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## Endogenous Threshold Stochastic Volatility Model: An Outlook Across the Globe for Stock Market Indices

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### Abstract

Asymmetries and heavy tails are well-known characteristics on compound daily returns stock market indices. The THSV-SMN –Threshold Stochastic Volatility Model with Scale Mixture of Normal Distributions– model has become an important tool for analysis regarding forecasting asset returns and Value at Risk and Expected Shortfall portfolio estimations in order to assess market risk. Therefore, under a Bayesian approach, we develop an extension on the model proposed by Abanto & Garrafa (2019). This extension allows for an endogenous threshold and will be studied under two theoretical frameworks: the use of order statistics and a random walk Metropolis–Hasting algorithm (RWMH). We test the model extension upon stock market indices across the globe along four regions (North America, LATAM, Europe and Asia) with our proposed RWMH algorithm and compare the results with the original (fixed threshold) model using goodness-of-fit and error prediction criteria. Evidence shows that stock markets indices differ both within and across regions, yet in most cases the extended model outperforms the original THSV-SMN. Thus, prudence and a personalized analysis per index are strongly recommended.

JEL Classification: C11, C15, C51, C52, C58

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# 1 Introduction

Financial econometrics have led the process of uncovering and studying financial asset returns and the main stylized fact highlights a significant non-Gaussian behavior. Thus, literature has been widely developed in the analysis of heavy-tailed distributions (Mandelbrot (1963), Fama (1965), Harvey & Shephard (1996), Liesenfeld & Jung (2000), Chib et al (2002)), distribution asymmetries (Black (1967), Christie (1982), So et al (2002), Chen et al (2008)) and volatility clustering (Taylor, (1982, 1994), among others).

In the current state of the literature, the Threshold Stochastic Volatility Model with Scale Mixture of Normal distributions (hereinafter, THSV-SMN) emerges in a time where efficient and computational feasible algorithms can be used to perform Markov chain Monte Carlo (hereinafter, MCMC) methods. The model has been consecutively extended since its first version, Taylor (1982, 1994), and encompasses a family of special cases such as the Stochastic Volatility (hereinafter, SV) model, the SV-Scale Mixture of Normal Distributions (hereinafter, SV-SMN) and the Threshold SV model (hereinafter, THSV). The THSV-SMN model studied in Abanto & Garrafa (2019) generalizes these extensions and allows us not only to incorporate both asymmetries and heavy tails distributions, but also comprehend a family of heavy-tailed distributions such as the t-Student, Slash and the Variance Gamma distributions, among others.

Our research aims to address and it conveys the following problem statement: what is the impact in goodness-of-fit and error prediction when we allow for an extension in the THSV-SMN model developed by Abanto & Garrafa (2019)? In order to do that, we have two main objectives: first, we propose an extension on the THSV-SMN model where we endogenously estimate a threshold for regime switching. This will be addressed by proposing two theoretical frameworks: the use of order statistics (hereinafter, OS) as in Chib (1998) and the use of a Random Walk Metropolis Hasting (hereinafter, RWMH) sampling algorithm as in Wirjanto et al (2016). Second, we intend to submit both the original and the extended THSV-SMN model to analyse stock market indices across four regions around the globe. We are interested in studying the goodness-of-fit with the Watanabe-Akaike information criteria –Watanabe, (2010, 2013)– and error prediction with the Mean Square Prediction Error –Abanto & Garrafa (2019)–. We also compute Kupiec (1995) likelihood ratio tests and Embrechts et al (2004) Expected Shortfall (hereinafter, ES) accuracy measure in order to assess Value at Risk (hereinafter, VaR) and ES metrics as in Abanto & Garrafa (2019).

Regarding our hypothesis, we expect to find a better fit when we compare results with the original heavy-tailed case with our extended model, yet we also expect to find heterogeneous behavior across markets. Because of that, goodness of fit can vary among the cases that have been studied. That is, we aim to complement the analysis by studying different stock market indices in order to control our results for plausible differences regarding markets behavior within and across regions.

Estimation results supports our hypothesis. The THSV-SMN model with endogenous threshold outperforms the fixed (zero) exogenous threshold in goodness-of-fit criteria using the WAIC statistic in the majority of our sample analysis. Results also have shown that persistence parameters in the compound return and the location parameters in the log volatility equations do behave differently according to its regime. Last, results indicate that stock market indices have clear differences across regions. Posterior modes of the parameters vary within and across regions. Yet, those specific and tailored cases can be covered by these following facts: (i) the lower variance in the log-volatility equation (ii) the close-to-zero location and close-to-unity persistence in the log-volatility equation, (iii) in our sample data analysis, the threshold do ranges close to zero and clearly differentiates the posterior densities in the compound return equation and (iv) posterior mean of the threshold locates slightly above zero and evidence appear to link this threshold value to the mean of the observed compound return data. Future research and analysis may address this issue.



The final report is organized as follows. Section 2 provides the theoretical framework and addresses the issue of SMN distributions (subsection 2.1), presents the THSV-SMN model (subsection 2.2), our proposed extension (subsection 2.3) and the methodological approach (subsections 2.4, 2.5 and 2.7). Section 3 refers to the empirical technique by describing our sample data-sets (subsection 3.1) and the estimation results (subsection 3.2 and 3.3). Finally, Section 4 contains our concluding remarks and provides details regarding the current agenda for future research.

## 2 The Heavy-Tailed Threshold Stochastic Volatility Model

### 2.1 SMN distributions

In this section we present excerpts from section 2 of Abanto & Garrafa (2019) and define the SMN distributions. We will focus in the cases where we retrieve the Normal and the Slash distributions. For a more detailed exposition regarding SMN distributions see Lange & Sinsheimer (1993) and Choy & Chang (2008).

A random variable  $X$  belongs to the SMN family if it can be expressed as

$$X = \mu + \kappa(\lambda)^{1/2} \epsilon$$

where  $\mu$  is a location parameter,  $\epsilon \sim \mathcal{N}(0, \sigma^2)$ ,  $\lambda$  is a positive random mixing variable<sup>2</sup> with cumulative distribution function (*cdf*)  $H(\cdot | \nu)$  and probability density function (*pdf*)  $h(\cdot | \nu)$ ,  $\nu$  is a scalar or parameter vector indexing the distribution of  $\lambda$  and  $\kappa(\cdot)$  is a positive weight function. We restrict our attention to the case where  $\kappa(\lambda) = 1/\lambda$  because its good mathematical properties<sup>3</sup>. Given  $\lambda$ , we have  $X | \lambda \sim \mathcal{N}(\mu, \lambda^{-1}\sigma^2)$  and the *pdf* of  $X$  yields:

$$f_{\text{SMN}}(X | \mu, \sigma^2, \nu) = \int_0^{\infty} \mathcal{N}(X | \mu, \lambda^{-1}\sigma^2) \cdot h(\lambda | \nu) d\lambda \quad (1)$$

By choosing an appropriate mixing density<sup>4</sup>  $h(\cdot | \nu)$ , a rich class of continuous symmetric distributions can be described by the density given in (1). They can accommodate thicker tails than the normal process. In that sense, note that when  $\lambda = 1$  (a degenerate random variable), we retrieve the normal distribution. Next, we present a heavy-tailed density based on the choice of the mixing density  $h(\cdot | \nu)$ .

- The Slash distribution  $X \sim \mathcal{S}(\mu, \sigma^2, \nu)$ ,  $\nu > 0$

This distribution includes the normal case when  $\nu \rightarrow \infty$ . Its pdf is given by:

$$f_{\text{SMN}}(X | \mu, \sigma^2, \nu) = \nu \int_0^1 \lambda^{\nu-1} \mathcal{N}(x | \mu, \sigma^2 \lambda^{-1}) du$$

<sup>2</sup>A mixing variable is defined as the probability distribution of a random variable that is derived from a collection of other random variables. First, a random variable is selected by chance from a collection according to given probabilities of selection, and then the value of the selected random variable is realized. These underlying random variables may be random real numbers, or they may be random vectors.

<sup>3</sup>Lange & Sinsheimer (1993) illustrate another examples of positive weight functions such as  $\kappa(\lambda) = \lambda^{(\nu/2)}, \forall \lambda \neq 0$  and uses it in the power-exponential density or  $\kappa(\lambda) = \kappa(0)_\epsilon + \lambda^{1/2} - \epsilon \ln(\epsilon + \lambda^{1/2})$  only when  $\nu = 1$  and with  $\epsilon$  defined as a small positive constant. The other cases, i.e.  $\nu \neq 1$ , are impossible to calculate and the weight function is define as its first derivative  $\kappa'(\lambda) = \nu / [2(\epsilon + \lambda^{1-\nu/2})]$ . Other examples also cannot be calculated and are defined as a first derivative, i.e.  $\kappa'(\lambda^2)$ , satisfying being completely monotone and with density  $\kappa(\lambda) = -2 \ln \left[ \int_0^{\infty} u^{k/2} \exp(-u\lambda^2/2) dH(u) \right]$ , where  $k$  is the order, i.e. size, of the  $\lambda$  vector. For those reasons, we follow Abanto & Garrafa (2019), Choy & Chang (2008) and Lange & Sinsheimer (1993) and restrict our attention to  $\kappa(\lambda) = 1/\lambda$

<sup>4</sup>A mixing density is defined as the probability density function of the outcome variable obtained from the underlying continuous random variables.

The distribution of  $\lambda$  is Beta i.e.  $\mathcal{B}e(\nu, 1)$  with density:

$$h(\lambda | \nu) = \nu \lambda^{\nu-1} \mathbb{I}_{(0,1)}$$

Thus, the slash distribution is equivalent to the following hierarchical form:

$$X | \mu, \sigma^2, \lambda \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{\lambda}\right), \quad \lambda | \nu \sim \mathcal{B}e(\nu, 1)$$

As pointed out by Abanto et al (2008) the Slash distribution has been mainly used in simulation studies because it represents an extreme situation. See for example Andrews et al. (1972), Gross (1973), and Morgenthaler and Tukey (1991).

## 2.2 THSV-SMN

The THSV-SMN model is a generalization of the discrete SV model proposed by Taylor (1980). It was then extended allowing for a threshold by So et al. (2002) with the objective to capture asymmetries and they called it the THSV model. Next, Abanto et al (2008) introduced SMN distributions in the SV model allowing for a generalization of distributions with heavy tails (SV-SMN). Last, Abanto-Valle & Garrafa (2019) combined those characteristics reaching the THSV-SMN model. We keep the notation and present a model with the extension we propose below:

$$y_t = \mu_{s_t} + \beta_{s_t} y_{t-1} + \exp(h_t/2) \lambda_t^{-1/2} \epsilon_t \quad (1a)$$

$$h_{t+1} = \alpha_{s_{t+1}} + \phi_{s_{t+1}} h_t + \sigma_{s_{t+1}} \eta_{t+1} \quad (1b)$$

$$\lambda_t \sim p(\lambda_t | \nu) \quad (1c)$$

$$s_t = \begin{cases} 1 & \text{if } y_{t-1} > r \\ 0 & \text{if } y_{t-1} \leq r \end{cases} \quad (1d)$$

Where (1a) is the compound return equation and the innovations  $\epsilon_t$  and  $\eta_{t+1}$  are independent and identically distributed (hereinafter, *iid*) with a Normal distribution  $\mathcal{N}(0, 1)$ . (1b) is the log-volatility equation. In (1c),  $\lambda_t$  is the scale factor,  $p(\lambda_t | \nu)$  is the mixing density and  $\nu$  the parameter that capture the heavy-tailness as in Abanto & Garrafa (2019); finally, in (1d) we propose an endogenous threshold  $r$ .

Furthermore, in this class of models, the normal (THSV-N), Student-t (THSV-T), Slash (THSV-S) and Variance Gamma (THSV-VG) distributions are special known cases. Abanto & Garrafa (2019) present the estimation results in four Stock Market Indices for all these models and they find that the Slash distribution ranks first in goodness-of-fit and prediction accuracy. That is the reason why in this research we focus our resources analyzing the Slash distribution as the best candidate to represent the heavy tail densities' family for stock market indices.

## 2.3 Extensions

As we have anticipated in the previous subsection, we develop an extension on the model proposed by Abanto & Garrafa (2019). By generalizing the model, we can describe a family of studies as we next trace:

- With  $r = 0$  we relate to the work of Abanto & Garrafa (2019) and So et al (2002) –with a slight different notation in the volatility–. The threshold,  $r = 0$ , at time  $t$  incorporates unexpected moves in prices due to bad or good news on  $t - 1$ . In that way, the parametric space  $\Theta = (\mu, \beta, \alpha, \phi, \sigma^2)'$  switch between the two regimes corresponding to the sign of  $y_{t-1}$ .

- With  $r \neq 0$  we refer to the work of Chen et al (2008). They formulate the threshold variable for regime switching in two cases (i) as an exogenous variable and (ii) as an endogenous variable that surrogates past information. In (i) they set the value with -0.2 referring studies in market returns and in (ii) they sample the threshold from an SV model.
- Also, with  $r \neq 0$  Wirjanto et al (2016) propose a parsimonious model without location parameters where the threshold differentiate the variances of the innovation in the compound return equation. As they refer, the threshold determine the marginal distribution of future returns. They sample the threshold  $r$  with a RWMH algorithm with a target density defined by the full conditional of  $r$ .
- In addition,  $r \neq 0$  refer to Chiou & Tsay in a yet not published paper. They define an endogenous threshold with a model selection criteria. They follow Tong (1990) to implement a self-exciting threshold autoregressive (SETAR) process. To update the threshold values they do not need the Metropolis-Hasting algorithm as in Wirjanto et al (2016). In that sense, a more efficient model obtains an accurate and tight posterior distribution. In order to do that, they use order statistics to re-arrange the regression setting on the observed return values. Finally, they use the Bayes factors to choose the best model with the importance-function approach suggested by Newton & Raftery (1994).
- With  $r = 0$  and with correlation between innovations we refer to Harvey & Shephard (1996). They estimate the correlation  $\rho$  recovering the information by carrying out inference conditional on the signs of the observations from a criterion function. Their state-space form is transformed to adopt a corresponding form with a linear and uncorrelated measurement and transition equation disturbances.

After the review of the studies, we next present two theoretical frameworks in order to estimate the endogenous threshold  $r$ . The first one uses order statistics and a transition probability matrix. The second includes a Random walk Metropolis–Hastings algorithm in order to sample from the full conditional density of  $r$ .

- Threshold with Order Statistics

The identification of a unique threshold relies on the findings highlighted from Chiou & Tsay. They show that, for stock market indices, a single threshold, i.e. two regimes, is the best specification to model regime switching. In that sense, let us introduce some notation. The threshold value,  $r = y_{t-1}$ , may assume the observed values  $y_0, y_1, \dots, y_{T-1}$ , where  $T$  denotes the sample size. Let  $y_{(0)} \leq y_{(1)} \leq \dots \leq y_{(T-1)}$  be the ordered values where  $t(n)$  denotes the time index of  $y_{(n)}$ , for  $n = 0, 1, \dots, T - 1$ . Now, equation (1d) can be re-written by using the  $n$ th order statistics, i.e.  $y_{(1)} \leq y_{(2)} \leq \dots y_{(T-1)}$  from the values of  $y_{t(1)} \leq y_{t(2)} \leq \dots y_{t(T-1)}$ :

$$\begin{aligned}
 y_{t(n)+1} &= \mu_0 + \beta_0 y_{t(n)} + \exp\left(\frac{h_{t(n+1)}}{2}\right) \lambda_{t(n+1)}^{-1/2} \epsilon_{t(n)+1}, & \text{if } 0 \leq n < \tau & \quad \wedge \quad \Theta_0 \subset \Theta \\
 y_{t(n)+1} &= \mu_1 + \beta_1 y_{t(n)} + \exp\left(\frac{h_{t(n+1)}}{2}\right) \lambda_{t(n+1)}^{-1/2} \epsilon_{t(n)+1}, & \text{if } n > \tau & \quad \wedge \quad \Theta_1 \subset \Theta
 \end{aligned}$$

We will also refer to the subsets  $\Theta_0$  and  $\Theta_1$  to simplify notation regarding the parameters for the regimes. The parametric space  $\Theta$  clearly satisfies  $\Theta = \Theta_0 \cup \Theta_1$  and  $\Theta_0 \cap \Theta_1 = \emptyset$ . Now we can update the threshold variable  $r$  following Chib (1998) and using the order statistics  $y_{t(n)+1}$ . Let  $\mathbf{K}_{2:T}$  be the binary state vector that follows a Markovian process with a transition probability matrix  $P_{2 \times 2}$ . A change point in  $K_t$  traduces in a break-jumping point, i.e a point with no return in the arranged time index  $T$ . Therefore, the threshold value as mentioned before is  $r = y_{t(\tau)}$ . With the transition probability matrix  $P$  given by:

$$P = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 0 & 1 \end{pmatrix}$$

The sampling scheme may be resume as



1.  $\Theta^{(1)}, P^{(1)} \mid \mathbf{y}_{1:T}, \mathbf{K}_{2:T}^{(0)}$
2.  $\mathbf{K}_{2:T}^{(j)} \mid \mathbf{y}_{1:T}, \Theta^{(j-1)}, P^{(j-1)}$
3.  $\Theta^{(j)}, P^{(j)} \mid \mathbf{y}_{1:T}, \mathbf{K}_{2:T}^{(j-1)}$
4. Set  $j = j + 1$  and return to step (2) until convergence is achieved.

Appendix 6 includes details and derivations for the implementation of the sampling scheme.

- Threshold with a random-walk MH algorithm

We follow ideas of Wirjanto et al. (2016) in order to sample the threshold parameter,  $r$ . Therefore, given that all other parameters and latent states in the model have been sampled, the full conditional of  $r$  can be expressed as follows:

$$f(r \mid \mathbf{y}_t, h_t, \Theta_{-r}) = \prod_{t=3, y_{t-1} \leq r}^{T-1} f(y_t \mid h_t, \Theta_{-r}) \cdot f(h_t \mid \Theta_{-r}) \prod_{t=3, y_{t-1} > r}^{T-1} f(y_t \mid h_t, \Theta_{-r}) \cdot f(h_t \mid \Theta_{-r}) \quad (2)$$

where

$$f(y_t \mid h_t, \Theta_{-r}) \propto \frac{1}{\exp(h_t/2)\lambda_t^{-1/2}} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_{s_t} - \beta_{s_t} y_{t-1})^2}{\exp(h_t)\lambda_t^{-1}}\right]$$

$$f(h_t \mid \Theta_{-r}) \propto \frac{1}{\sigma_{s_t}} \exp\left[-\frac{1}{2} \frac{(h_t - \alpha_{s_t} - \phi_{s_t} h_{t-1})^2}{\sigma_{s_t}^2}\right]$$

$\Theta_{-r}$  defines the vector of parameters excluding  $r$ . The full conditional distribution of  $r$  is not in the form of any standard distribution. Thus, we use a random-walk Metropolis Hastings algorithm with the target density given in (2) in order to simulate  $r$ , where the proposal distribution is a univariate normal distribution with the mean value given by the previous iteration, and the variance is adjusted so that the acceptance rate can be tuned to be at least between 20% and 50% and such rates are commonly used in Bayesian literature as pointed out by Wirjanto et al. (2016). In addition, in order to ensure that there is enough data in each of the two regimes, we require that  $r$  has a uniform prior distribution over the interval between the first and the third quartiles of the observed data. This prior distribution assumption for the threshold value has also been used in Chen et al. (2008).

## 2.4 Parameter Estimation via MCMC

The Bayesian estimation in the THSV-SMN class of models must incorporate an estimation for the unknown process log-volatility  $h_t$ . We follow the proposal developed by Abanto & Garrafa (2019) which consists in an algorithm based on MCMC simulation to make the analysis feasible.

