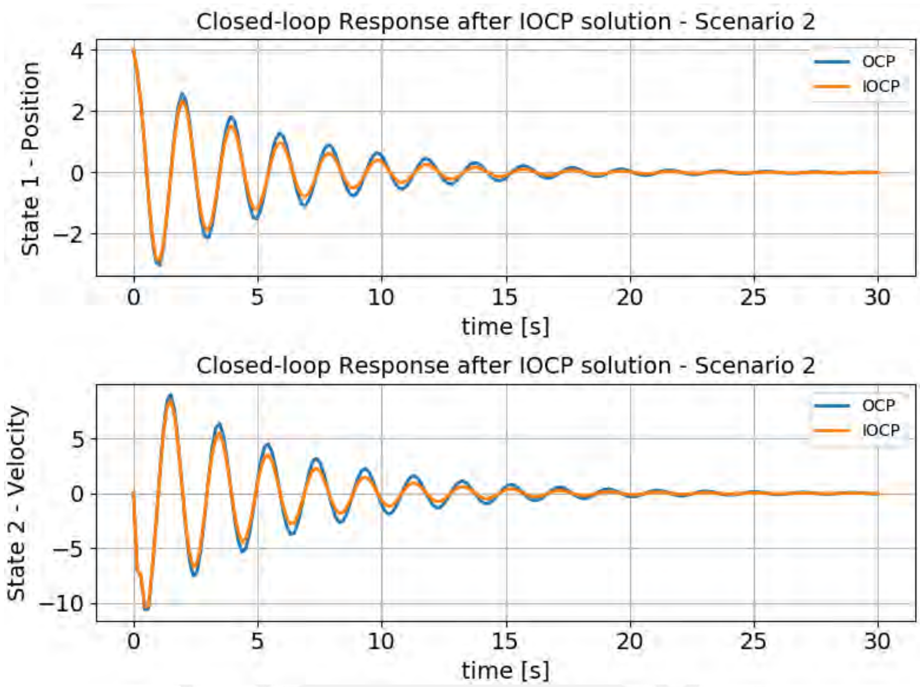


Figure 21 - System response after IOC for scenario 2

As can be seen from the above, a stable but oscillatory response is obtained, which can be addressed with additional constraints of the regional linear matrix inequalities type to handle the desired specifications for the system, i.e., to obtain a robust system. The configuration of the weight parameters allows a stable response and a settling time of 20 seconds.

4.4.1.6 Inverse optimal control for scenario 3

In this case after solving the inverse optimization problem, the values of the Q and R parameters of the cost function are estimated.

State weighting matrix:

$$Q = \begin{bmatrix} 0.01557152 & -9.58289553e-10 & -7.93586856e-11 & 7.68095299e-11 \\ -9.58289553e-10 & 1.16562375 & 1.39997267e-10 & -1.36192121e-10 \\ -7.93586856e-11 & 1.39997267e-10 & 995.13 & -2.23823261e-11 \\ 7.68095299e-11 & -1.36192121e-10 & -2.23823261e-11 & 1.12597187 \end{bmatrix}$$

Input weighting matrix

$$R = [31.94249366]$$

Likewise, the response of the mass-spring-damper system to a unit step input with parameters Q and R obtained from the inverse optimal control problem and with an initial condition of $x_0 = (4, 0, 0, 0)$ is as follows