Following their notation, let  $\mathbf{h}_{1:T} = (h_1, \dots, h_T)'$  be the vector of unknown log-volatilities,  $\lambda_{1:T} = (\lambda_1, \dots, \lambda_T)'$  the mixing variables and  $\Theta = (\psi_0, \psi_1, \varphi_0, \varphi_1, \sigma_0^2, \sigma_1^2, \nu, r, p_{11})'$  the complete parameter vector of the THSV-SMN class of models, where  $\psi_{s_t} = (\mu_{s_t}, \beta_{s_t})'$  and  $\varphi_{s_t} = (\alpha_{s_t}, \phi_{s_t})'$  for  $s_t = 0, 1$ . We draw random samples from the posterior density  $p(\Theta, \mathbf{h}_{1:T}, \lambda_{1:T} \mid \mathbf{y}_{1:T})$  using Gibbs sampling. The algorithm is described below:

## Algorithm 1

1. Set  $j = 0$ , the starting values for the parameters  $\Theta^{(j)}$  and the latent  $\mathbf{h}_{1:T}^{(j)}$  and  $\lambda_{1:T}^{(j)}$ .
2. Generate  $\Theta^{(j+1)} \sim p(\Theta | \mathbf{h}_{1:T}^{(j)}, \lambda_{1:T}^{(j)}, \mathbf{K}_{1:T}^{(j)}, \mathbf{y}_{1:T})$ .
3. Update  $\mathbf{K}_{1:T}^{(j+1)} | \mathbf{y}_{1:T}, \Theta^{(j+1)}$ .
4. Draw  $\lambda_{1:T}^{(j+1)} \sim p(\lambda_{1:T} | \Theta^{(j+1)}, \mathbf{h}_{1:T}^{(j)}, \mathbf{K}_{1:T}^{(j+1)}, \mathbf{y}_{1:T})$ .
5. Update  $\mathbf{h}_{1:T}^{(j+1)} \sim p(\mathbf{h}_{1:T} | \Theta^{(j+1)}, \lambda_{1:T}^{(j+1)}, \mathbf{K}_{1:T}^{(j+1)}, \mathbf{y}_{1:T})$ .
6. Set  $j = j + 1$  and return to step (2) until convergence is achieved.

Given the fact that the Gibbs sampling algorithm requires to sample the parameters  $\Theta$  and the latent quantities ( $\mathbf{h}_{1:T}$  and  $\lambda_{1:T}$ ) from their full conditionals, we will use the Multi-Move Procedure (hereinafter, MMP) following Abanto & Garrafa (2019) in order to sample the log-volatility chain. The reason we use the MMP algorithm, as stated by Abanto et al. (2010, 2014), is because Multi-move algorithms are computationally efficient and convergence is achieved faster than using a single-move. Omori et al. (2007) and Kim et al. (1998) are also main references in the use of Multi-move algorithms.

### Multi-move sampler for the log-volatilities

To sample the latent variables  $\mathbf{h}_{1:T}$  we use the simulation smoothing method proposed by McCausland et al. (2011). We do so from the states in a linear Gaussian state space-model defined by two equations. For that, we need to transform (1a).

Let  $z_t = \ln[(y_t - \mathbf{r}_t \phi_{s_t})^2]$ , where  $\mathbf{r}_t = (1 \ y_{t-1})'$ . With the log-square transformation, (1a) can be expressed as:

$$z_t = -\ln \lambda_t + h_t + \ln \epsilon_t^2 \quad (2)$$

where  $\ln \epsilon_t^2 \sim \ln \chi_1^2$ . With (2) and (1b) we have a linear state-space model, yet a non-Gaussian one. So, as in Abanto & Garrafa (2019) we follow the ideas of Kim et al. (1998) and Omori et al. (2007) to approximate  $\ln \epsilon_t^2$  by a finite mixture of normal distributions as follows:

$$f(\ln \epsilon_t^2) \approx \sum_{i=1}^Q q_i \cdot f(\ln \epsilon_t^2 | k_t = i)$$

where  $k_t$  is a discrete mixing variable,  $\ln \epsilon_t^2 | k_t = i \sim \mathcal{N}(\vartheta_i, \varpi_i^2)$  and  $q_i = \Pr(k_t = i)$ . Now equation (2) can be written as:

$$z_t = -\ln \lambda_t + h_t + \xi_t \quad (3)$$

where  $\xi_t | k_t = i \sim \mathcal{N}(\vartheta_i, \varpi_i^2)$ .

For the number  $Q$  of normal distributions and for the parameters of the mixture of normal distributions  $\vartheta_i$  and  $\varpi_i^2$ ,  $\forall i = 1, \dots, 10$ , we use the results found by Omori et al. (2007) where  $Q = 10$  gives a satisfactory approximation of the  $\ln \chi_1^2$  density and the discrete mixing variable  $k_t$  can be drawn from the conditional probability:

$$\Pr(k_t = i \mid z_t, \lambda_t, h_t) \propto q_i \cdot \exp\left[-\frac{1}{2\sigma^2} (z_t + \ln \lambda_t - h_t - \vartheta_t)^2\right], \quad \forall i = 1, \dots, 10 \wedge t = 1, \dots, T$$

Section 6 shows more details in the `updmix` function. Now, with (3) and (1b) we have the following linear Gaussian state space model

$$\begin{aligned} z_t &= -\ln \lambda_t + h_t + \xi_t \\ h_{t+1} &= \alpha_{s_{t+1}} + \phi_{s_{t+1}} h_t + \sigma_{s_{t+1}} \eta_t \end{aligned}$$

and we could use MMP algorithm developed by McCausland et al. (2011).

## MMP

IF  $\mathbf{h}_{1:T} \mid \mathbf{z}_{1:T} \sim \mathcal{N}(\Omega^{-1}\mathbf{c}, \quad \Omega^{-1})$ , WHERE  $\Omega$  IS DEFINED IN SECTION 6, THEN

$$\begin{aligned} h_t \mid h_{t+1}, \mathbf{y}_{1:T} &\sim \mathcal{N}(m_t - \Sigma_t \cdot \Omega_{t,t+1} \cdot h_{t+1}, \quad \Sigma_t) \\ \mathbf{E}[\mathbf{h}_{1:T} \mid \mathbf{z}_{1:T}] &= (\mu_1, \dots, \mu_T)' \end{aligned}$$

WHERE  $\Sigma_1 = \Omega_{11}^{-1}$ ,  $m_1 = \Sigma_1 c_1$ ,  $\Sigma_t = (\Omega_{t,t} - \Omega_{t-1,t}^2 \cdot \Sigma_{t-1})^{-1}$ ,  $m_t = \Sigma_t (c_t - \Omega_{t-1,t} \cdot m_{t-1})$ ,  $\mu_T = m_T$  AND  $\mu_t = m_t - \Sigma_{t,t+1} \cdot \mu_{t+1}$ .

Now we can describe the simulation process for the  $j$ -th MCMC iteration:

### Algorithm - MMP

1. Simulate  $h_T^{(j)} \sim \mathcal{N}(m_T^{(j)}, \quad \Sigma_T^{(j)})$
2. For  $t = T - 1, T - 2, \dots, 1$ , simulate  $h_t^{(j)} \sim \mathcal{N}(m_t^{(j)} - \Sigma_t^{(j)} \cdot \Omega_{t,t+1}^{(j)} \cdot h_{t+1}^{(j)}, \quad \Sigma_t^{(j)})$

## 2.5 Forecasting returns and volatility

We also follow Abanto & Garrafa (2019) in the process of forecasting the compound return and the log-volatilities. The  $K$ -step ahead prediction densities can be computed using the following recursive procedure:

$$\begin{aligned} p(y_{T+K} \mid \mathbf{y}_{1:T}) &= \int [p(y_{T+K} \mid h_{T+K}, \lambda_{T+K}) \cdot p(\lambda_{T+K} \mid \Theta) \\ &\quad \times p(h_{T+K} \mid \Theta, \mathbf{y}_{1:T}) \cdot p(\Theta \mid \mathbf{y}_{1:T})] \cdot dh_{T+K} d\lambda_{T+K} d\Theta, \\ p(h_{T+K} \mid \Theta, \mathbf{y}_{1:T}) &= \int p(h_{T+K} \mid \Theta, h_{T+K-1}) \cdot p(h_{T+K-1} \mid \Theta, \mathbf{y}_{1:T}) \cdot dh_{T+K-1} \end{aligned}$$

To initialize the recursion, we use  $h_T^{(i)}$  and  $\Theta^{(i)}$  for  $i = 1, \dots, N$  from the MCMC output. Given the  $N$  draws, we sample  $h_{T+K}^{(i)}$  from  $p(h_{T+K} \mid \Theta^{(i)}, h_{T+K-1}^{(i)})$  and  $\lambda_{T+K}^{(i)}$  from  $p(\lambda_{T+K} \mid \Theta^{(i)})$  by using (1b) and (1c). Finally, using (1a) we sample  $y_{T+K}^{(i)}$  from  $p(y_{T+K} \mid \Theta^{(i)}, h_{T+K}^{(i)}, \lambda_{T+K}^{(i)})$ .

## 2.6 Value at Risk and Expected Shortfall

- Value at Risk

To evaluate the performance of the model on *VaR* metrics, the likelihood ratio test introduced in Kupiec (1995) is used to test the null hypothesis which states that the expected proportion of the number of "beyond VaR" or "violations" during the test periods is equal to  $\alpha$  (the percentile level). The violation is formulated by:

$$I_t(\alpha) = I[y_{T+1} < \hat{VaR}_t(\alpha)]$$

$$I_t(\alpha) = I[y_{T+1} > \hat{VaR}_t(\alpha)]$$

for the left and right tail, respectively, where  $I[\cdot]$  is an indicator function and  $\hat{VaR}_t(\alpha)$  is the estimated *VaR* at level  $\alpha$ , which can be obtained by simulation using the  $k$ -step ahead densities described below. Next, let  $x_\alpha$  be the number of violations, that is,  $x_\alpha = \sum_{t=T+1}^{T+m} I_t(\alpha)$  and  $\hat{\alpha} = x_\alpha/m$ . The unconditional test of Kupiec (1995) is a likelihood ratio test with the  $\chi^2_1$ -distributed test statistic defined as:

$$\text{LRuc} = 2(\log[\hat{\alpha}^{x_\alpha} (1 - \hat{\alpha})^{m-x_\alpha}] - \log[\alpha^{x_\alpha} (1 - \alpha)^{m-x_\alpha}])$$

- Expected Shortfall

The Expected Shortfall (ES) is formally defined by:

$$\text{ES}_t(\alpha) = E[y_t | y_t < \text{VaR}_t(\alpha)]$$

$$\text{ES}_t(\alpha) = E[y_t | y_t > \text{VaR}_t(\alpha)]$$

Following Aas and Haff (2006) and Nakajima (2013), we compute the measure developed by Embrechts et al. (2004) for evaluating the performance of the predicted ES, denoted by  $\hat{ES}_t(\alpha)$ . We define

$$\delta_t(\alpha) = y_t - \hat{ES}_t(\alpha)$$

as an excess of return. Let  $\delta_\alpha$  be the  $\alpha$ th quantile of  $[\delta_t(\alpha)]_{t=T+1}^{T+m}$ . Next, we define:

$$S_t(\alpha) = I[\delta_t(\alpha) < \delta_\alpha]$$

$$S_t(\alpha) = I[\delta_t(\alpha) > \delta_\alpha]$$

for the left and right tail. Then, let be  $s_\alpha = \sum_{t=T+1}^{T+m} S_t(\alpha)$ . Now we have all the inputs to compute Embrechts et al. (2004) measure given by:

$$D(\alpha) = \frac{1}{2} [|D_1(\alpha)| + |D_2(\alpha)|]$$

where

$$D_1(\alpha) = \frac{1}{x_\alpha} \sum_{I_t(\alpha)=1} \delta_t(\alpha)$$

$$D_2(\alpha) = \frac{1}{s_\alpha} \sum_{S_t(\alpha)=1} \delta_t(\alpha)$$

As discussed in Aas and Haff (2006) and Nakajima (2013),  $D_1(\alpha)$  is the standard back-testing measure for expected shortfall estimates. Its weakness is that it strongly depends on the VaR estimates without adequately reflecting the correctness of these values.  $D_2(\alpha)$  is computed to correct it because it measures an average difference between the return and the estimated ES for the  $\alpha$ -level tail of that difference from all the tested periods. A smaller  $D(\alpha)$  implies a more precise prediction of ES.

## 2.7 Bayesian Model Comparison

To compare the goodness of fit for the estimated models we cannot use the standard Deviance Information Criterion (DIC) because our complete posterior distribution of  $\Theta$  is not Multivariate Normal, Ando (2010, 2011). To overcome this issue, Abanto & Garrafa (2019) rely on the Watanabe-Akaike Information Criterion (hereinafter, WAIC) Watanabe (2010, 2013). The WAIC is defined as:

$$\text{WAIC} = -2 (\text{lppd} - p_{\text{WAIC}}) \quad (4)$$

where lppd is the log point-wise predictive density defined by:

$$\text{lppd} = \sum_{t=1}^T \ln \int p(y_t | \Theta, \mathbf{h}_{1:T}) \cdot p(\mathbf{h}_{1:T} | \mathbf{y}_{1:T}) d\Theta d\mathbf{h}_{1:T} \quad (5)$$

and

$$p_{\text{WAIC}} = 2 \sum_{t=1}^T \left[ \ln E_{\Theta, \mathbf{h}_{1:T} | \mathbf{y}_{1:T}} [p(y_t | \Theta, \mathbf{h}_{1:T})] - E_{\Theta, \mathbf{h}_{1:T} | \mathbf{y}_{1:T}} [\ln p(y_t | \Theta, \mathbf{h}_{1:T})] \right] \quad (6)$$

To estimate (5) and (6) we use the MCMC output with the computations given by:

$$\begin{aligned} \widehat{\text{lppd}} &= \sum_{t=1}^T \ln \left( \frac{1}{I} \sum_{i=1}^I p(y_t | \Theta_i) \right) \\ \widehat{p}_{\text{WAIC}} &= 2 \sum_{t=1}^T \left( \ln \left( \frac{1}{I} \sum_{i=1}^I p(y_t | \Theta_i) \right) - \frac{1}{I} \sum_{i=1}^I \ln p(y_t | \Theta_i) \right) \end{aligned}$$

where  $i = 1, \dots, I$  is the  $i$ -th iteration. The minimum value of the WAIC indicates evidence of the best fit.

## 3 Empirical Application

### 3.1 Data

We use daily adjusted closing price data for all our markets sample. Also, data-sets were downloaded from the Yahoo Finance repository. Given the adjusted closing price  $P_t$  on day  $t$  and apply the following transformation in order to obtain our compound return data-sets:

$$y_t = 100 \times (\ln P_t - \ln P_{t-1})$$

The period of analysis ranges from January 5th, 1998 – April 30th, 2022. The starting point follows Abanto & Garrafa (2019) and end-to-date point has been chosen with the last closing month at the recollection. We have selected four regions for the analysis: North America, Latam, Europe and Asia. Within region, four Stock Market Indices were chosen. Election criteria was defined by: (i) availability in the main source, (ii) Stock Market Index that have been already studied were preferred and (iii) well-known Indices in the region.

The African region and country Peru were excluded due to: (i) liquidity, (ii) relative peer-to-peer volume of transactions and (iii) availability in the main data source. Table 1 presents a complete description of the sample analysis. We use the country code according to ISO-3166 Alpha-3 criteria. SMX Code Reference names are for abbreviations within the document. Tables 2, 3, 4 and 5 proffer the summary statistics for each region. For the mode, we compute the Meanshift mode estimator developed by Fukunaga, K. and Hostetler, L. (1975).



Table 1: Regions — Sample Analysis

Region	Country	ISO-3166	Stock Market Index	SMX Code Reference
North America				
	United States	USA	Standard & Poor's 500	S&P500
	United States	USA	National Association of Securities Dealers Automated Quotation	Nasdaq
	United States	USA	Dow Jones Industrial Average	DowJones
	Canada	CAN	S&P/TSX Composite Index	TSX
LATAM				
	Brasil	BRA	IBOVESPA	Bovespa
	Mexico	MEX	IPC Mexico	IPC
	Argentina	ARG	S&P Merval	Merval
	Chile	CHL	S&P/CLX IPSA	IPSA
Europe				
	Belgium	BEL	Euronext Bruselas	BFX
	Germany	GER	DAX Performance Index	DAX
	England	GBR	Financial Times Stock Exchange	FTSE 100
	France	FRA	Cotation Assistée en Continu	CAC 40
Asia				
	Singapore	SGP	Straits Times Index	STI
	Japan	JPN	Nikkei 225	Nikkei
	South Korea	KOR	Korea Composite Stock Price Index	KOSPI
	Hong Kong	HKG	Hang Seng Index	Hang Seng

From the data, we describe the following main characteristics:

- All regions show evidence of leptokurtic distributions. This has been intensively described in Liesenfeld & Jung (2000), Chib et al (2002), So et al (2002), Jacquier et al (2004) and Abanto et al (2010).
- North America exhibits left sided asymmetries in all its markets. In LATAM, only Merval supports the statement (with even a higher degree). This is our first indicator that markets have a non-homogeneous behaviour.
- In LATAM, IPC (MEX) and IPSA (CHL) distance themselves from their peers in extreme values. Not all markets in LATAM show the greatest extreme values.
- In the EURO region, we highlight the within narrower volatility (given by the SD) compared with other regions. The region also exhibits leptokurtic distributions and left-sided asymmetries.
- Regarding Asia (and among all regions), Singapore exhibits the only negative mode value.
- Markets with the higher mean are Kospi (KOR), Merval (ARG) and Nasdaq (USA).

Table 2: Summary statistics — North America

Return	size	mean	median	mode	min	max	SD	Skewness	Kurtosis	N° Returns $\geq 0$
S&P500	6,120	0.02356	0.06371	0.04319	-12.7652	10.9572	1.23928	-0.39131	13.20159	3,283
Nasdaq	6,120	0.03343	0.11055	0.10878	-13.1491	13.2546	1.59957	-0.16810	8.94648	3,339
DowJones	6,120	0.02319	0.04972	0.03465	-13.8418	10.7643	1.18699	-0.37850	15.31859	3,246
TSX (CAN)	6,111	0.01841	0.07449	0.07853	-13.1758	11.2945	1.11506	-0.91821	18.99115	3,320

Table 3: Summary statistics — LATAM

Return	size	mean	median	mode	min	max	SD	Skewness	Kurtosis	N° Returns $\geq 0$
Bovespa (BRA)	6,014	0.03857	0.07802	0.08824	-17.2082	28.8324	1.95694	0.24777	17.74931	3,118
IPC (MEX)	6,102	0.03755	0.05212	0.01960	-10.3409	12.1536	1.35251	0.08664	9.03298	3,194
Merval (ARG)	4,947	0.08257	0.12633	0.10869	-47.6922	16.1165	2.35173	-1.64371	35.1129	3,165
IPSA (CHL)	4,345	0.03339	0.04515	0.01049	-7.2362	11.8033	0.96232	0.02869	13.33454	2,270

Table 4: Summary statistics — EURO

Return	size	mean	median	mode	min	max	SD	Skewness	Kurtosis	N° Returns $\geq 0$
BFX (BEL)	6,203	0.00777	0.04136	0.01738	-15.3275	9.3339	1.25385	-0.37676	12.13572	3,244
DAX (GER)	6,167	0.01882	0.07888	0.13420	-13.0548	10.7974	1.49208	-0.17273	8.39919	3,277
FTSE 100	6,143	0.00586	0.04832	0.09382	-11.5117	9.3842	1.18935	-0.30902	10.16639	3,190
CAC 40 (FRA)	6,205	0.01216	0.05073	0.02969	-13.0983	10.5945	1.43304	-0.20573	8.95752	3,225

Table 5: Summary statistics — ASIA

Return	size	mean	median	mode	min	max	SD	Skewness	Kurtosis	N° Returns $\geq 0$
STI (SGP)	6,086	0.01328	0.02446	-0.01003	-9.1534	12.8737	1.22369	-0.05135	11.67528	3,116
Nikkei 225	5,961	0.00981	0.04121	0.01841	-12.1110	13.2345	1.48005	-0.31280	8.78926	3,070
KOSPI (KOR)	6,205	0.03194	0.07221	0.05462	-12.8047	11.2843	1.63890	-0.19682	8.99814	3,212
Hang Seng	5,997	0.01194	0.05170	0.06896	-13.5820	13.4068	1.54820	0.08494	10.30979	3,103

Next, we assess the data-sets and evaluate the linear and quadratic dependence with the auto-correlation function (hereinafter, ACF). Figure 1 and 2 prints the ACF plots for the compound returns and the quadratic compound returns in our sample analysis.

Figure 1: Stock Market Indices' ACF

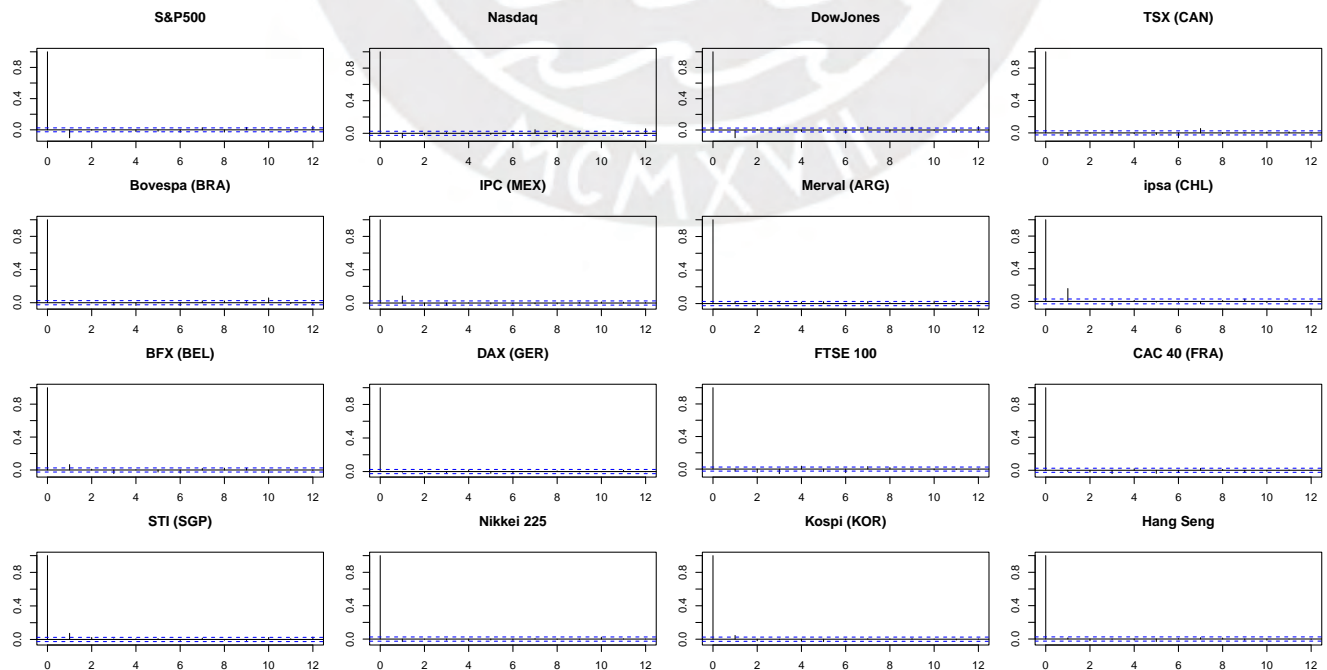
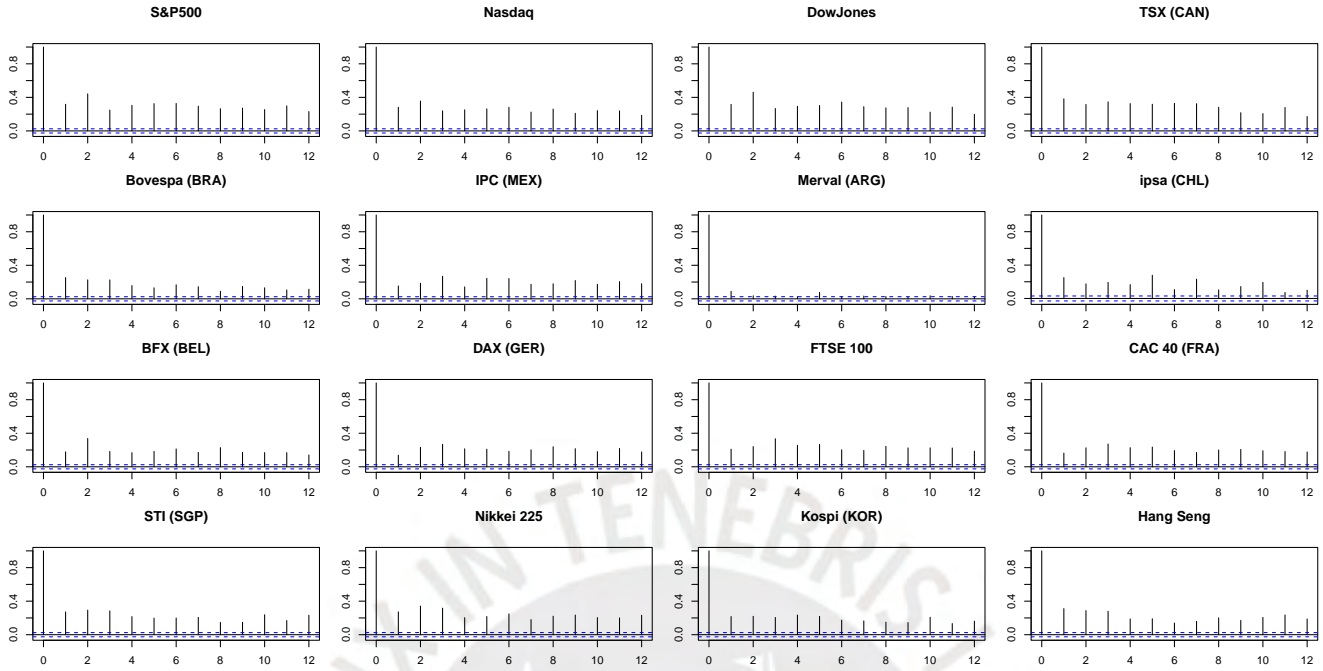


Figure 2: Squared Stock Market Indices' ACF



As expected, evidence supports the widely studied stylized fact of the nonlinear dependence of stock market indices: while the ACF on the compound returns cannot show evidence on linear dependence, the ACF plots in the quadratic compound returns clearly reports dependence. Nevertheless, IPSA (CHL) and Merval (ARG) stand out: the first one shows evidence of a first order linear dependence in the compound return ACF plot. The second one, even when still reflects evidence of a non-linear dependence, it only holds for the first lag. So far, we recall that we are not only observing differences in the summary statistics but also in the linear and quadratic auto correlation dependence. We also have computed the Durbin-Watson statistic in order to test for first-order serial auto correlation. The statistic is defined bellow and Table 6 presents the results for our sample analysis:

$$DW = \frac{\sum_{t=2}^n (x_t - x_{t-1})^2}{\sum_{t=1}^n x_t^2} \quad (7)$$

With the  $DW$  statistic we test the null hypothesis that stands the following statement: there is not first order serial auto-correlation. For our compound return data-sets, in all cases we cannot reject the null hypothesis, as expected. Yet, in the quadratic compound returns, even when we clearly reject the null for the majority of the stock market indices, IPC (MEX), Merval (ARG) and IPSA (CHL) for Latam, and BFX (BEL) and DAX (GER) for Euro cannot reject the null hypothesis.

Finally, Figure 3 plot the densities of the compound returns data-sets group by region. These densities were estimated using the R base `density` function: its algorithm disperses the mass of the empirical distribution function over a regular grid of at least 512 points and then uses the fast Fourier transform to convolve this approximation with a discretized version of the kernel and then uses linear approximation to evaluate the density at the specified points, Venables, W. N. & Ripley, B. D. (2002). We prefer this strategy instead of plotting the histograms because overlaying panes are not that clear. This visualization may also anticipate evidence of heterogeneous behavior between stock market indices.

Table 6: Durbin Watson Statistics

North America				
	S&P500	Nasdaq	DowJones	TSX (CAN)
$y_t$	2.2009	2.1073	2.1966	2.0666
$y_t^2$	<b>1.2609</b>	<b>1.2742</b>	<b>1.2775</b>	<b>1.1692</b>
LATAM				
	Bovespa (BRA)	IPC (MEX)	Merval (ARG)	IPSA (CHL)
$y_t$	2.0307	1.8286	1.9463	1.6821
$y_t^2$	<b>1.4081</b>	1.5054	1.7680	<b>1.3859</b>
EURO				
	BFX (BEL)	DAX (GER)	FTSE 100 (GBR)	CAC 40 (FRA)
$y_t$	1.8688	2.0154	2.0553	2.0308
$y_t^2$	1.5080	1.5173	<b>1.4242</b>	<b>1.4868</b>
ASIA				
	STI (SGP)	Nikkei 225 (JPN)	KOSPI (KOR)	Hang Seng (HKG)
$y_t$	1.8519	2.0595	1.9088	1.9726
$y_t^2$	<b>1.3311</b>	<b>1.2897</b>	<b>1.3915</b>	<b>1.2431</b>

Note.

Bold numbers indicate we reject  $H_0$ : there is not first order serial auto-correlation.

### 3.2 Results

We have submitted the extended THSV-SMN original model (fixed-zero threshold) and the extended model with our proposed RWMH methodology. First, in order to set up the priors for the parameters, we follow Abanto & Garrafa (2019) and for most cases we use non-informative priors. Those are defined as:

$$\begin{aligned}
 \psi_i &\sim \mathcal{N}_{2[\beta_i < 1]}(\bar{\psi}_i, \mathbf{S}_{\psi_i}) & \bar{\psi}_i &= (0, 0)' & \mathbf{S}_{\psi_i} &= \text{diag}(100, 100) \\
 \varphi_i &\sim \mathcal{N}_{2[\phi_i < 1]}(\bar{\varphi}_i, \mathbf{S}_{\varphi_i}) & \bar{\varphi}_i &= (0, 0.98)' & \mathbf{S}_{\varphi_i} &= \text{diag}(100, 100) \\
 \sigma_i^2 &\sim \mathcal{IG}(5, 0.5) \quad \forall i = 0, 1 & r &= 0
 \end{aligned}$$

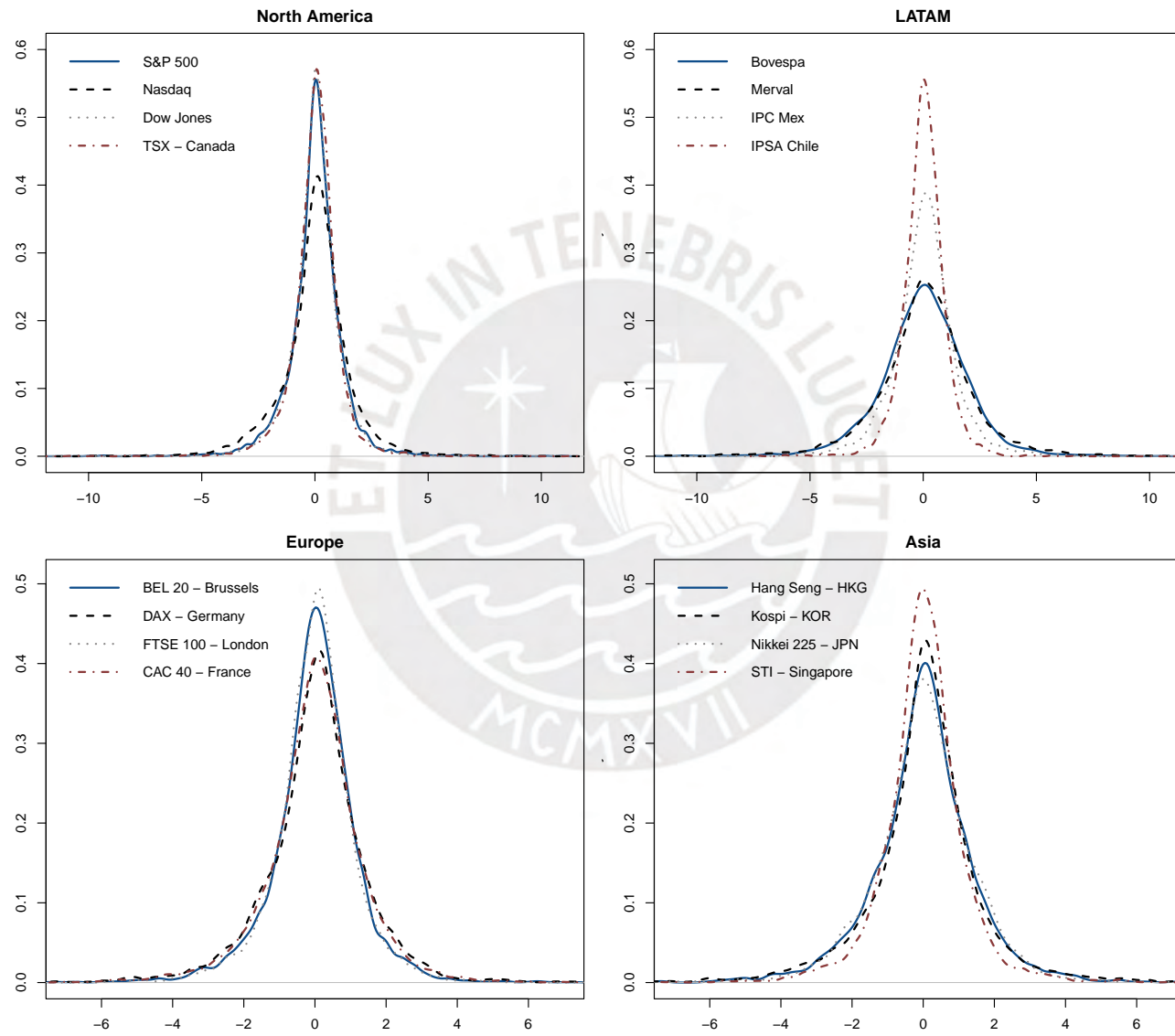
Second, the full posterior conditional distributions are detailed in Appendix 6. Third, for each data-set we have selected 40,000 MCMC iterations where the first 10,000 draws correspond to a burn-in period. Fourth, we have computed the posterior mode, the 95% credible intervals and the convergence diagnostic (CD statistics). In the latter, the null hypothesis states the sequence is stationary and we cannot reject the null with a 95% confidence if  $CD \in (-1.96 : 1.96)$ . Finally, Tables 7, 8, 9 and 10 refer details on the estimation results.

Regarding the estimation results, we present key-points we highlight under our analysis:

- Location ( $\mu_0$  and  $\mu_1$ ) and persistence ( $\beta_0$  and  $\beta_1$ ) parameters in the compound return process vary both in magnitude and sign among stock market indices and regions. Likewise summary statistics in Tables 2, 3, 4 and 5 this is evidence that markets do behave differently within and across regions.
- Posterior modes and 95% credibility intervals of  $\phi_0$  and  $\phi_1$  are close to unity. This results reflect a high degree of persistence in the log-volatility process and were also found by Harvey & Shephard (1996), Chen et al (2008), Wirjanto et al (2016) and Abanto & Garrafa (2019).

## Data densities

Figure 3: Compound returns densities





- $\sigma_0^2$  and  $\sigma_1^2$ , on the other hand, are close to zero. This indicates that the log-volatility process has a low degree of random variability.
- The threshold impact relies on both the compound return and the log-volatility equations: in the first one over its persistence ( $\beta_0$  and  $\beta_1$ ) and, in the second one, over its location ( $\alpha_0$  and  $\alpha_1$ ). Even with the two regimes, location ( $\mu_0$  and  $\mu_1$ ), persistence ( $\phi_0$  and  $\phi_1$ ) and variance ( $\sigma_0^2$  and  $\sigma_1^2$ ) show no significant differences on the posterior modes. Threshold posterior mean is positive located slightly close to the mean of the compound returns data mean.
- Location parameters  $\alpha_0$  and  $\alpha_1$  in the log-volatility process are also close to zero. This results could also indicate that a random walk process with no drift and lower variability may be another process to properly describe the log volatility behavior. This also has been mentioned in Harvey et al (1994) and studied in Harvey & Shephard (1996). Nevertheless, authors conclude that allowing  $h_t$  to follow a more complex linear process does not affect the basic statistical issues.

In addition we have selected the S&P500 Stock Market Index in order to present (i) posterior densities and (ii) box-plots. Figures 4, 5, 6, 7, 8, 10 and 9 account for those items. Furthermore, Figure 11 presents the log-volatility posterior mode and the 95% confidence interval. It also presents the  $h_t$  ratio computed by  $\exp(h_t^1 - h_t^0)$  where  $h_t^1$  is the log volatility chain with the endogenous threshold and  $h_t^0$  is the log volatility chain with the fixed zero threshold. Finally, Figure 12 shows the absolute compound returns for our representative market and the instant volatility computed by  $\exp(h_t/2) \cdot \lambda_t^{-1/2}$ . A key-point we highlight at first glance: estimation results supports the existence of a custom-specific threshold over the S&P500.

After the individual results of our representative stock market index case study, next in Figures 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 and 25 we present the joint posterior densities group by region for each estimated parameter. The joint results clearly illustrate how stock market indices do behave differently within and across regions when we study the same parameter across its peers. LATAM and Asia, i.e. regions with a higher degree of variability in the compound return data, show the most asymmetric behavior. We also have taken into consideration plotting the comparison for regime switching grouped by region, but keeping in mind that each pane would result in eight overlays instead of four and that the resulting figures were clumsy and impractical, we decided not to include them.