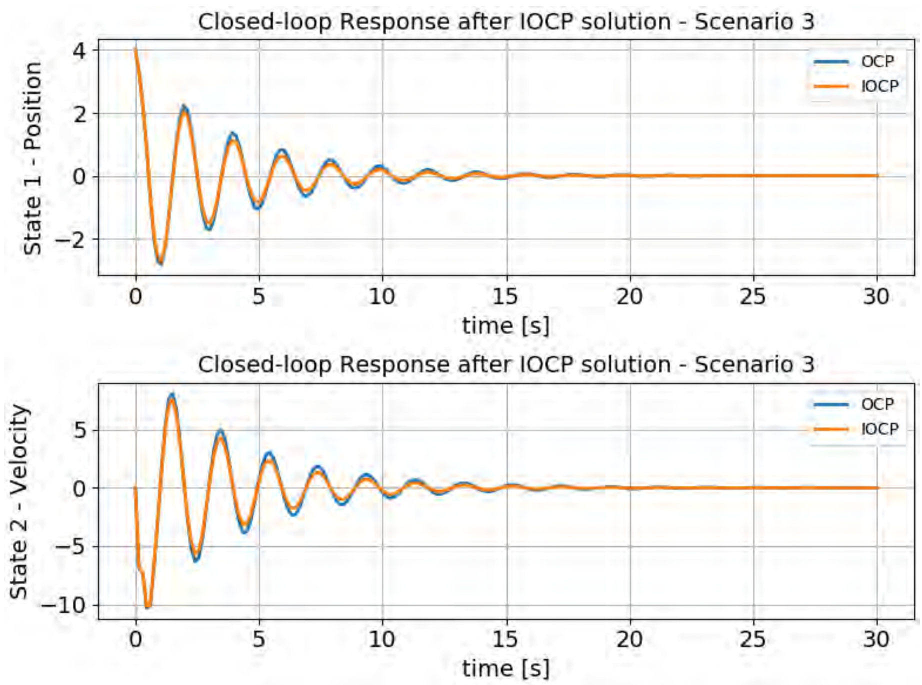


Figure 22 - System response after IOC for scenario 3

As can be seen from the above, a stable but oscillatory response is obtained, which can be addressed with additional constraints of the regional linear matrix inequalities type to handle the desired specifications for the system. The configuration of the weight parameters allows a stable response and a settling time of 15 seconds.

4.4.2 Inverse optimal control with transient response enhancement constraints

It is known that the transient response of a linear system is related to the location of its poles. By limiting the poles to a specific region, a satisfactory transient response can be guaranteed. Limiting the closed-loop poles to the region $S(\alpha, r, \theta)$ guarantees a minimum decay rate α which is the ratio of the magnitude between successive peaks in the response, a minimum damping ratio $\zeta = cos\theta$ to ensure minimum underdamped response and a maximum undamped natural frequency $w_d = rsin\theta$ that also contributes to a minimum underdamped response and is therefore inversely proportional to the damping ratio. This, in turn, limits the maximum overshoot, delay time, rise time, and settling time [25].

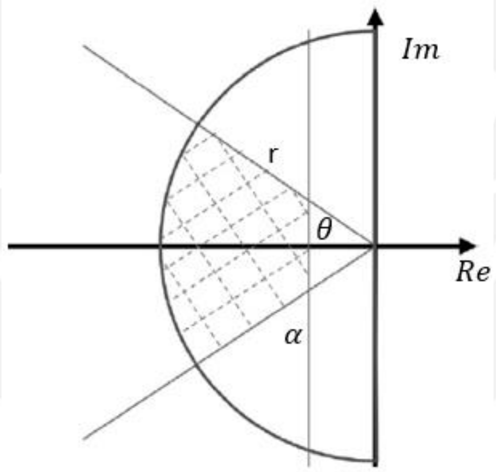


Figure 23 - D-stability region

Therefore, according to the above description of pole location, the following constraints are added to the inverse optimal control problem solved in section 4.4.1.5.

$$
\begin{gathered}
2\alpha P + AP + PA^T < 0 \\
\begin{bmatrix} -rP & AP \\ PA^T & -rP \end{bmatrix} < 0 \\
\begin{bmatrix} (AP + PA^T)sin\theta & (AP - PA^T)cos\theta \\ (PA^T - AP)cos\theta & (AP + PA^T)sin\theta \end{bmatrix}
\end{gathered}
\tag{4.15}
$$

The defined region results from the intersection of other regional linear matrix inequalities as shown in the figure above [26]. In this sense, according to the relationship described between the parameters of the $S(\alpha, r, \theta)$ region, it is desirable to have a small value for α, a high value for r and a low value for θ. Therefore, the following parameters are assigned to constrain the location of the closed-loop poles in the complex plane sector, i.e., $\alpha = 1$, $r = 10$ and $\theta = 80°$. Now, since the percentage of overshoot (PO) is a function of only the damping ratio ζ, the dominant pole approximation [41] is used to approximate the percentage overshoot, equation 4.16, from a second order system to a fourth order system.

$$
PO = 100e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \tag{4.16}
$$

Likewise, the numerical value of the above parameters, the expected percentage of overshoot (PO) will be 57.47%.