Figure 4: (S&P 500) —  $\mu_0$  and  $\mu_1$

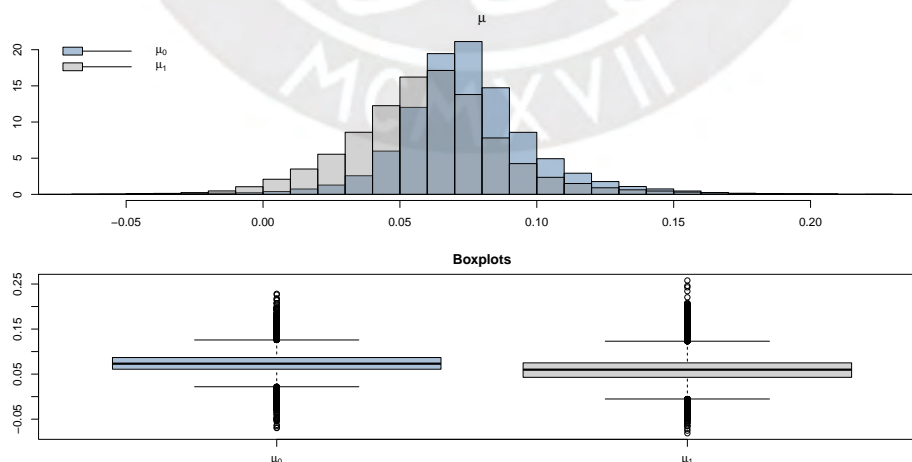


Figure 5: (S&P 500) —  $\beta_0$  and  $\beta_1$

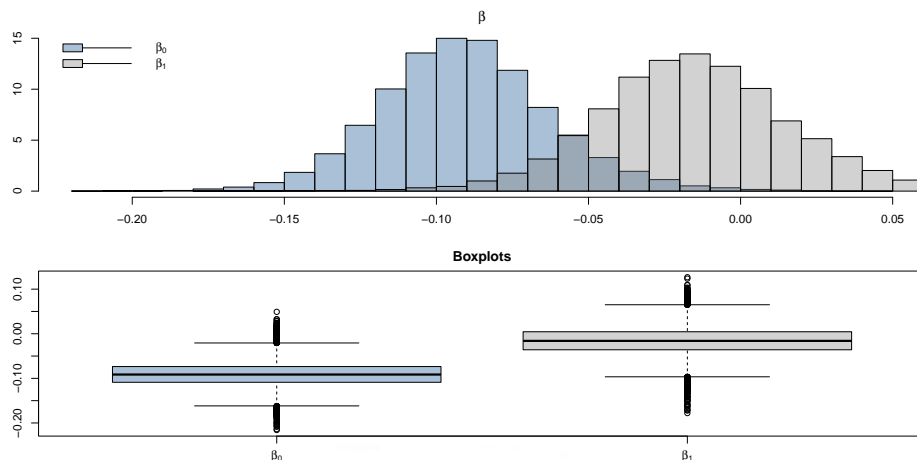


Figure 6: (S&P 500) —  $\alpha_0$  and  $\alpha_1$

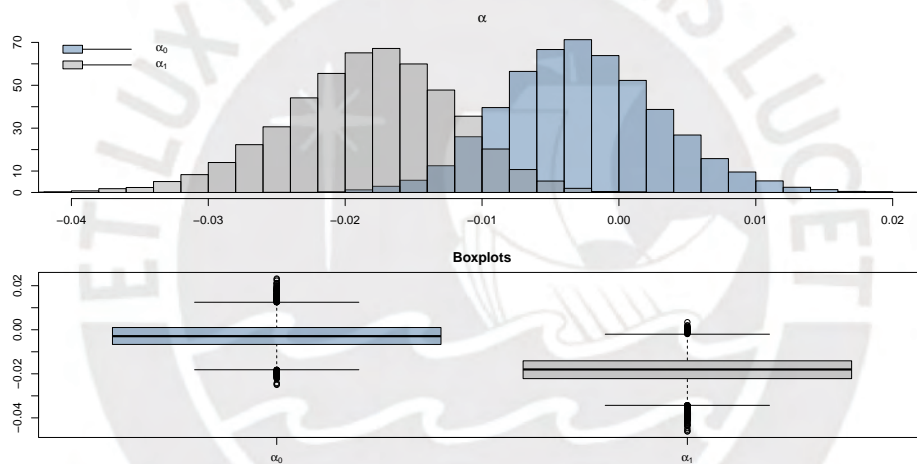


Figure 7: (S&P 500) —  $\phi_0$  and  $\phi_1$

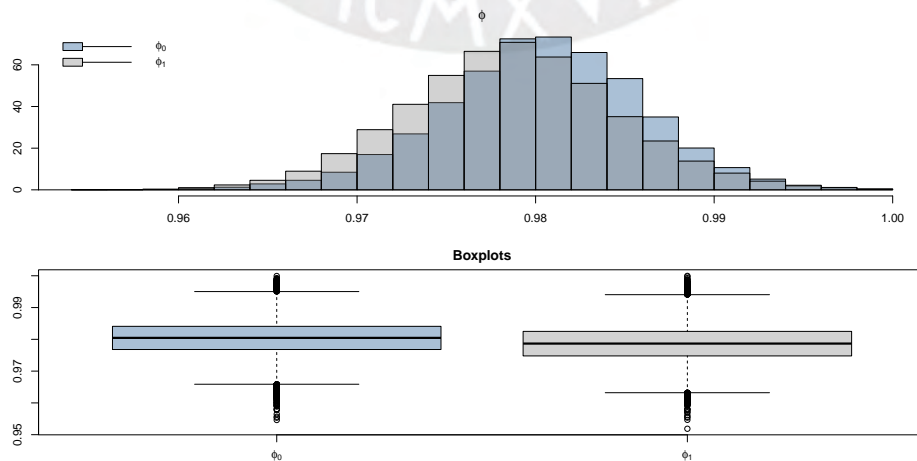


Figure 8: (S&P 500) —  $\sigma_0^2$  and  $\sigma_1^2$

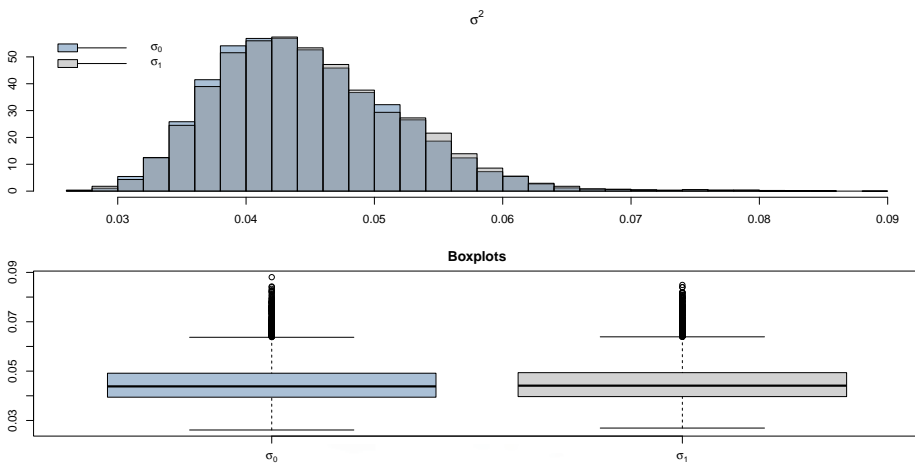


Figure 9: (S&P 500) —  $r$

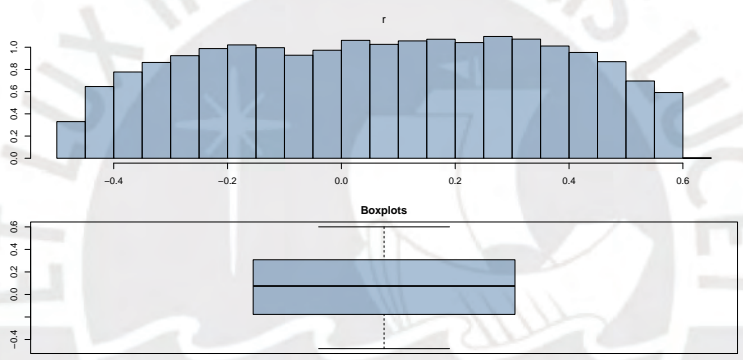
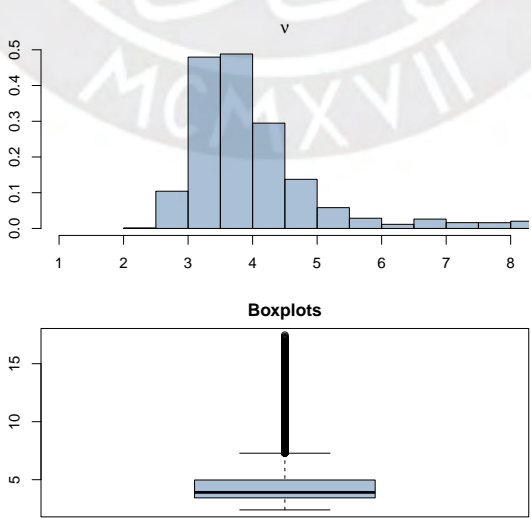


Figure 10: (S&P 500) —  $\nu$



## Estimation results — THSV-S

Table 7: Estimation results — North America — THSV-S

Parameter	S&P500 (US) THSV-S		Nasdaq (US) THSV-S		Dow Jones (US) THSV-S		TSX (CAN) THSV-S	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
	$\mu_0$	0.077143 (0.0350 : 0.1136) 0.6941	0.071325 (0.0300 : 0.1353) -0.3565	0.130044 (0.0756 : 0.1832) 0.9744	0.107274 (0.0076 : 0.1674) 0.5259	0.061210 (0.0183 : 0.0953) -0.0050	0.063872 (-0.0237 : 0.0959) -0.2281	0.061664 (0.0230 : 0.0964) -0.8580
$\mu_1$	0.055394 (0.0179 : 0.0893) -1.4350	0.061398 (0.0025 : 0.1247) 0.9278	0.090336 (0.0388 : 0.1327) 0.0146	0.109861 (0.0199 : 0.1714) -0.7169	0.071239 (0.0327 : 0.1036) -1.0120	0.068310 (0.0228 : 0.1368) -0.7479	0.056638 (0.0228 : 0.0913) -0.1511	0.055041 (0.0016 : 0.1087) 1.3330
$\beta_0$	-0.091490 (-0.1429 : -0.0381) 1.8050	-0.097160 (-0.1433 : -0.0305) -1.8060	-0.026391 (-0.0798 : 0.0265) 1.5400	-0.037524 (-0.1068 : 0.0125) -0.6931	-0.077979 (-0.1350 : -0.0287) 1.2240	-0.073315 (-0.1565 : -0.0311) -2.2810	0.014783 (-0.0355 : 0.0704) -0.1476	0.020409 (-0.0342 : 0.0816) -0.7035
$\beta_1$	-0.008954 (-0.0612 : 0.0391) 1.7490	-0.017147 (-0.0784 : 0.0474) -0.1501	-0.002625 (-0.0561 : 0.0454) 1.0340	-0.021378 (-0.0759 : 0.0470) -0.3596	-0.027903 (-0.0762 : 0.0234) 1.1920	-0.028956 (-0.0931 : 0.0262) 1.7300	0.052377 (-0.0084 : 0.1018) -0.5656	0.050403 (-0.0144 : 0.1188) -1.6090
$\alpha_0$	-0.003304 (-0.0138 : 0.0076) -1.4620	-0.002113 (-0.0131 : 0.0100) -7.6490	0.007042 (-0.0020 : 0.0153) -1.3570	0.005260 (-0.0026 : 0.0163) 1.6600	-0.004713 (-0.0165 : 0.0057) -0.9395	-0.005183 (-0.0162 : 0.0065) -2.0610	-0.004275 (-0.0145 : 0.0051) 0.7895	-0.004425 (-0.0141 : 0.0063) -1.7330
$\alpha_1$	-0.018235 (-0.0295 : -0.0074) -0.4448	-0.017443 (-0.0310 : -0.0068) -2.1470	-0.003384 (-0.0108 : 0.0040) -1.2990	-0.002321 (-0.0145 : 0.0040) 1.5790	-0.020520 (-0.0323 : -0.0090) -0.4014	-0.019298 (-0.0332 : -0.0088) -0.9535	-0.015801 (-0.0264 : -0.0066) 2.0760	-0.015577 (-0.0268 : -0.0058) -2.2100
$\phi_0$	0.981696 (0.9697 : 0.9903) -0.0428	0.979762 (0.9676 : 0.9906) 6.2550	0.987479 (0.9777 : 0.9945) 1.1300	0.987192 (0.9782 : 0.9956) 0.0224	0.980108 (0.9683 : 0.9893) -0.1625	0.979526 (0.9689 : 0.9905) 1.6890	0.986336 (0.9757 : 0.9938) 0.2982	0.985170 (0.9753 : 0.9944) 0.8248
$\phi_1$	0.977752 (0.9676 : 0.9874) -0.2923	0.978087 (0.9660 : 0.9902) 5.1850	0.983827 (0.9758 : 0.9918) 1.3290	0.986076 (0.9765 : 0.9946) -0.5259	0.977041 (0.9667 : 0.9872) -0.2548	0.977415 (0.9673 : 0.9899) 1.5210	0.982272 (0.9730 : 0.9904) 0.5735	0.982144 (0.9726 : 0.9921) 0.6199
$\sigma_0^2$	0.044214 (0.0333 : 0.0560) 0.0670	0.043103 (0.0341 : 0.0624) -5.1950	0.031643 (0.0238 : 0.0412) 0.2195	0.030256 (0.0230 : 0.0397) 1.2170	0.044710 (0.0333 : 0.0566) 0.1142	0.042502 (0.0312 : 0.0555) -1.1290	0.034153 (0.0233 : 0.0437) -0.3921	0.033999 (0.0265 : 0.0439) -0.3988
$\sigma_1^2$	0.043764 (0.0335 : 0.0561) 0.0750	0.045080 (0.0343 : 0.0631) -5.2610	0.031026 (0.0239 : 0.0412) 0.2083	0.030821 (0.0231 : 0.0399) 1.4770	0.043383 (0.0335 : 0.0568) 0.0841	0.042501 (0.0313 : 0.0558) -1.0830	0.034148 (0.0233 : 0.0437) -0.4075	0.034833 (0.0264 : 0.0438) -0.3934
$\nu$	3.405908 (2.9495 : 6.0661) -0.6687	3.697091 (2.9560 : 15.2593) -3.8410	4.338820 (3.4043 : 8.7556) -0.6115	3.918407 (3.1710 : 8.4834) 1.2790	3.458358 (2.9180 : 6.1752) -0.3132	3.558294 (2.7255 : 6.6051) -1.3970	4.628053 (3.7390 : 15.1603) -0.1157	6.264360 (4.5573 : 36.6905) -3.7980
$r$	0 - -	0.074954 (-0.4304 : 0.5587) -0.2892	0 - -	0.075840 (-0.5753 : 0.7367) -0.7548	0 - -	0.054215 (-0.4244 : 0.5262) 0.5006	0 - -	0.043854 (-0.4013 : 0.5115) -0.1913

Note. First row: posterior mode. Second row: posterior 95% credible interval in parenthesis. Third row: CD statistics.  
Index (1) stands for the original THSV-SMN, (2) stands for the extended THSV-SMN model with endogenous threshold,

Table 8: Estimation results — LATAM — THSV-S

Parameter	Bovespa (BRA)		IPC MEXICO		Merval (ARG)		IPSA (CHL)	
	THSV-S		THSV-S		THSV-S		THSV-S	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
$\mu_0$	0.038556 (-0.0489 : 0.1121) -0.4658	0.056476 (-0.1032 : 0.1281) 2.6220	0.021441 (-0.0276 : 0.0699) -0.52080	0.040150 (-0.1162 : 0.0726) 1.8440	-0.042499 (-0.1267 : 0.0546) 0.09962	0.056381 (-0.3242 : 0.1243) 0.1187	0.009242 (-0.0389 : 0.0526) 0.9692	0.031134 (-0.1463 : 0.0610) 1.5940
$\mu_1$	0.056351 (-0.0261 : 0.1346) -0.2434	0.063918 (-0.1069 : 0.1310) -1.6370	0.054488 (0.0069 : 0.1009) 2.9690	0.055960 (-0.0234 : 0.1112) 0.3099	0.161185 (0.0792 : 0.2509) 2.7190	0.136001 (0.0644 : 0.3210) 0.4748	0.071588 (0.0295 : 0.1201) -0.2682	0.073683 (0.0290 : 0.1756) 0.14230
$\beta_0$	-0.061492 (-0.1160 : -0.0011) -0.6494	-0.045886 (-0.1330 : 0.0029) 1.9040	0.022905 (-0.0279 : 0.0795) -1.0830	0.040012 (-0.0719 : 0.0832) 1.8470	-0.057686 (-0.1143 : -0.0061) 0.3779	-0.040158 (-0.1730 : 0.0309) 0.1334	0.120988 (0.0525 : 0.1900) 1.5000	0.147526 (-0.0382 : 0.2035) 1.5800
$\beta_1$	0.018837 (-0.0355 : 0.0761) -0.6747	0.023650 (-0.0311 : 0.1063) 2.85	0.082007 (0.0304 : 0.1318) -1.5110	0.082552 (0.0305 : 0.1487) 0.4731	0.058443 (0.0057 : 0.1084) -2.4900	0.071361 (-0.0116 : 0.1158) -0.9213	0.149526 (0.0876 : 0.2159) 1.5600	0.148402 (0.0454 : 0.2189) -0.2033
$\alpha_0$	0.012743 (0.0045 : 0.0240) -1.199	0.013189 (0.0027 : 0.0234) 3.9990	0.000377 (-0.0058 : 0.0079) -1.6770	0.000555 (-0.0060 : 0.0086) -1.847	0.026963 (0.0125 : 0.0450) -1.3140	0.027000 (0.0111 : 0.0476) 0.2740	-0.011695 (-0.0255 : 0.0020) 3.2230	-0.012440 (-0.0278 : 0.0035) 1.4510
$\alpha_1$	0.010083 (0.0019 : 0.0197) -1.0440	0.010098 (-0.0015 : 0.0196) 2.2550	-0.006597 (-0.0136 : 0.0000) -4.5120	-0.006247 (-0.0148 : 0.0004) -2.382	0.022081 (0.0097 : 0.0376) -1.3050	0.023143 (0.0056 : 0.0390) 0.2061	-0.025700 (-0.0447 : -0.0142) 1.6970	-0.027166 (-0.0476 : -0.0141) 0.0910
$\phi_0$	0.984674 (0.9744 : 0.9932) 2.1660	0.984387 (0.9748 : 0.9953) -3.0140	0.988689 (0.9748 : 0.9961) -0.5269	0.988406 (0.9799 : 0.9966) -0.4375	0.970817 (0.9546 : 0.9843) 1.8140	0.970715 (0.9534 : 0.9855) -0.5744	0.973190 (0.9562 : 0.9871) -0.9419	0.972129 (0.9548 : 0.9901) -1.1010
$\phi_1$	0.979933 (0.9698 : 0.9891) 1.7910	0.980188 (0.9700 : 0.9911) -2.6860	0.984316 (0.9742 : 0.9910) -0.7469	0.985255 (0.9749 : 0.9929) 0.2772	0.966023 (0.9505 : 0.9794) 1.5790	0.967039 (0.9481 : 0.9800) -0.3773	0.962423 (0.9440 : 0.9770) -0.8917	0.962955 (0.9428 : 0.9794) -0.7844
$\sigma_0^2$	0.019102 (0.0147 : 0.0266) -1.5200	0.018926 (0.0137 : 0.0267) 2.4140	0.023181 (0.0162 : 0.0310) 0.6474	0.021640 (0.0168 : 0.0301) 0.0191	0.041699 (0.0291 : 0.0601) -1.5200	0.042031 (0.0302 : 0.0638) 0	0.041158 (0.0292 : 0.0575) 0.8793	0.039847 (0.0300 : 0.0595) 0.7033
$\sigma_1^2$	0.019216 (0.0146 : 0.0264) -1.4140	0.019577 (0.0137 : 0.0266) 2.4780	0.023110 (0.0161 : 0.0306) 0.6128	0.021065 (0.0167 : 0.0299) 0.0590	0.042154 (0.0289 : 0.0596) -1.5650	0.040753 (0.0303 : 0.0636) 0.3425	0.040654 (0.0289 : 0.0567) 0.9397	0.040342 (0.0298 : 0.0585) 0.6513
$\nu$	4.416235 (3.6340 : 6.5000) -0.0228	4.022336 (3.2676 : 7.2825) 1.0820	3.594941 (2.8142 : 5.5591) -1.1700	3.606456 (2.8870 : 5.6645) -0.6882	2.590226 (2.2238 : 3.1517) 0.3342	2.610205 (2.2311 : 3.1536) -0.6603	7.367838 (4.8186 : 42.0700) 1.636	8.538462 (5.3043 : 34.1569) 1.9220
$r$	0 - -	0.034324 (-0.8861 : 0.9994) 0.9471	0 - -	0.064120 (-0.5471 : 0.6674) 1.5340	0 - -	0.006715 (-0.9228 : 1.1023) 0.1582	0 - -	0.044785 (-0.4163 : 0.5140) 1.5010

Note. First row: posterior mode. Second row: posterior 95% credible interval in parenthesis. Third row: CD statistics. Index (1) stands for the original THSV-SMN, (2) stands for the THSV-SMN model with extensions.



Table 9: Estimation results — EURO — THSV-S

Parameter	BEL 20 (BEL) THSV-S		DAX (GER) THSV-S		FTSE 100 (GBR) THSV-S		CAC 40 (FRA) THSV-S	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
$\mu_0$	0.080628 (0.0374 : 0.1254) -1.0680	0.062905 (0.0259 : 0.1498) 1.5890	0.107558 (0.0557 : 0.1635) 0.6646	0.096169 (0.0370 : 0.1815) 1.4370	0.027008 (-0.0152 : 0.0713) -0.3025	0.038612 (-0.0447 : 0.0850) -0.9852	0.095417 (0.0504 : 0.1497) 0.8993	0.083470 (0.0135 : 0.1550) -0.8342
$\mu_1$	0.024721 (-0.0167 : 0.0666) -0.0934	0.043260 (-0.0639 : 0.0784) 2.0240	0.071115 (0.0189 : 0.1168) -2.1700	0.082578 (-0.0728 : 0.1179) 1.1800	0.042499 (-0.0018 : 0.0797) 0.1039	0.038765 (-0.0280 : 0.0998) 0.0598	0.033863 (-0.0125 : 0.0815) -0.6576	0.060527 (-0.0939 : 0.0951) 0.0126
$\beta_0$	0.048014 (-0.0094 : 0.1016) 0.3239	0.031274 (-0.0182 : 0.1083) 2.1230	-0.000007 (-0.0517 : 0.0548) 1.5590	-0.009638 (-0.0593 : 0.0572) 1.2230	-0.046178 (-0.1019 : 0.0081) -0.5147	-0.040206 (-0.1071 : 0.0161) -2.5420	-0.010646 (-0.0625 : 0.0411) -0.4049	-0.020646 (-0.0769 : 0.0391) -0.1943
$\beta_1$	0.071654 (0.0202 : 0.1240) -0.1493	0.057621 (0.0106 : 0.1518) -1.8980	-0.001909 (-0.0535 : 0.0461) -0.3889	-0.012548 (-0.0555 : 0.0880) -0.7757	-0.010308 (-0.0636 : 0.0439) -0.4928	-0.010870 (-0.0690 : 0.0579) -0.6005	0.015906 (-0.0342 : 0.0669) 1.8230	0.004966 (-0.0462 : 0.1084) 0.1582
$\alpha_0$	0.001399 (-0.0078 : 0.0103) 4.2290	-0.000911 (-0.0085 : 0.0105) -0.3382	0.003902 (-0.0036 : 0.0118) -5.7550	0.003470 (-0.0035 : 0.0141) -2.2160	-0.000442 (-0.0093 : 0.0080) -7.7590	-0.001240 (-0.0098 : 0.0090) -2.9700	0.002353 (-0.0046 : 0.0105) 1.9890	0.002224 (-0.0045 : 0.0121) -0.0862
$\alpha_1$	-0.011895 (-0.0218 : -0.0033) 3.2740	-0.012561 (-0.0249 : -0.0042) 0.0984	-0.004823 (-0.0118 : 0.0020) 0.0363	-0.004735 (-0.0158 : 0.0022) -1.3520	-0.012982 (-0.0223 : -0.0040) -1.5730	-0.012622 (-0.0227 : -0.0036) -1.2830	-0.006871 (-0.0147 : 0.0002) 1.3330	-0.006890 (-0.0166 : 0.0003) 1.1530
$\phi_0$	0.982076 (0.9708 : 0.9911) 1.0180	0.982117 (0.9719 : 0.9933) -1.3550	0.985880 (0.9765 : 0.9944) 3.2650	0.985185 (0.9756 : 0.9942) 1.6740	0.984234 (0.9730 : 0.9923) 4.3850	0.984469 (0.9730 : 0.9937) 3.0020	0.985713 (0.9753 : 0.9933) -2.2520	0.985670 (0.9757 : 0.9941) 2.2900
$\phi_1$	0.977258 (0.9653 : 0.9859) 0.8524	0.977251 (0.9668 : 0.9891) -2.4530	0.983614 (0.9736 : 0.9910) 4.1770	0.983333 (0.9737 : 0.9931) 0.4565	0.979399 (0.9697 : 0.9889) 3.9060	0.980786 (0.9697 : 0.9918) 2.9000	0.982728 (0.9732 : 0.9909) -1.0640	0.984132 (0.9738 : 0.9931) 2.8660
$\sigma_0^2$	0.034739 (0.0284 : 0.0497) -1.0390	0.035276 (0.0273 : 0.0461) 1.6990	0.025168 (0.0191 : 0.0371) -3.0460	0.026772 (0.0201 : 0.0369) -0.9555	0.030714 (0.0219 : 0.0421) -3.2250	0.030067 (0.0209 : 0.0436) -2.3160	0.028325 (0.0196 : 0.0377) 0.9606	0.025591 (0.0203 : 0.0359) -1.6740
$\sigma_1^2$	0.035542 (0.0281 : 0.0494) -1.0220	0.034882 (0.0272 : 0.0456) 1.5260	0.025204 (0.0190 : 0.0371) -3.0420	0.026691 (0.0201 : 0.0370) -1.0580	0.030954 (0.0218 : 0.0420) -3.0000	0.0293546 (0.0209 : 0.0438) -2.2270	0.027435 (0.0196 : 0.0381) 0.8266	0.025318 (0.0203 : 0.0361) -1.7560
$\nu$	5.177608 (3.7058 : 15.7965) 0.6123	4.919699 (3.5602 : 8.3856) -0.2922	3.218609 (2.6356 : 4.3890) -2.4650	3.201330 (2.6341 : 4.6525) -0.8468	5.199634 (3.3983 : 13.9329) -2.0140	3.961265 (3.2588 : 14.4082) -1.6140	2.985812 (2.6002 : 3.8539) 1.6720	2.994778 (2.5152 : 3.9383) 0.0819
$r$	0 - -	0.048120 (-0.5125 : 0.6001) -2.0900	0 - -	0.053080 (-0.6103 : 0.7064) -1.3860	0 - -	0.046499 (-0.5003 : 0.5624) -1.0390	0 - -	0.041038 (-0.5971 : 0.6795) -0.0926

Note. First row: posterior mode. Second row: posterior 95% credible interval in parenthesis. Third row: CD statistics.

Index (1) stands for the original THSV-SMN, (2) stands for the THSV-SMN model with extensions.

Table 10: Estimation results — ASIA — THSV-S

Parameter	STI Index (SGP) THSV-S		Nikkei 225 (JPN) THSV-S		KOSPI (KOR) THSV-S		Hang Seng Index (HKG) THSV-S	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
$\mu_0$	0.043753 (0.0047 : 0.0840) -0.2116	0.043147 (-0.0108 : 0.0944) 1.7780	0.027439 (-0.0388 : 0.0810) 1.8180	0.024949 (-0.1364 : 0.0763) -0.3030	0.030869 (-0.0128 : 0.0856) -1.6700	0.039095 (-0.0417 : 0.0953) 1.2810	-0.010777 (-0.0717 : 0.0447) 1.8620	0.023278 (-0.1576 : 0.0652) 1.4870
$\mu_1$	0.017652 (-0.0203 : 0.0585) -1.1150	0.028630 (-0.0884 : 0.0564) 2.8170	0.011874 (-0.0418 : 0.0750) 0.0485	0.030090 (-0.0562 : 0.1581) -0.3815	0.038263 (-0.0123 : 0.0820) 0.2244	0.041908 (-0.0249 : 0.1178) 0.1732	0.053394 (-0.0036 : 0.1051) -0.1896	0.039864 (-0.0584 : 0.1040) -1.0510
$\beta_0$	0.023148 (-0.0338 : 0.0723) -2.6260	0.016761 (-0.0400 : 0.0747) 2.1420	-0.076789 (-0.1347 : -0.0260) -0.0860	-0.085461 (-0.1743 : -0.0302) 1.4370	-0.030385 (-0.0796 : 0.0187) -2.5700	-0.028902 (-0.0860 : 0.0232) -0.9265	-0.044531 (-0.0974 : 0.0081) 1.2060	-0.017345 (-0.1286 : 0.0242) 1.6040
$\beta_1$	0.046229 (-0.0049 : 0.1015) -0.9139	0.044804 (-0.0029 : 0.1468) -3.6260	0.040451 (-0.0143 : 0.0913) -1.3180	0.023591 (-0.0494 : 0.0962) 0.4239	0.058254 (0.0095 : 0.1079) -0.0003	0.056462 (-0.0054 : 0.1107) 0.0178	0.038491 (-0.0129 : 0.0907) -0.1657	0.046197 (-0.0086 : 0.1096) 1.5820
$\alpha_0$	-0.001702 (-0.0097 : 0.0059) -10.3200	-0.003011 (-0.0107 : 0.0058) -1.0140	0.005665 (-0.0018 : 0.0147) -1.1660	0.005656 (-0.0021 : 0.0167) -0.7296	0.001514 (-0.0041 : 0.0074) -2.9380	0.001135 (-0.0043 : 0.0088) 1.6340	0.001481 (-0.0034 : 0.0067) -0.5053	0.001593 (-0.0035 : 0.0071) 3.6720
$\alpha_1$	-0.009529 (-0.0185 : -0.0021) -7.7080	-0.011060 (-0.0206 : -0.0027) 0.2364	-0.001258 (-0.0084 : 0.0063) -1.5520	-0.001230 (-0.0115 : 0.0069) -1.4140	-0.002401 (-0.0083 : 0.0026) -3.6630	-0.002733 (-0.0104 : 0.0027) -2.2680	-0.001987 (-0.0069 : 0.0027) -1.7360	-0.002152 (-0.0080 : 0.0028) 1.1270
$\phi_0$	0.987015 (0.9790 : 0.9950) 7.1990	0.988747 (0.9799 : 0.9962) 1.1750	0.978071 (0.9655 : 0.9891) 0.2341	0.978212 (0.9652 : 0.9894) 1.0310	0.992942 (0.9872 : 0.9983) -0.1888	0.992842 (0.9866 : 0.9983) -2.4290	0.992039 (0.9848 : 0.9977) -0.5307	0.991823 (0.9849 : 0.9980) -1.6310
$\phi_1$	0.984079 (0.9747 : 0.9910) 5.4970	0.984721 (0.9753 : 0.9929) 0.2195	0.978928 (0.9659 : 0.9884) -0.4006	0.977990 (0.9647 : 0.9896) 1.5760	0.993160 (0.9869 : 0.9975) 0.5401	0.992607 (0.9863 : 0.9983) -1.3530	0.990045 (0.9836 : 0.9956) -1.9230	0.989954 (0.9835 : 0.9967) -1.1790
$\sigma_0^2$	0.026101 (0.0192 : 0.0365) -4.8940	0.025627 (0.0189 : 0.0343) -0.4917	0.025377 (0.0171 : 0.0370) -0.0884	0.024945 (0.0186 : 0.0366) -0.9047	0.016155 (0.0116 : 0.0210) -0.2732	0.016207 (0.0118 : 0.0233) 1.4950	0.011473 (0.0089 : 0.0159) -0.7858	0.011409 (0.0087 : 0.0156) 1.0990
$\sigma_1^2$	0.026103 (0.0191 : 0.0363) -4.8280	0.025670 (0.0188 : 0.0344) -0.5231	0.025116 (0.0170 : 0.0370) -0.0511	0.025055 (0.0185 : 0.0367) -0.8877	0.015973 (0.0116 : 0.0210) -0.2323	0.016683 (0.0118 : 0.0234) 0.0118	0.011396 (0.0089 : 0.0158) 1.0500	0.011767 (0.0087 : 0.0158) 1.0630
$\nu$	4.249733 (3.2710 : 21.7040) -3.1840	4.016712 (3.1229 : 8.5218) -0.9172	3.244536 (2.7480 : 5.1465) -0.8876	-0.6498 (2.6685 : 7.1599) -1.0380	2.683087 (2.3329 : 3.5295) -3.0820	2.869879 (2.3190 : 3.7426) -0.2709	2.848309 (2.4412 : 3.7208) -1.2290	2.884411 (2.4105 : 3.6075) 3.0530
$r$	0 - -	0.041005 (-0.4870 : 0.5437) -2.4650	0 - -	0.078646 (-0.6490 : 0.7547) 0.0262	0 - -	0.057573 (-0.5882 : 0.7129) 0.5027	0 - -	0.033746 (-0.6351 : 0.7099) 1.6620

Note. First row: posterior mode. Second row: posterior 95% credible interval in parenthesis. Third row: CD statistics.  
Index (1) stands for the original THSV-SMN, (2) stands for the THSV-SMN model with extensions.