Therefore, after solving the inverse optimization problem, the values of the Q and R parameters of the cost function are estimated.

State weighting matrix:

$$
Q = \begin{bmatrix} 12.23693647 & -4.01035014e-08 & -1.39623261e-09 & 4.21642949e-10 \\ -4.01035014e-08 & 17.19132863 & 2.49090590e-09 & -7.62045378e-10 \\ -1.39623261e-09 & 2.49090590e-09 & 15.60922611 & -5.19247679e-11 \\ 4.21642949e-10 & -7.62045378e-10 & -5.19247679e-11 & 14.46232981 \end{bmatrix}
$$

Input weighting matrix

$$R = [0.052353302]$$

Likewise, the response of the mass-spring-damper system to a unit step input with parameters Q and R obtained from the inverse optimal control problem and with an initial condition of $x_0 = (4, 0, 0, 0)$ is as follows.

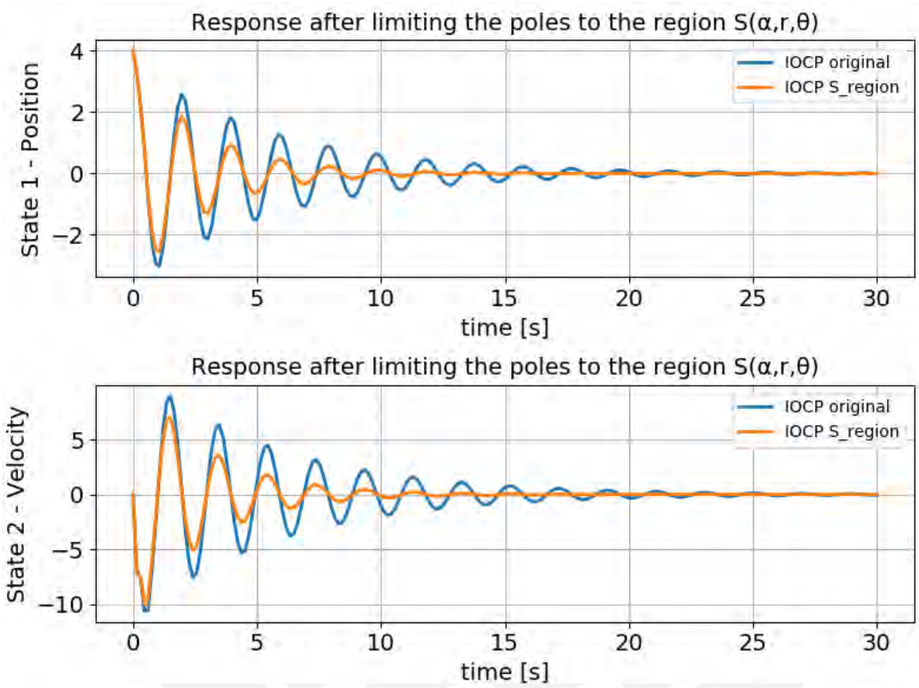


Figure 24 - System response after limiting the poles to the region S(α, r, θ)

From the above it can be observed that the amplitude of the overshoot was reduced and also the settling time was reduced from 20 to 12 seconds approximately, which represents an improvement in the transient response with respect to the initial response of the system.

On the other hand, a similar approach can be used, this time not using the constraints of regions of linear matrix inequalities but imposing a constraint specifically on the percentage of overshoot with the following constraints [27].

$$
\begin{aligned}
&A^T P + PA < 0 \\
&\begin{bmatrix} P & PB \\ B^T P & \zeta I \end{bmatrix} > 0 \\
&\begin{bmatrix} P & C^T \\ C & \zeta I \end{bmatrix} > 0
\end{aligned}
\tag{4.17}
$$

In this case, unlike the previous one, an overshoot percentage of 40 is desired. Therefore, after solving the inverse optimization problem, the values of the Q and R parameters of the cost function are estimated for this case.

State weighting matrix:

$$
Q = \begin{bmatrix} 16.15880802 & -1.10408037e-06 & -4.54753078e-08 & -5.97228443e-09 \\ -1.10408037e-06 & 19.84785721 & 8.18308876e-08 & 1.05253765e-08 \\ -4.54753078e-08 & 8.18308876e-08 & 12.21807782 & -5.44460466e-09 \\ -5.97228443e-09 & 1.05253765e-08 & -5.44460466e-09 & 19.66248128 \end{bmatrix}
$$

Input weighting matrix

$$
R = [0.041500481]
$$

Likewise, the response of the mass-spring-damper system to a unit step input with parameters Q and R obtained from the inverse optimal control problem and with an initial condition of $x_0 = (4, 0, 0, 0)$ is as follows.

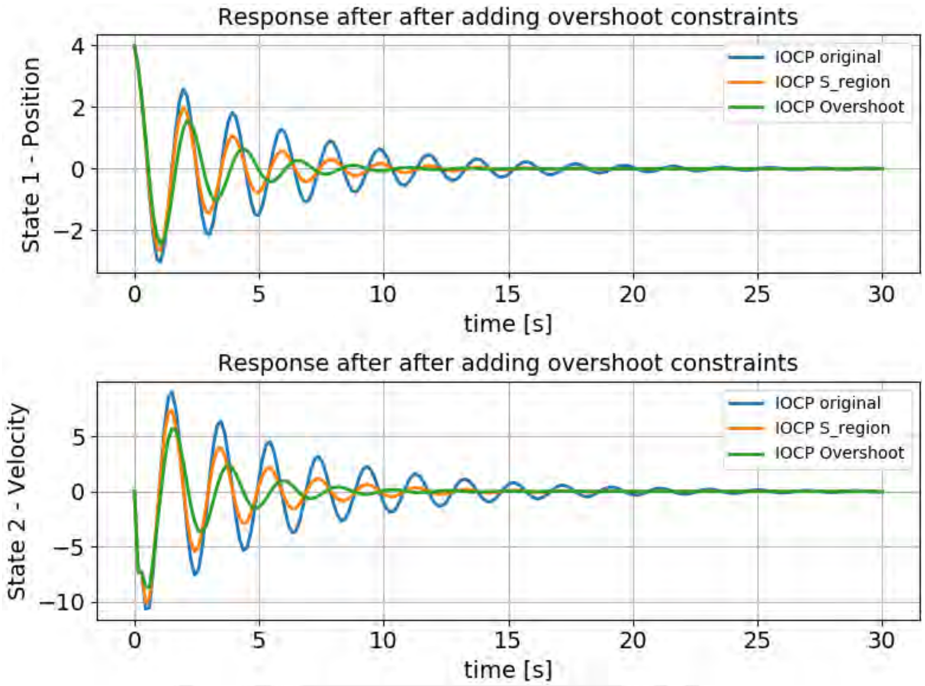


Figure 25 - System response after adding overshoot constraints

According to the above, and corroborating the desired overshoot percentage specification, it can be observed that the overshoot amplitude was reduced compared to the previous case demonstrating its effectiveness. However, it is more flexible to improve the transient response of the system by restricting the poles to a specific region as in the previous case, since in this way it is possible to control the decay rate, damping ratio and undamped natural frequency, which together regulate the transient response of the system.