Figure 11: (S&P 500) — log-volatilities  $h_t$  and 95% CI

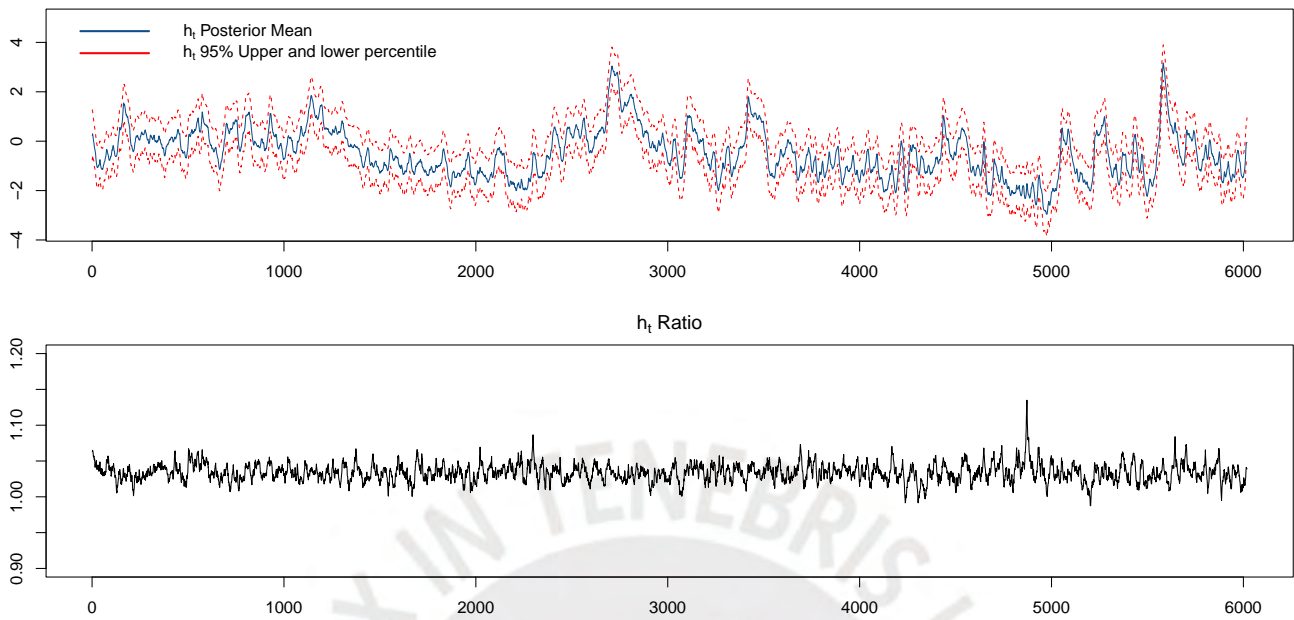


Figure 12: (S&P 500) — Absolute returns and instant volatility

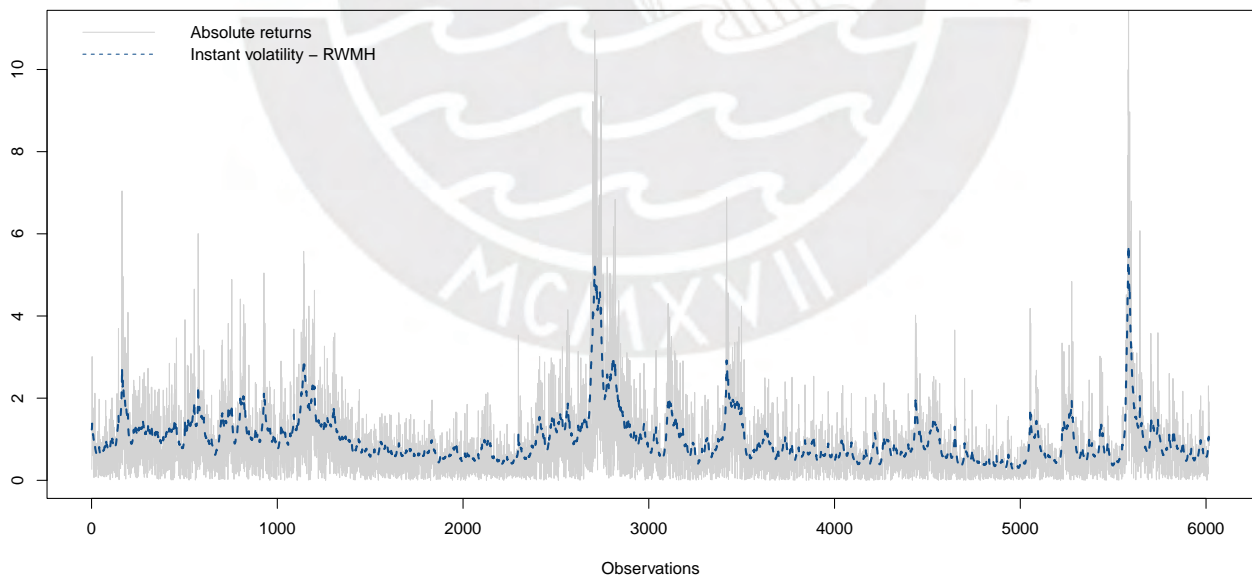


Figure 13:  $\mu_0$  posterior densities

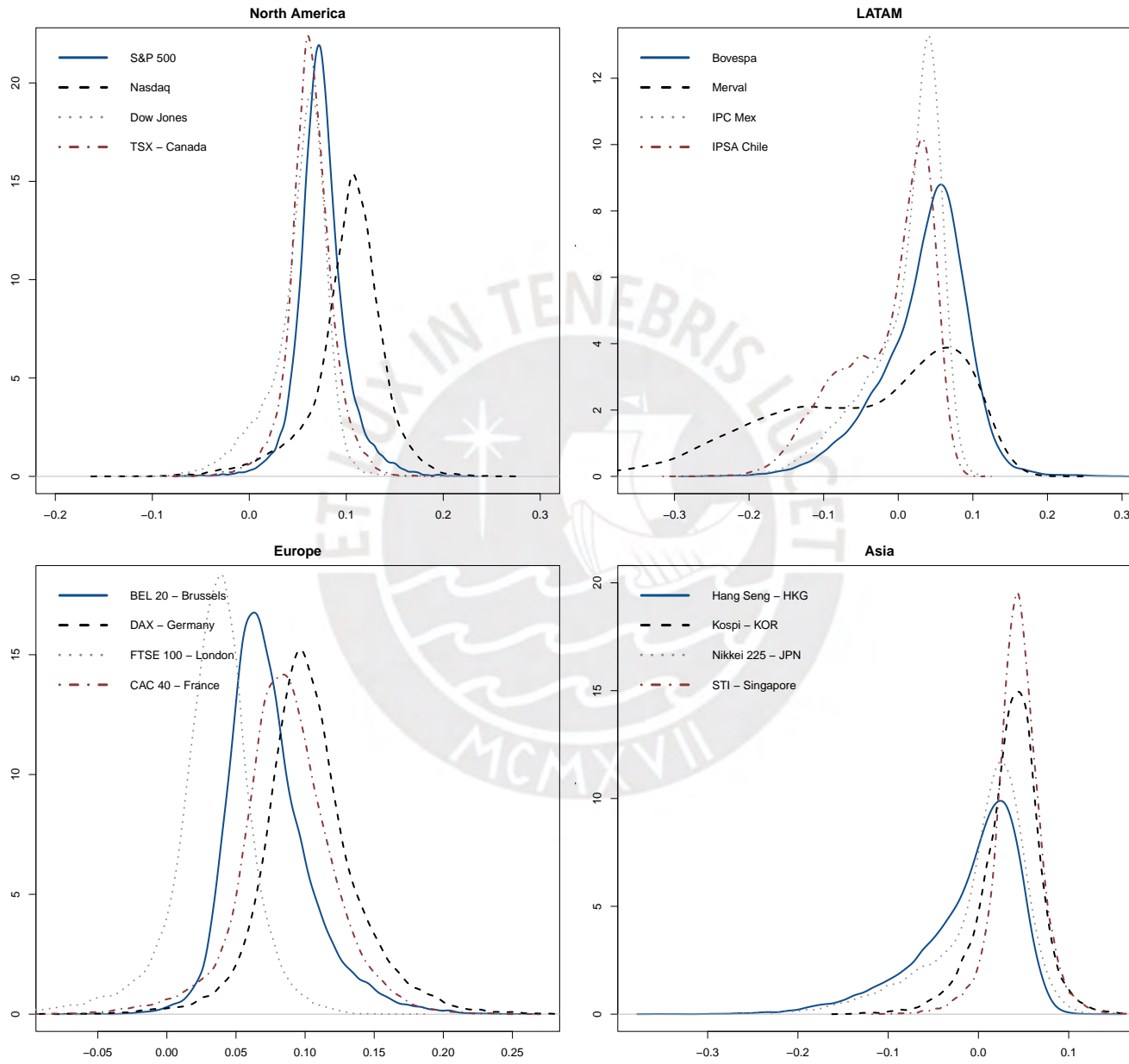


Figure 14:  $\mu_1$  posterior densities

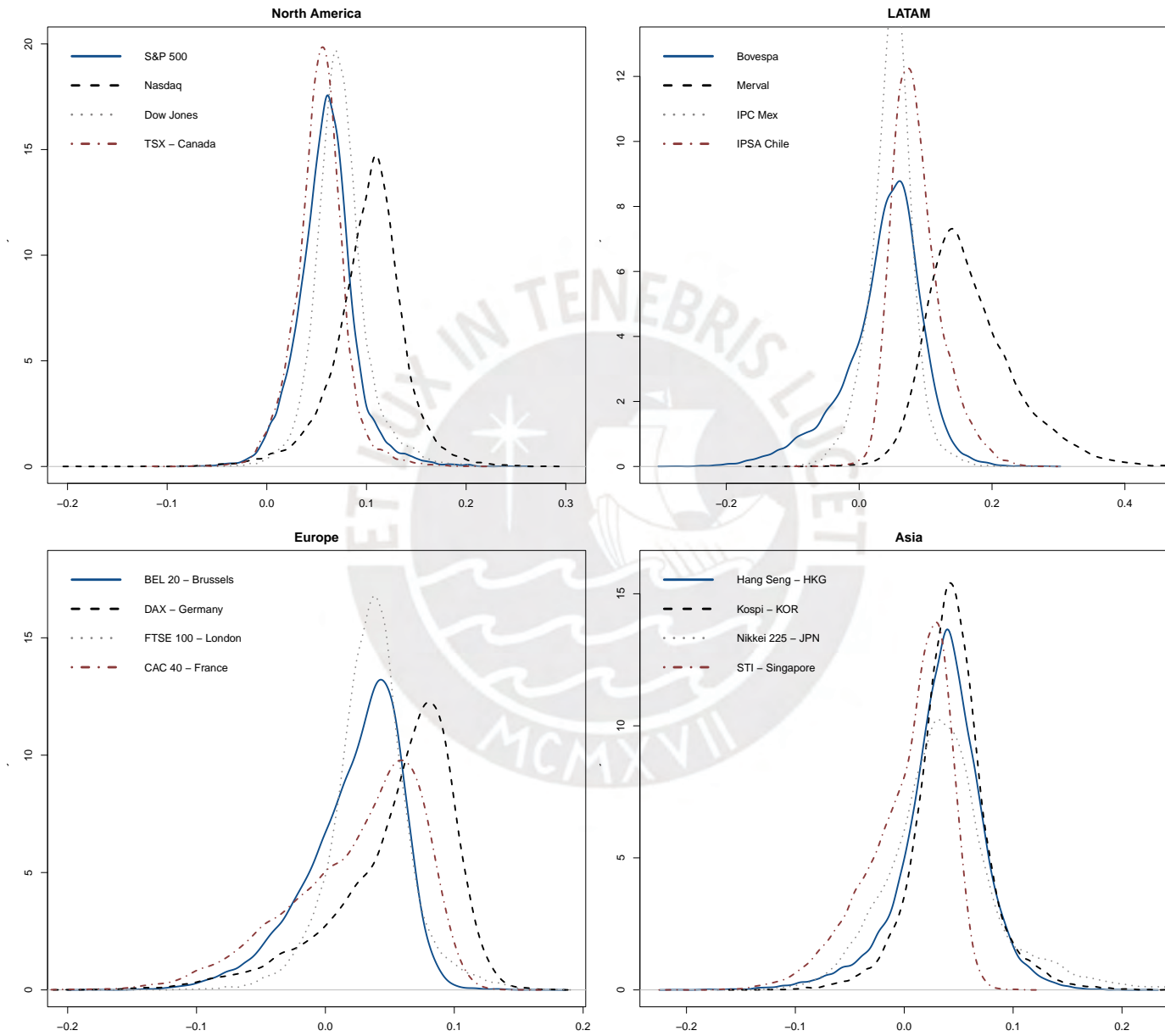




Figure 15:  $\beta_0$  posterior densities

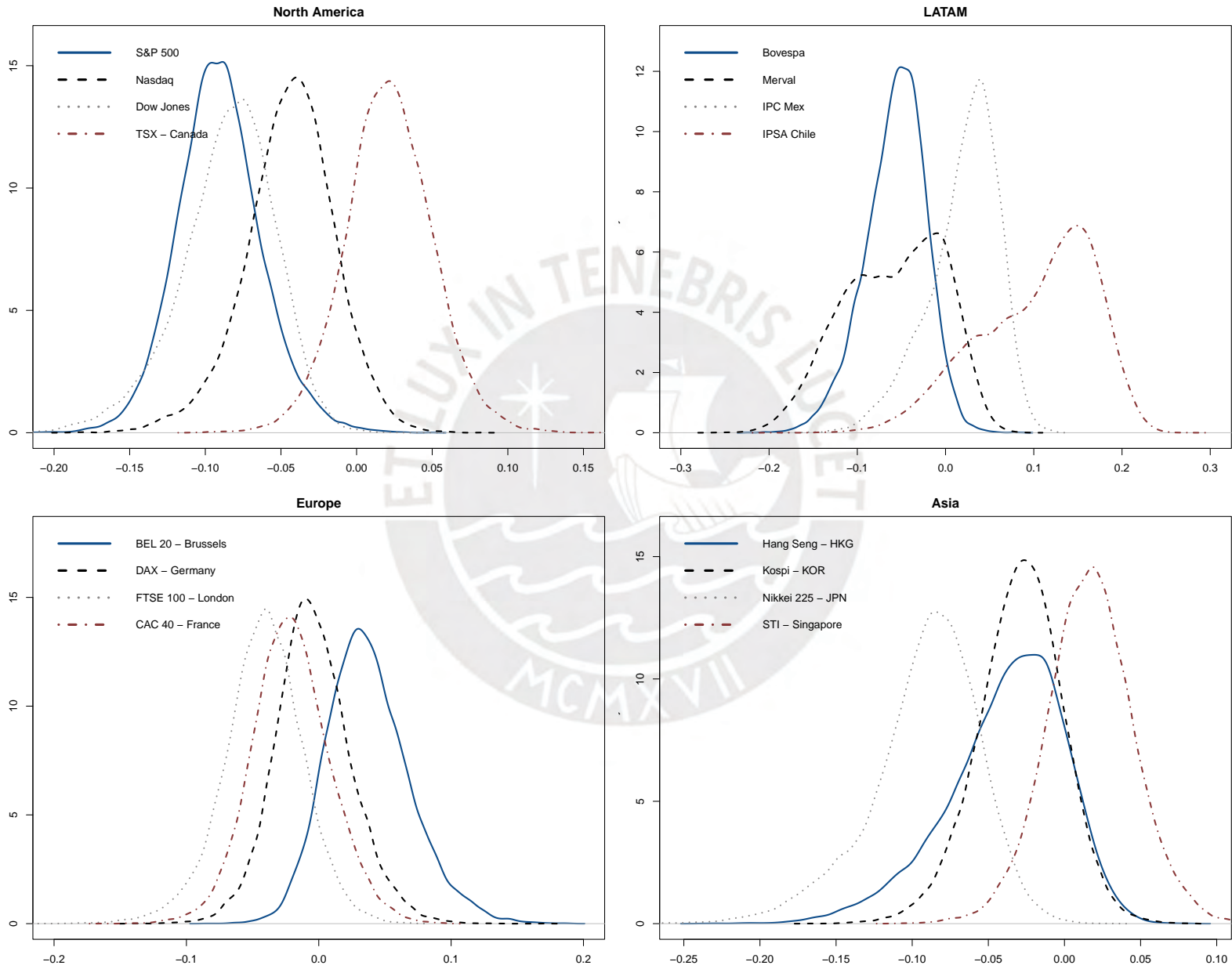


Figure 16:  $\beta_1$  posterior densities

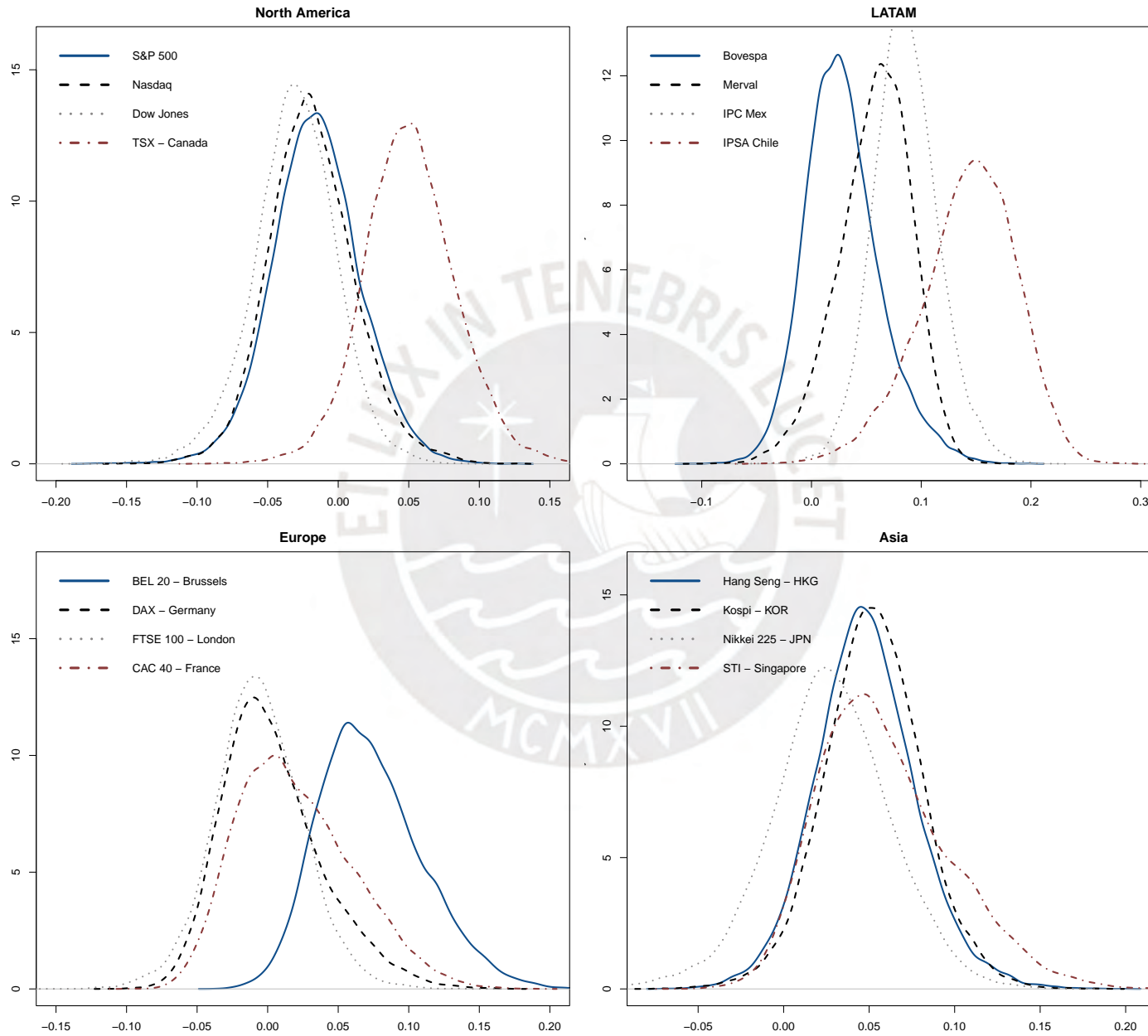


Figure 17:  $\alpha_0$  posterior densities

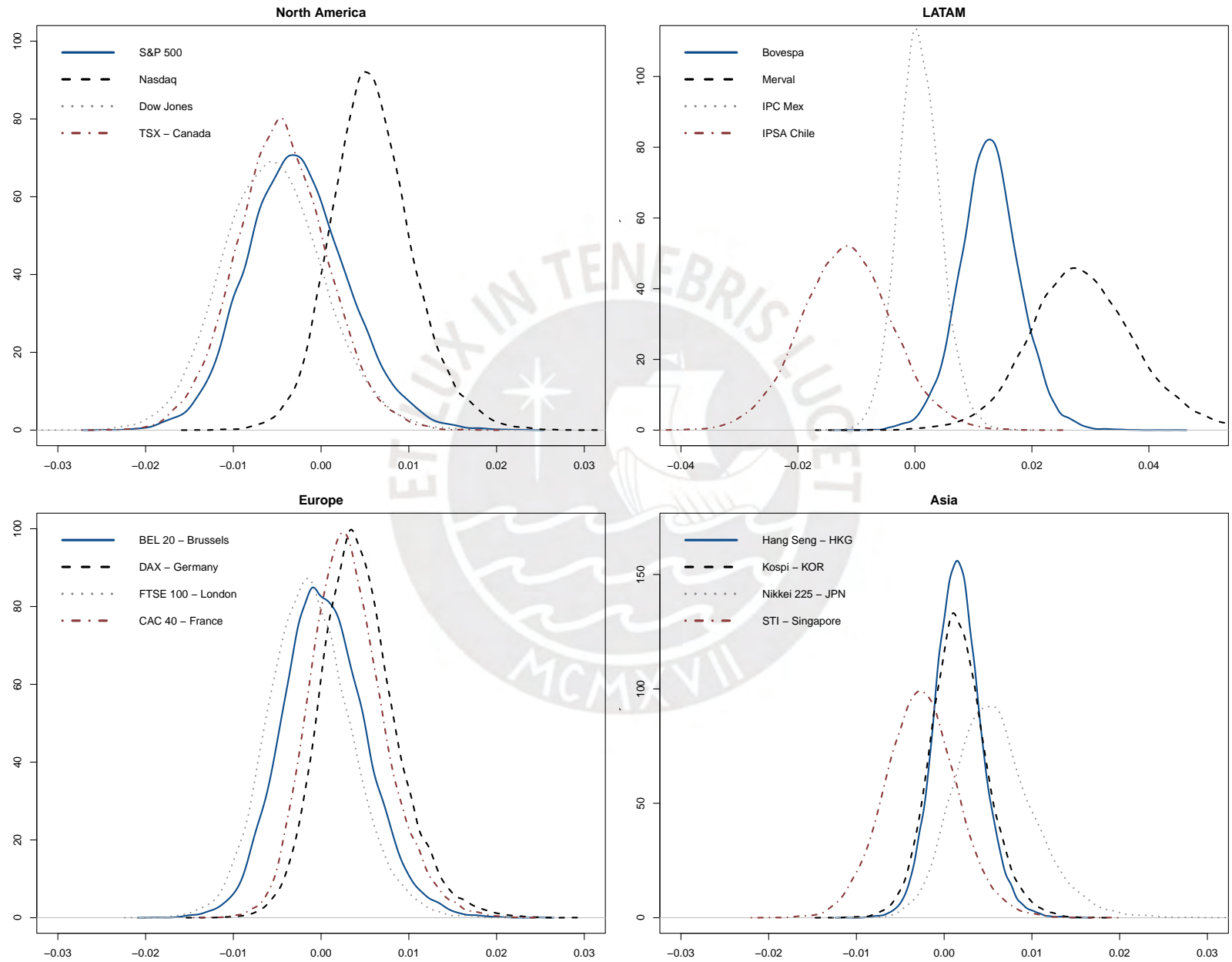


Figure 18:  $\alpha_1$  posterior densities

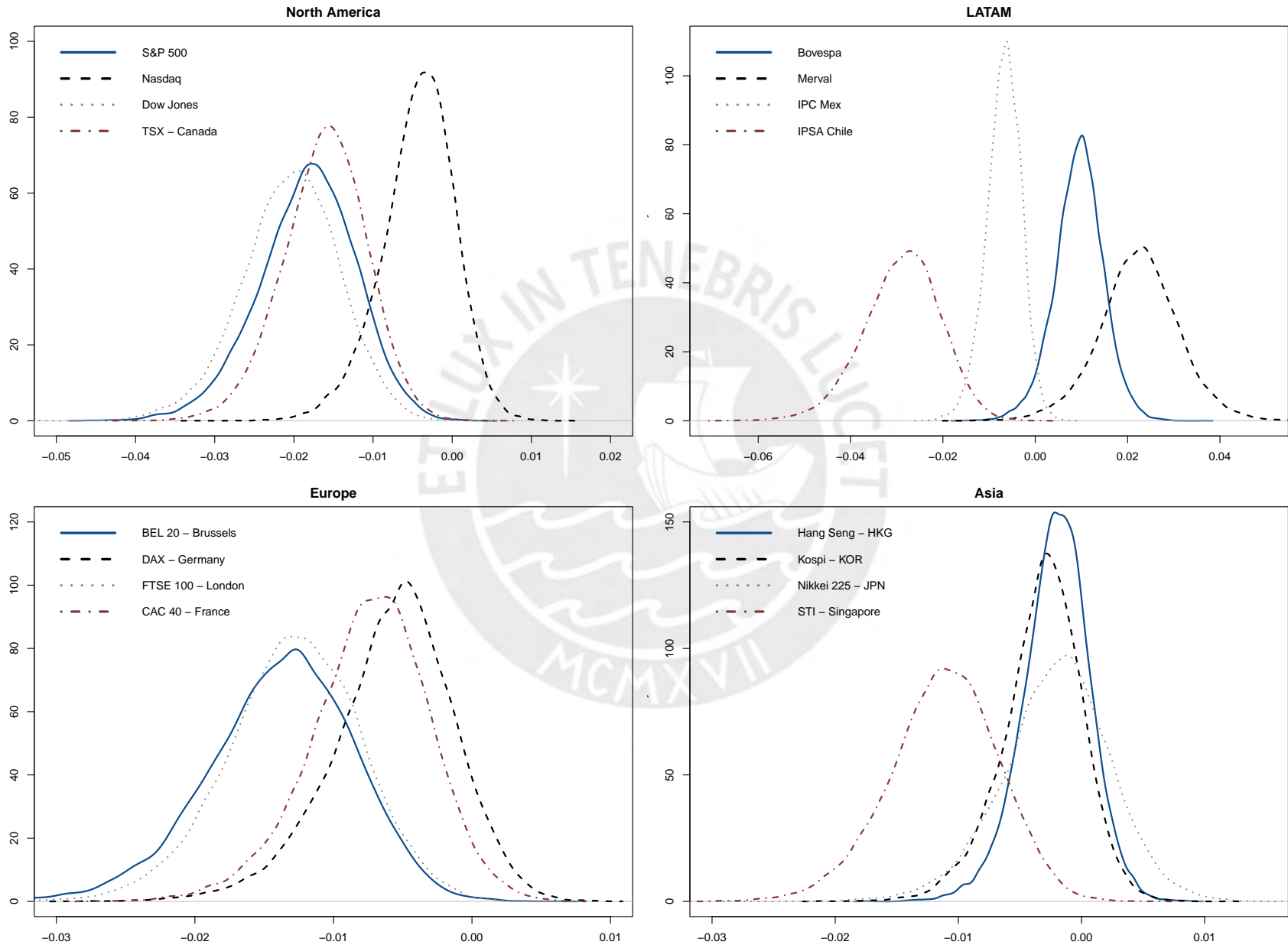


Figure 19:  $\phi_0$  posterior densities

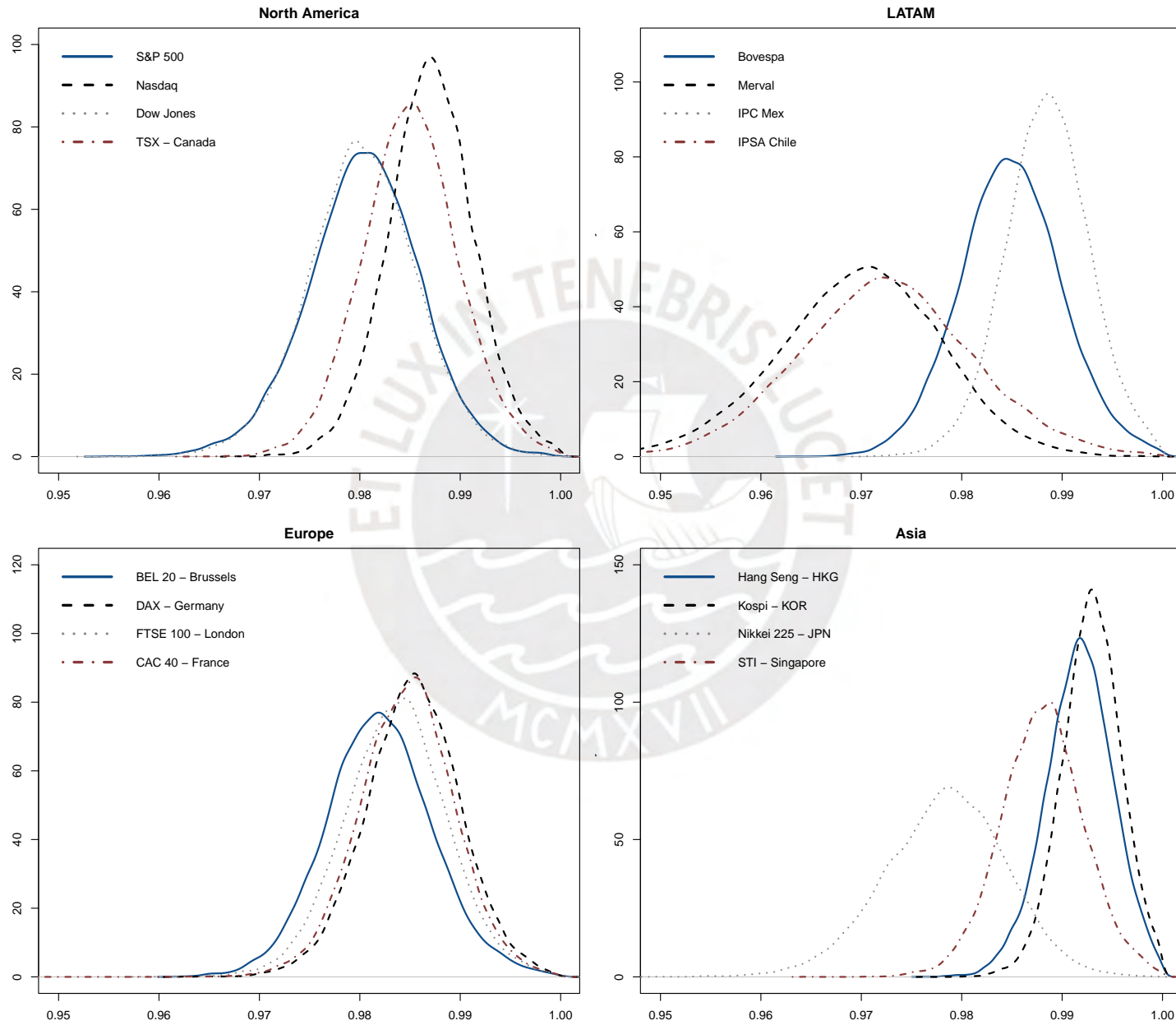


Figure 20:  $\phi_1$  posterior densities

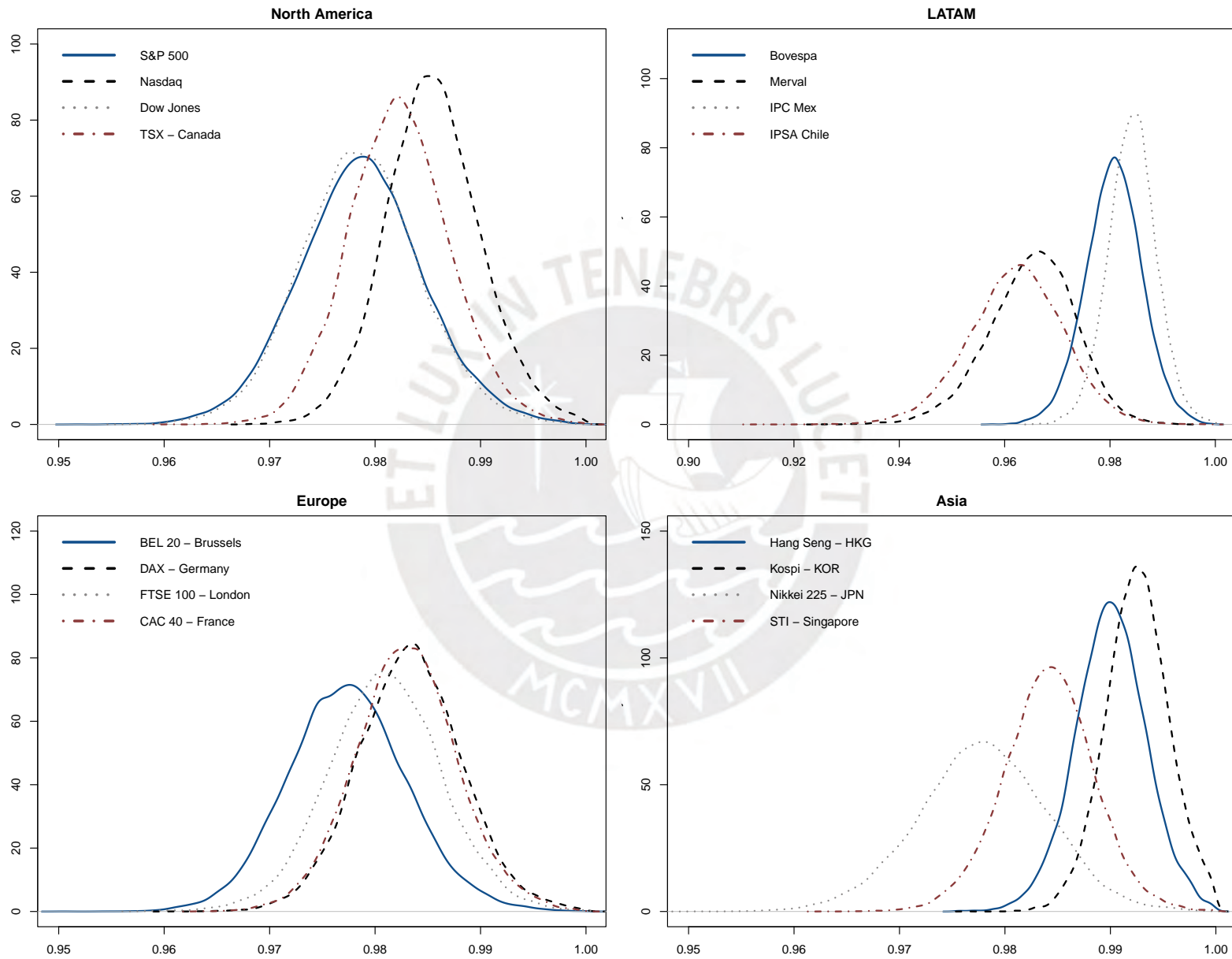




Figure 21:  $\sigma_0^2$  posterior densities

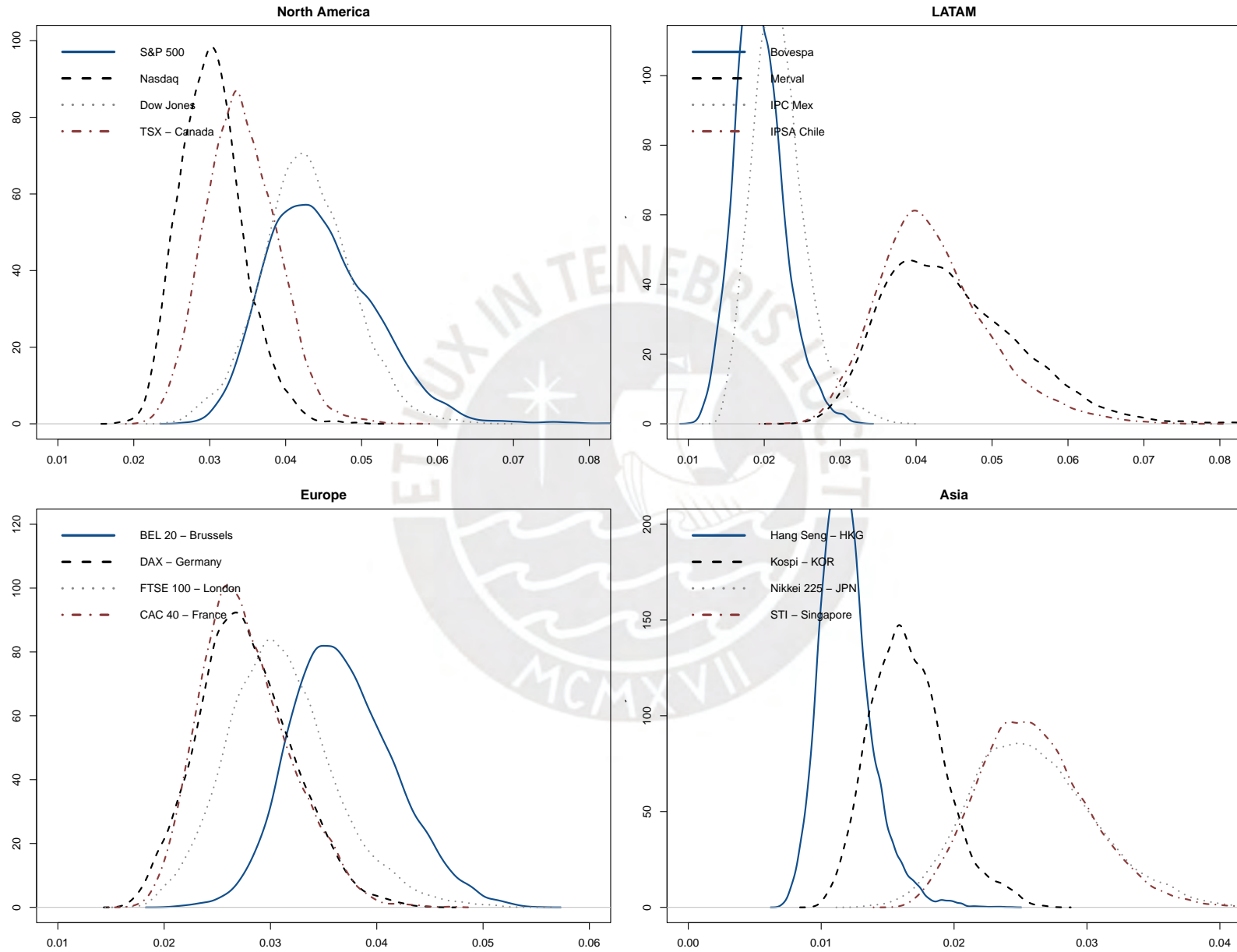


Figure 22:  $\sigma_1^2$  posterior densities

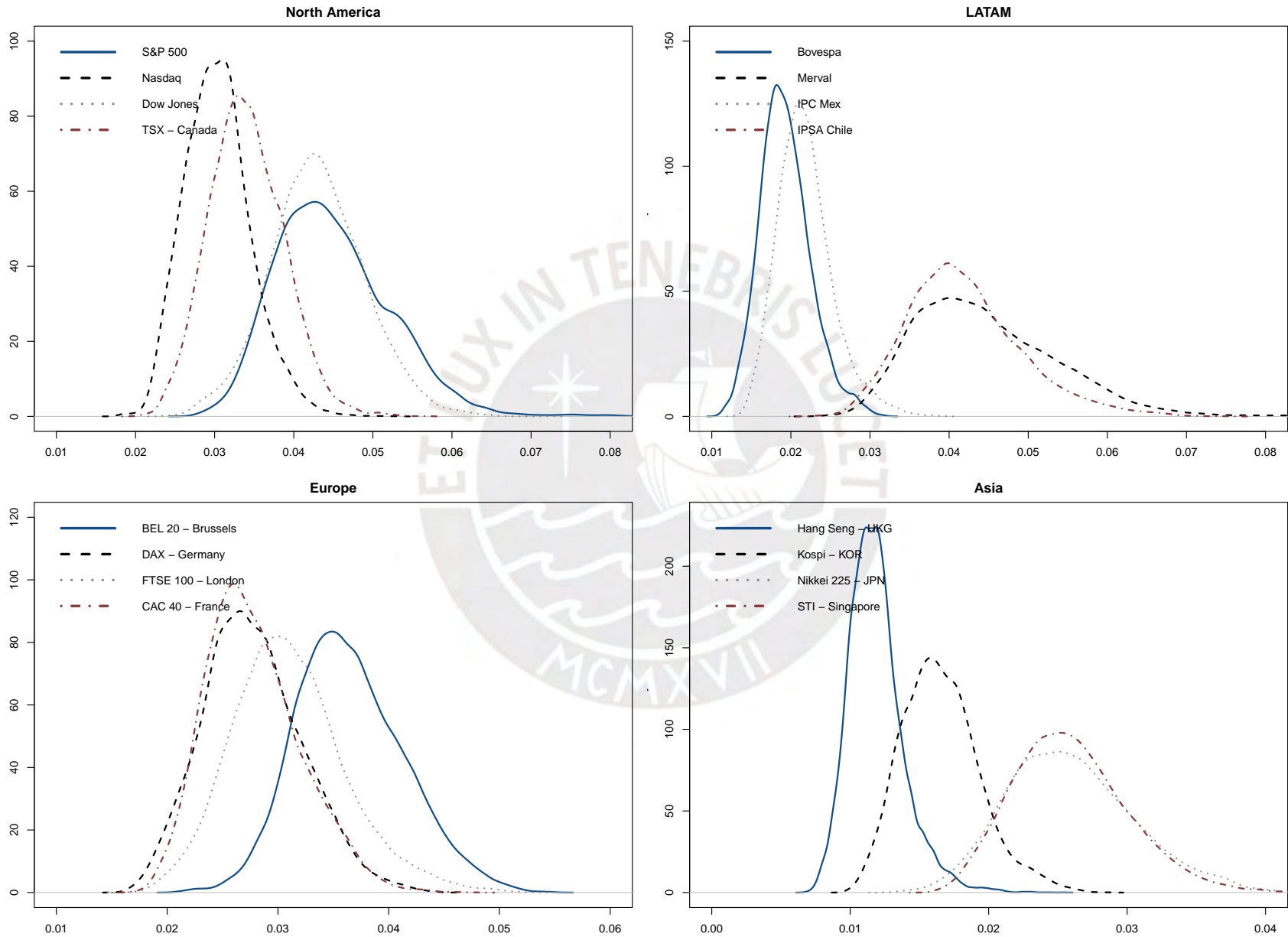