4.4.3 State deviation penalty approach for obtaining Q and R parameters.

As mentioned above, there are situations in which having the gain controller K in advance may be difficult, challenging or impractical. In this sense, if the system response is available in the form of state trajectories or observed trajectories, it is possible to formulate an optimization problem in which the deviation from the observed state trajectory is penalized, while imposing constraints on the well-posedness, i.e., constraints on the unknown parameters Q and R, constraints on the system dynamics as well as constraints on the system stability. These restrictions are shown below

$$
\begin{gathered}
\dot{x} = Ax + Bu \\
A^T P + PA \leq 0 \\
\begin{bmatrix} R & B^T P \\ PB & A^T P + PA + Q \end{bmatrix} \geq 0 \\
Q = Q^T, Q \geq 0 \\
R = R^T, R \geq 0 \\
P = P^T, P \geq 0
\end{gathered}
\tag{4.18}
$$

In this case, in order to estimate the parameters Q and R, it is sought to penalize the differences between observed and obtained states, i.e., $\|x - \hat{x}\|$, where x is the state parameter that controls the set of feasible solutions, $\hat{x}$ is an observed state $x_{observed}$ that represents a priori belief or estimate. Likewise, the system dynamics constraint couples the state variables x and the control variable u. On the other hand, linear matrix inequalities (LMI) are proposed for a well-posedness of the problem, specifically regarding the system stability, the semidefinite positivity for P, Q and R and their relation through the block matrix shown.

In this sense, the motion dynamics of a mobile robot is proposed to simulate and record its trajectory in order to serve as input data, i.e., observed trajectories for the inverse optimal control problem.

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

It is worth mentioning that the system is stable, controllable and observable. Also shown are the following weight matrices Q and R

$$Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad\qquad R = [3]$$

After performing forward optimal control, the following movement trajectories or observed states are obtained

$$x_{observed} = \begin{bmatrix} 1 & 0.97 & 0.83 & 0.6 & 0.36 & 0.18 & 0.07 & 0.03 & 0.02 & 0.03 \\ 0 & -0.15 & -0.36 & -0.44 & -0.38 & -0.26 & 0.13 & -0.03 & 0 & 0 \\ 0 & -0.40 & -0.29 & -0.02 & 0.19 & 0.26 & 0.20 & 0.11 & 0.03 & -0.01 \end{bmatrix}$$

$$eingevalues(A_{cl}) = [-0.6913 \quad -0.6350 + 1.2815\text{j} \quad -0.6350 - 1.2815\text{j}]$$

The trajectories obtained after solving the forward problem are plotted below.

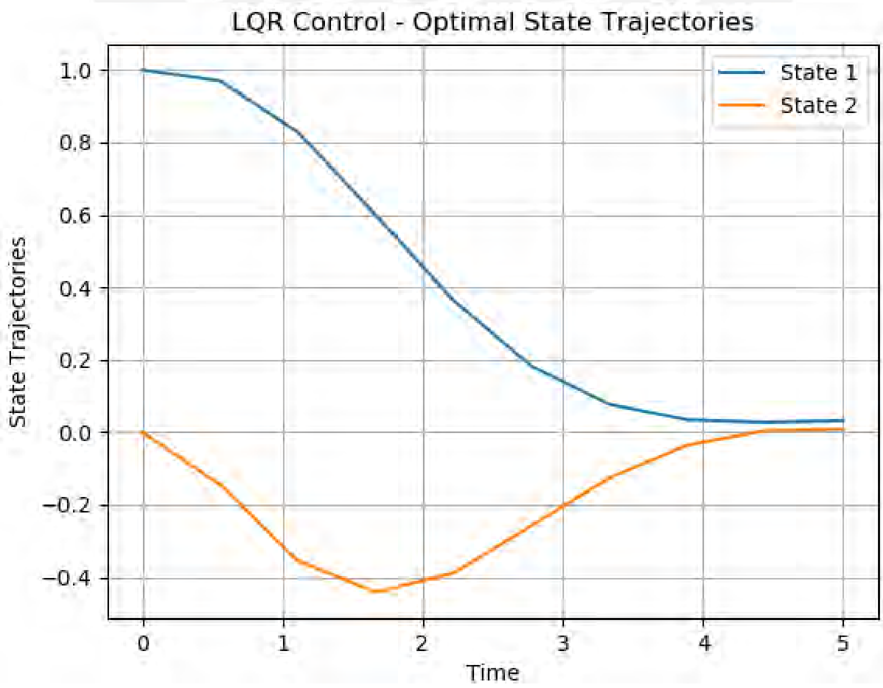


Figure 26 - Trajectories obtained after solving the forward problem

Now, after solving the optimization problem shown above for the inverse optimal control, we obtain the following matrices Q and R.

$$Q = \begin{bmatrix} 3.77958848 & 0.03334946 & 0.79676056 \\ 0.03334946 & 3.84160924 & 0.28659584 \\ 0.79676056 & 0.28659584 & 4.27348977 \end{bmatrix}$$

$$R = [3.75938141]$$

Likewise, the system response after solving the inverse optimal control problem and reconstructing the trajectories are shown below.

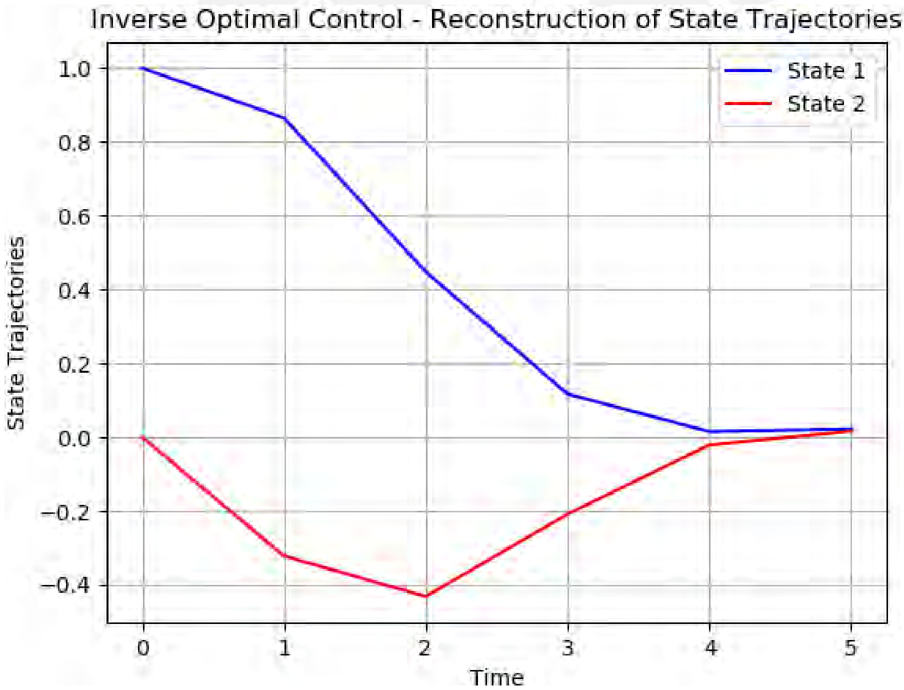


Figure 27 - Reconstruction of State Trajectories

The following is a comparison between the trajectories obtained from the solution of the forward optimal control problem and the trajectories reconstructed after the solution of the inverse optimal control problem.