Figure 23: Scale factor  $\nu$  posterior densities

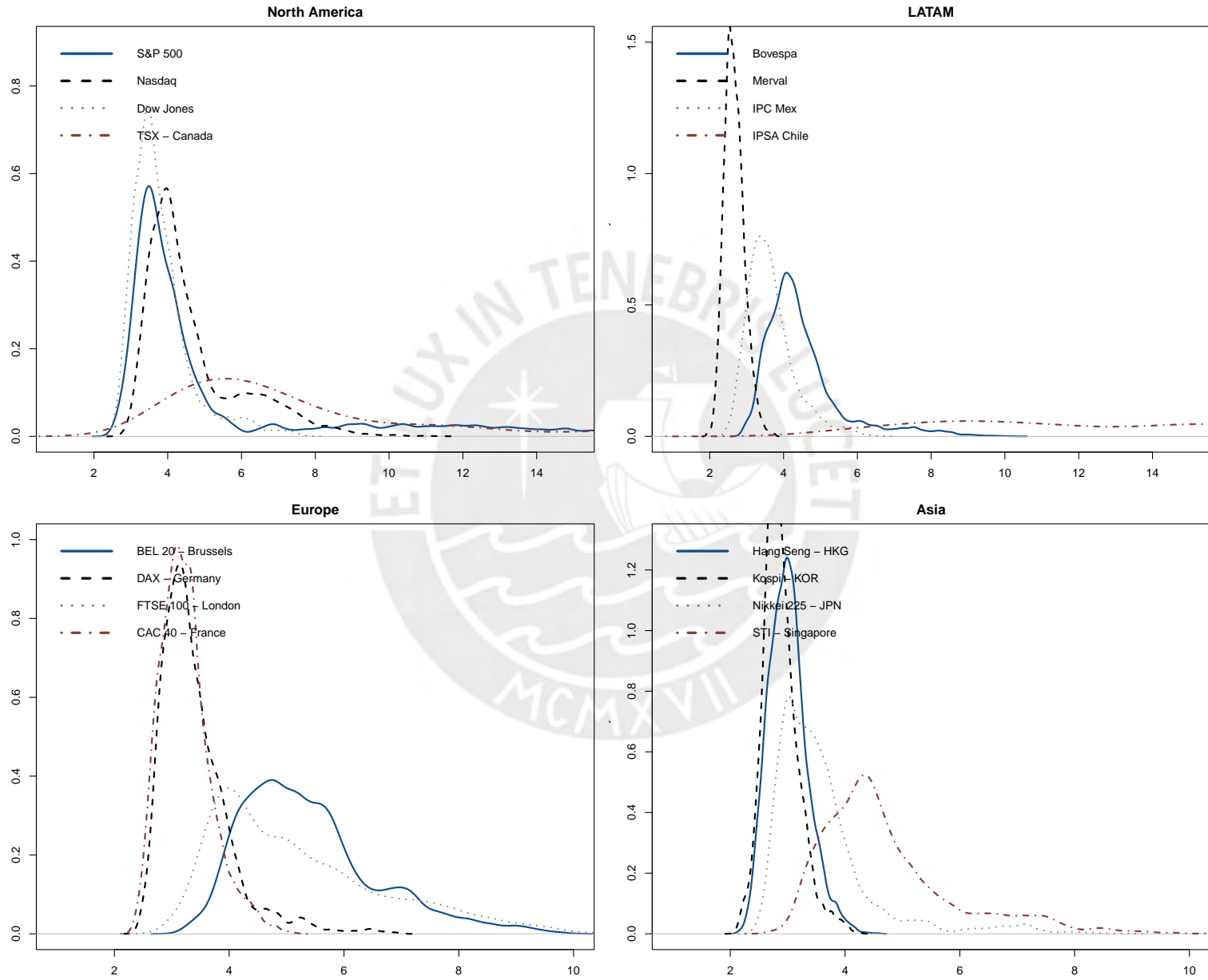


Figure 24: Threshold  $r$  posterior densities

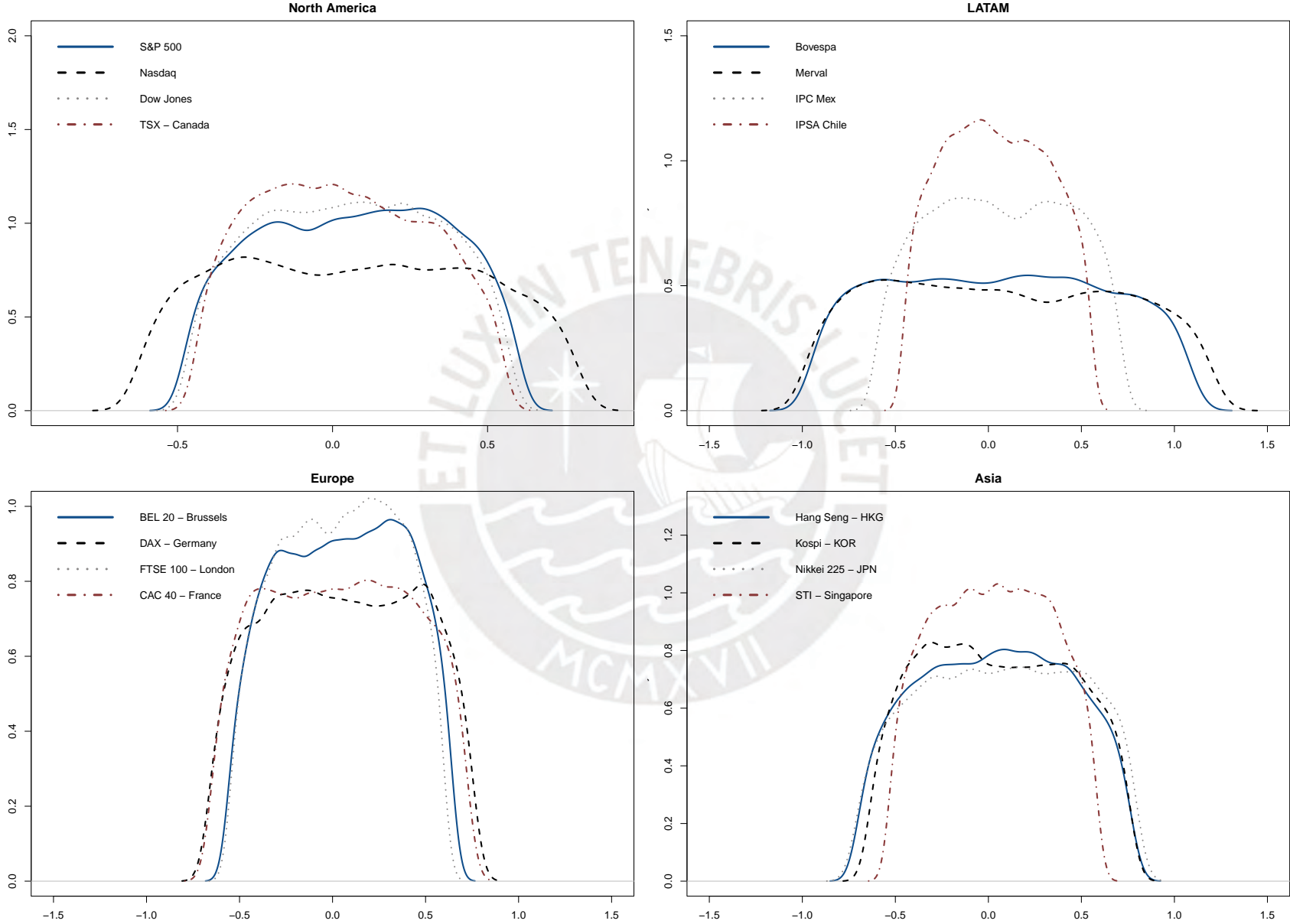
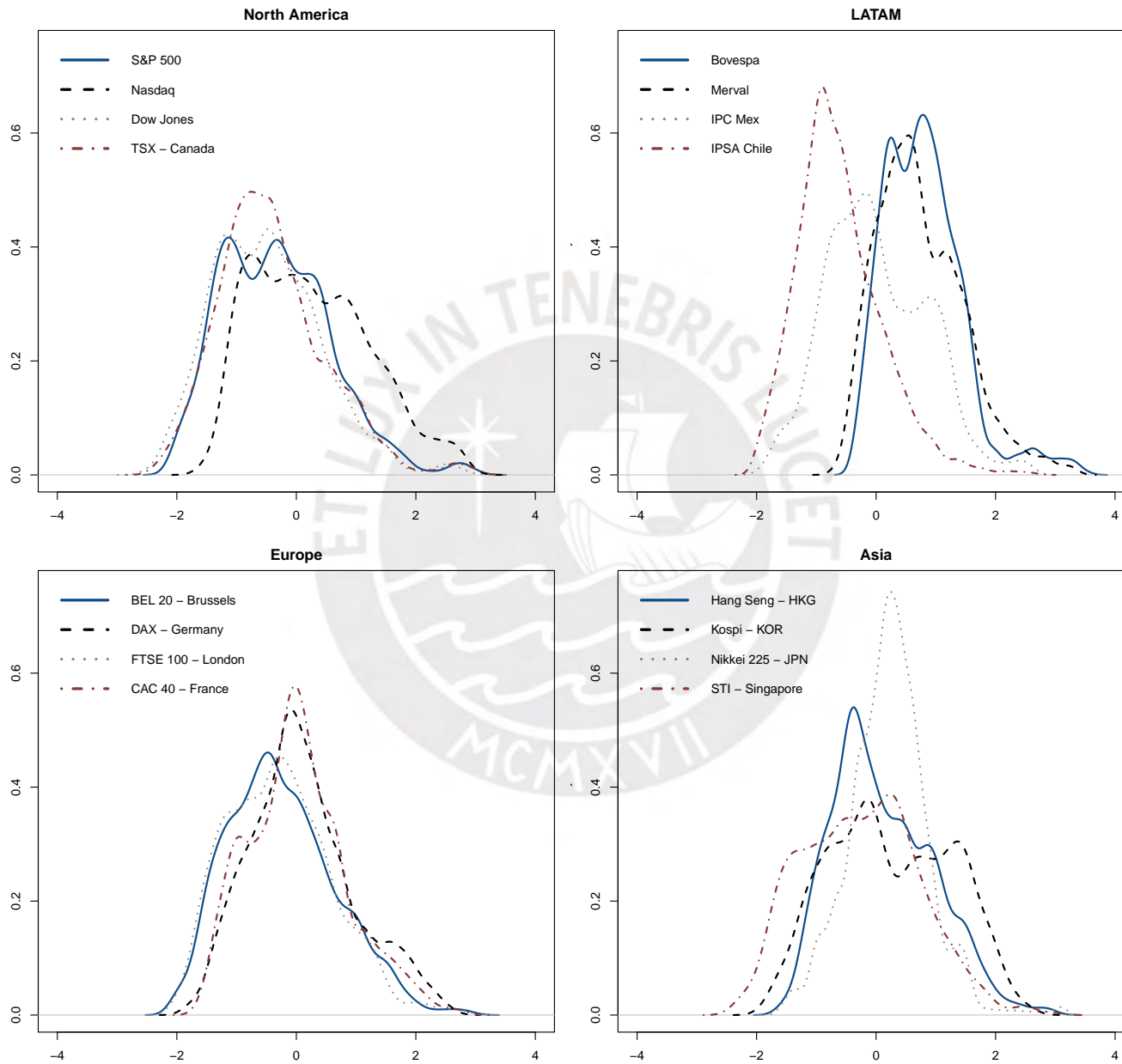


Figure 25: log volatility chain  $h_t$  posterior densities



After the estimation results, in Table 11 we present the WAIC statistics. Results vary from region. For instance, in the Euro region evidence support our hypothesis: by a goodness-of-fit criteria, the THSVS model with endogenous threshold has a better performance than the original THSVS model –which assumes a fixed exogenous threshold in zero–. On the contrary, in the Asia region, evidence indicate that the fixed-zero threshold results in a better fit. North America and Latam regions are mixed: on each of them, two of the four stock market indices have a better fit with the endogenous threshold. Here, in addition to data analysis and estimations results, performance metrics like goodness-of-fit criteria also indicates that not all indices have the same expected behavior.