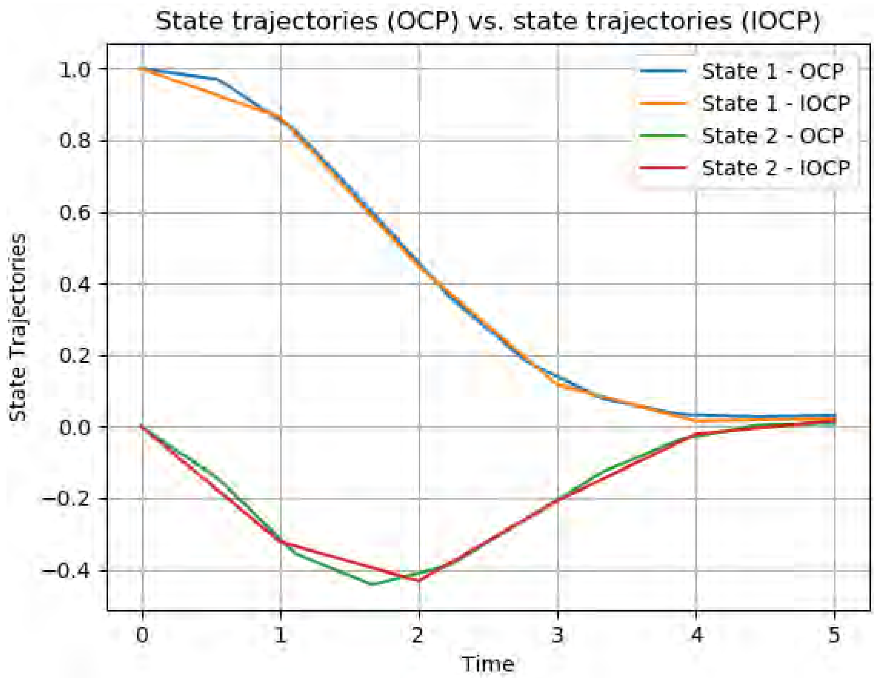


Figure 28 - Comparison between original and reconstructed state trajectories

In order to establish a measure of the difference between trajectories, correlation is used, which represents a measure of the similarity between two signals. Therefore, to calculate the correlation between the trajectories, the Python module $\mathrm{scipy.stats}$ for statistical functions is used, which ranges from 0 (no correlation) to 1 (perfectly correlated). In this case a value of 0.89 is obtained, indicating that the trajectories are highly correlated. The differences between the figures obtained are mainly due to the inaccuracy in the estimation of the Q and R parameters in the inverse optimal control. Likewise, the criteria of well-posedness such as stability and feasibility in the solution are met, since the solution of the optimization problem exists and as for uniqueness, different sets of Q and R matrices are obtained for each set of different values of the observed data. However, the penalty on the deviation of the observed states may introduce a relaxation in the conditions of the problem that could be addressed by the inclusion of other constraints that more strictly penalize the deviation in these states. Finally, after using the Python *time* library, the computational cost calculation results in 4,064 seconds, which corresponds to the high computational demand of the two-level approach used, but can be improved by analyzing other global and local minima search designs for the optimization algorithm.

Chapter 5
Conclusions and future research

5.1 Conclusions

In the present work, the existence, uniqueness and stability conditions that contribute to a well-posedness to the inverse optimal control problem were studied. These conditions were proposed using linear matrix constraints (LMI) and convex optimization according to the guidelines of semidefinite programming (SDP).

The approach of the constraints as convex formulations made it possible to take advantage of existing computational methods to retrieve the solution of the inverse optimal control problem of the Linear Quadratic Regulator.

Three analysis scenarios were addressed: the writing of well-posedness conditions for the solution of the inverse problem, the inclusion of additional constraints to improve the transient response of the system, and the approach of penalizing the deviation of the states to obtain the Q and R parameters.

The simulation of the system was carried out, obtaining transient responses close to the original, also when applying the pole region restrictions an improvement in the transient response of the system was obtained, in that sense the weight matrices found comply with the existence conditions since they originated feasible problems, injectivity since for different observed data different weight matrices were obtained and stability since the system responses were stable in all cases.

5.2 Future research

The inverse optimal control approach studied here serves as a basis for its application to other forms of optimal regulators. Therefore, future research may aim at recovering general forms of objective functions beyond the linear quadratic regulator approach analyzed here.

The general assumption in inverse control problems that the optimal solution is available in advance can be discarded in future research as this would more closely resemble real scenarios.

The optimization algorithm for parameter estimation can be improved by going deeper into the alignment of the observed trajectories with the optimal trajectories and using this alignment as the basis for the recovery of the cost function.

Further research can also be done to optimize directly on the cost function parameters to minimize the discrepancy between the observed and optimal trajectories and thus have greater resolution in the parameters obtained.

Finally future work can apply inverse optimal control in robotic arm motion that starts from the observation of an agent and the subsequent imitation representing the motion reconstructed from that observation.

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