Table 11: Model Comparison — WAIC Statistics

North America				
	S&P500	Nasdaq	DowJones	TSX (CAN)
THSVS (1)	<b>102,901</b>	125,368	99,813	<b>94,135</b>
THSVS (2)	103,030	<b>125,338</b>	<b>99,767</b>	94,144
LATAM				
	Bovespa (BRA)	IPC (MEX)	Merval (ARG)	IPSA (CHL)
THSVS (1)	149,464	<b>117,295</b>	160,511	<b>92,232</b>
THSVS (2)	<b>149,426</b>	117,311	<b>160,488</b>	92,258
EURO				
	BFX (BEL)	DAX (GER)	FTSE 100 (GBR)	CAC 40 (FRA)
THSVS (1)	109,263	124,247	106,024	<b>120,917</b>
THSVS (2)	<b>109,037</b>	<b>124,235</b>	<b>106,030</b>	120,955
ASIA				
	STI (SGP)	Nikkei 225 (JPN)	KOSPI (KOR)	Hang Seng (HKG)
THSVS (1)	104,988	<b>128,780</b>	<b>126,078</b>	<b>128,071</b>
THSVS (2)	<b>104,870</b>	128,793	126,101	128,151

Note.

THSVS (1) refers to the original Slash THSV model.

THSVS (2) refers to the extended model with endogenous threshold.

### 3.3 Forecasting and VaR

We will use two metrics regarding accuracy prediction i.e. Mean Square Predictive Error (hereinafter, MSPE). The first one follows Abanto & Garrafa (2019) and is computed by:

$$MSPE_1 = \frac{1}{m} \cdot \frac{1}{N} \left[ \sum_{t=T+1}^{T+m} \sum_{i=1}^N (y_t^{(i)} - y_t)^2 \right]$$

$y_t^{(i)}$  is obtained with a simulation from the MCMC procedure.  $m$  is the number of projections (from  $T + 1$  to  $T + m$ ) and  $N$  is the number of simulations ( $i$ ). The second one will be defined as follows:

$$MSPE_2 = \sum_{t=T+1}^{T+m} \frac{1}{m} (\bar{y}_t^{(i)} - y_t)^2$$

$\bar{y}_t^{(i)}$  is the mean of the  $N$  simulations for each of the one-step-ahead projections. The difference we highlight between  $MSPE_1$  and  $MSPE_2$  consists in the following: for a single one-step-ahead projection,  $MSPE_1$  computes the squared deviations from all the  $N$  simulations. In contrast,  $MSPE_2$  directly computes the squared deviation from the posterior mean of the  $N$  draws. Because of that, generally  $MSPE_1$  will be greater than  $MSPE_2$  as it uses extreme values from the  $N$  simulations in the squared deviations.



For MSPE metrics, VaR's Kupiec (1995) likelihood ratio test and Embrechts et al (2004) Expected Shortfall accuracy measure statistic we use  $m = 100$ . That is, we use a 100 out-of-sample observations in order to estimate the metrics. Table 12 shows the results for the MSPE and Tables 13, 14, 15 and 16 presents the results for VaR and ES risk metrics estimates.

Table 12: Forecasting — MSPE

	MSPE	THSV - N (1)	THSV - N (2)	THSV - S (1)	THSV - S (2)
North America					
S&P500	(I)	3.6725	<b>3.6711</b>	3.7540	3.7967
	(II)	1.8659	<b>1.8489</b>	1.8878	1.8955
Nasdaq	(I)	<b>7.2357</b>	7.2868	7.2693	7.3336
	(II)	3.7929	<b>3.7550</b>	3.7663	3.7603
DowJones	(I)	<b>2.4229</b>	2.4665	2.5651	2.5373
	(II)	1.2071	1.2061	<b>1.1941</b>	1.2159
TSX (CAN)	(I)	1.4652	1.4667	<b>1.4553</b>	1.4899
	(II)	0.6859	0.6914	0.6821	<b>0.6757</b>
LATAM					
Bovespa (BRA)	(I)	<b>2.6621</b>	2.6833	2.7575	2.7491
	(II)	1.2295	1.2260	1.2233	<b>1.2226</b>
IPC (MEX)	(I)	<b>2.2468</b>	2.2536	2.3185	2.3332
	(II)	1.0865	1.0795	<b>1.0655</b>	1.1034
Merval (ARG)	(I)	6.9718	<b>6.8718</b>	7.1290	7.1552
	(II)	3.5470	3.4837	3.4654	<b>3.4179</b>
Ipsa (CHL)	(I)	0.9235	<b>0.9188</b>	0.9233	0.9338
	(II)	0.3857	<b>0.3849</b>	0.3904	0.3914
EURO					
BFX (BEL)	(I)	3.3983	3.3219	<b>3.3025</b>	3.3396
	(II)	1.7023	1.6541	<b>1.6322</b>	1.6904
DAX (GER)	(I)	5.4978	5.4827	5.5662	<b>5.3808</b>
	(II)	2.8945	2.9001	2.9637	<b>2.8590</b>
FTSE 100 (GBR)	(I)	2.5281	2.5351	<b>2.5085</b>	2.5597
	(II)	1.2619	1.2462	<b>1.2331</b>	1.2657
CAC 40 (FRA)	(I)	<b>5.1163</b>	5.1753	5.1374	5.2740
	(II)	<b>2.6175</b>	2.7148	2.6581	2.7694
ASIA					
STI (SGP)	(I)	1.3272	1.3275	<b>1.2995</b>	1.3206
	(II)	0.5908	<b>0.5871</b>	0.5896	0.5975
Nikkei 225	(I)	<b>4.0573</b>	4.1177	4.2665	4.1593
	(II)	2.0528	2.0961	2.0768	<b>2.0373</b>
Kospi (KOR)	(I)	<b>2.5014</b>	2.5567	2.5945	2.6114
	(II)	1.1666	1.1901	<b>1.1592</b>	1.2094
Hang Seng (HGK)	(I)	7.0735	7.1216	7.0511	<b>7.0262</b>
	(II)	4.0883	4.0730	4.0365	<b>3.9482</b>

Note.

(1) Stands for the original fixed-zero threshold THSV-SMN model.

(2) Stands for the RWMH threshold THSV-SMN model.

(I) Stands for the MSPE<sub>1</sub> metric.

(II) Stands for the MSPE<sub>2</sub> metric.

The main reason we use the additional prediction accuracy metric relies in controlling for sensibility. Steady and far superior models will result outperforming the other ones in both metrics. Nonetheless, evidence shows that it does not holds for all cases. While in the Euro region we clearly identify a single (but not unique) model that outperform the other ones, for North America, Latam and Asia that statement does not holds true. Results indicate that for prediction accuracy we do not only have to take into consideration models but also the sensibility of the metric. The ranking of the models therefore will depend on the nature of the study and the preference in the sensibility

of the metric. Results on prediction accuracy criteria are the most heterogeneous so far. Depending on the stock market index, evidence shows the following key-items for prediction accuracy: (i) there are markets where the THSV-N model outperforms the THSV-S, (ii) the best performing model is not the same within a region, (iii) the sensibility of the metric yields in different best-performing models and (iv) the endogenous THSV-S has the best performing score in the four regions: TSX (CAN) for North America, Bovespa (BRA) and Merval (ARG) for Latam, DAX (GER) for Euro and Hang Seng (HGK) for Asia are the stock market indices that obtain the highest performing scores with the endogenous THSV-S model.

Now let us assess the risk metrics statistics. Due to the intensive use of computational resources, we first highlight the fact that, in order to compute these risk metrics, we had to run the models  $K \cdot N \cdot H = 6,400$  times, where  $K$  are the one hundred recursive one-step-ahead rolling estimations,  $N$  equals the number of our markets sample and  $H$  the number of models being tested. Each one of them, properly estimated with their total number of 40,000 simulations.

In Tables 13, 14, 15 and 16 we present the results in four models: the originals fixed-zero threshold THSV-N and THSV-S, and the endogenous THSV-N and THSV-S with our RWMH methodology. In addition, we use four different levels of  $\alpha \in [0.01, 0.05, 1 - 0.95, 1 - 0.99]$  that account for the, left and right tail, 1% (99%) and 5% (95%) percentile. We validate the 100 one-step-ahead rolling projections with real validation-data and, for the Violation Rate, we expect to have a  $\hat{\alpha}$  close to the true  $\alpha$ . This Violation Rate accounts for the % of the observations in the validation-data sets that falls outside the given  $\alpha$  percentile. For Kupiec's VaR  $LR$  we present the p-value of the statistic: a significant p-value, i.e.  $< 0.05$ , will give us evidence to reject Kupiec's null hypothesis ( $\hat{\alpha} = \alpha$ ). Finally, for the Embrecht's et al ES measure, the lowest  $D$  will be preferred.

In the North America region we highlight the following key-points: first, for the left-tailed VaR at 1% violations range from 1 up to 3 cases. In consequence,  $LR$  tests indicate us we have no evidence to reject  $\hat{\alpha} = \alpha$ . Also,  $D$  statistics rank first the E-THSV-N<sup>5</sup> for S&P 500, the F-THSV-N for Nasdaq, the E-THSV-S for DowJones and for TSX (CAN), F-THSV-N, F-THSV-S and E-THSV-N tie. Second, for the left-tailed VaR at 5% violations range from 5 up to 11 cases. in the  $LR$  test, the F-THSV-N perform as expected with evidence that fail to reject the null and  $D$  statistic rank first the E-THSV-N for S&P 500, the F-THSV-N for Nasdaq, the E-THSV-S for DowJones and the F-THSV-N for TSX (CAN). Third, for the right-tailed VaR at 1% and 5%, we are well aware that market risk analysis does not focus its attention in assessing unexpected profits, but let us look into the statistical properties: in the 5% right-tailed VaR, violations ranges from 2 to 4. Regarding  $LR$  test, in all models and markets we have no evidence to reject the null and for the ES accuracy measure  $D$  statistic, the E-THSV-N model ranks first for the S&P 500, the E-THSV-N for Nasdaq, the E-THSV-S for DowJones and the F-THSV-N and F-THSV-S for TSX (CAN). Finally, in the 1% right-tailed VaR, violations ranges from 0 to 2. Regarding  $LR$  test, in all models and markets we also have no evidence to reject the null and for the ES accuracy measure  $D$  statistic, we have multiple ties in all the models for the S&P 500, Nasdaq and DowJones. For TSX (CAN), the F-THSV-N model ranks first.

For Latam, in the left-tailed 1% VaR, violations ranges from 0 to 4. Regarding  $LR$  test, except for the F-THSV-N model in Bovespa (BRA), in all models and markets we have no evidence to reject the null and for the ES accuracy measure  $D$  statistic, the E-THSV-S model ranks first for Bovespa (BRA), the E-THSV-N for IPC (MEX), the E-THSV-N for Merval (ARG) and the all the models tie for Ipsa (CHL). In addition, in the left-tailed 5% VaR, violations ranges from 3 to 10. Regarding  $LR$  test, in all models and markets we have no evidence to reject the null and for the ES accuracy measure  $D$  statistic, the F-THSV-N, E-THSV-N and the F-THSV-S models tie for Bovespa (BRA), the E-THSV-S for IPC (MEX), the E-THSV-S and F-THSV-S for Merval (ARG) and the F-THSV-N for Ipsa (CHL). Next, in the right-tailed 5% VaR, violations ranges from 2 to 7. Regarding  $LR$  test, except for the E-THSV-N in Merval (ARG) and the E-THSV-N for Ipsa (CHL), in all models and markets we have no evidence to reject the null and for the ES accuracy measure  $D$  statistic, the E-THSV-S model ranks first for Bovespa (BRA), the F-THSV-S for IPC (MEX), the E-THSV-N for Merval (ARG) and the F-THSV-S and the E-THSV-S for Ipsa (CHL).

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<sup>5</sup>Prefixes F- and E- will indicate "Fixed" and "Endogenous" in order to differentiate the models. Because of that, as an example, F-THSV-N and E-THSV-S will denote the fixed-zero THSV normal model and the endogenous RWMH THSV slash model.

Table 13: Value at Risk and Expected Shortfall metrics - USA &amp; CANADA

	Violation Rate				Kupiec's VaR LR test				Embrecht's et al ES measure			
	(0.01)	(0.05)	(0.95)	(0.99)	LRuc (0.01)	LRuc (0.05)	LRuc (0.95)	LRuc (0.99)	D (0.01)	D (0.05)	D (0.95)	D (0.99)
North America												
S&P 500												
Normal I	0.03	0.11	0.03	0.00	0.0228	0.0166	0.3229	0.1562	0.5508	0.2991	0.0457	1.2686
Normal II	0.03	0.10	0.03	0.00	0.1047	0.0421	0.3229	0.1562	0.4367	0.3092	0.0314	1.2686
Slash I	0.02	0.09	0.02	0.00	0.3763	0.0971	0.1191	0.1562	0.2102	0.2222	0.0543	1.2686
Slash II	0.03	0.10	0.02	0.00	0.1047	0.0421	0.1191	0.1562	0.5463	0.2913	0.0675	1.2686
Nasdaq												
Normal I	0.03	0.10	0.06	0.00	0.1047	0.0421	0.6559	0.1562	0.1809	0.2877	0.0335	1.8494
Normal II	0.01	0.11	0.04	0.00	1.0000	0.0166	0.6350	0.1562	0.4293	0.3513	0.0009	1.8494
Slash I	0.01	0.08	0.07	0.00	1.0000	0.2036	0.3855	0.1562	0.4293	0.3163	0.0554	1.8494
Slash II	0.01	0.11	0.05	0.00	1.0000	0.0166	1.0000	0.1562	0.4293	0.3346	0.0093	1.8494
DowJones												
Normal I	0.02	0.12	0.04	0.00	0.3763	0.0060	0.6350	0.1562	0.0132	0.2901	0.0665	1.2409
Normal II	0.01	0.07	0.03	0.00	1.0000	0.3855	0.3229	0.1562	0.0000	0.2434	0.0599	1.2409
Slash I	0.03	0.07	0.02	0.00	0.1047	0.3855	0.1191	0.1562	0.2987	0.2444	0.0432	1.2409
Slash II	0.01	0.09	0.03	0.00	1.0000	0.0971	0.3229	0.1562	0.2991	0.0599	0.0000	1.2409
TSX (CAN)												
Normal I	0.03	0.05	0.03	0.02	0.1047	0.0000	0.3229	0.1209	0.1070	0.0311	0.0239	0.0553
Normal II	0.03	0.06	0.04	0.00	0.1047	0.6559	0.6350	0.1562	0.1070	0.0495	0.0318	0.9326
Slash I	0.03	0.06	0.03	0.01	0.1047	0.6559	0.3229	0.1382	0.1070	0.0680	0.0239	0.1107
Slash II	0.01	0.09	0.03	0.00	0.3763	0.0000	0.6350	0.1562	0.1605	0.0486	0.0318	0.9326

Note.

Normal I stands for the original fixed-zero threshold THSV-SMN (normal) model.

Normal II stands for the endogenous RWMH threshold THSV-SMN (normal) model.

Slash I stands for the original fixed-zero threshold THSV-SMN (slash) model.

(Slash II stands for the endogenous RWMH threshold THSV-SMN (slash) model.

Table 14: Value at Risk and Expected Shortfall metrics - LATAM

	Violation Rate				Kupiec's VaR LR test				Embrecht's et al ES measure			
	(0.01)	(0.05)	(0.95)	(0.99)	LRuc (0.01)	LRuc (0.05)	LRuc (0.95)	LRuc (0.99)	D (0.01)	D (0.05)	D (0.95)	D (0.99)
LATAM												
Bovespa (BRA)												
Normal I	0.04	0.06	0.02	0.00	0.0228	0.6559	0.1191	0.1562	0.2121	0.0881	0.0393	1.1981
Normal II	0.02	0.06	0.02	0.01	0.3763	0.6559	0.1191	0.2390	0.1112	0.0881	0.0745	0.0000
Slash I	0.02	0.06	0.03	0.00	0.3763	0.6559	0.3229	0.1562	0.3630	0.0881	0.0167	1.1981
Slash II	0.02	0.07	0.03	0.00	0.3763	0.3855	0.3229	0.1562	0.0872	0.1032	0.0111	1.1981
IPC (MEX)												
Normal I	0.02	0.07	0.06	0.00	0.3763	0.3855	0.6559	0.1562	0.2246	0.1191	0.0656	1.0591
Normal II	0.01	0.08	0.06	0.00	1.0000	0.2036	0.6559	0.1562	0.0000	0.1039	0.0746	1.0591
Slash I	0.02	0.08	0.06	0.01	0.3763	0.2036	0.6559	0.0583	0.1788	0.1404	0.0456	0.2281
Slash II	0.00	0.07	0.05	0.00	0.1562	0.3855	1.0000	0.1562	1.1903	0.0883	0.0686	1.0591
Merval (ARG)												
Normal I	0.00	0.08	0.06	0.02	0.1562	0.2036	0.6559	0.0228	2.3246	0.3535	0.2590	0.5220
Normal II	0.03	0.10	0.06	0.02	0.1047	0.3855	0.0421	0.6559	0.0228	0.4020	0.1421	1.5912
Slash I	0.02	0.07	0.06	0.01	0.3763	0.3855	0.6559	0.0583	0.4298	0.2973	0.2014	0.0000
Slash II	0.02	0.07	0.04	0.02	0.3763	0.3855	0.6350	0.0593	0.4298	0.2973	0.0311	0.5220
IPSA (CHL)												
Normal I	0.02	0.03	0.02	0.00	0.3763	0.3229	0.1191	0.1562	0.0725	0.0017	0.1143	0.8099
Normal II	0.02	0.05	0.01	0.01	0.3763	1.0000	0.0261	1.0000	0.0725	0.0800	0.1789	0.0000
Slash I	0.02	0.04	0.02	0.00	0.3763	0.6350	0.1191	0.1562	0.0725	0.0284	0.0188	0.8099
Slash II	0.02	0.05	0.02	0.00	0.3763	1.0000	0.1191	0.1562	0.0725	0.0197	0.0188	0.8099

Note.

Normal I stands for the original fixed-zero threshold THSV-SMN (normal) model.

Normal II stands for the endogenous RWMH threshold THSV-SMN (normal) model.

Slash I stands for the original fixed-zero threshold THSV-SMN (slash) model.

Slash II stands for the endogenous RWMH threshold THSV-SMN (slash) model.

Table 15: Value at Risk and Expected Shortfall metrics - EURO

	Violation Rate				Kupiec's VaR LR test				Embrecht's et al ES measure			
	(0.01)	(0.05)	(0.95)	(0.99)	LRuc (0.01)	LRuc (0.05)	LRuc (0.95)	LRuc (0.99)	D (0.01)	D (0.05)	D (0.95)	D (0.99)
Europe												
BFX (BEL)												
Normal I	0.03	0.08	0.02	0.02	0.1047	0.2036	0.1191	0.3763	0.6827	0.2133	0.8085	0.3824
Normal II	0.02	0.08	0.03	0.02	0.3763	0.2036	0.3229	0.1209	0.5363	0.2819	0.2501	0.3824
Slash I	0.01	0.06	0.02	0.02	1.0000	0.6559	0.1191	0.3763	0.0000	0.3462	0.8085	0.3824
Slash II	0.02	0.09	0.02	0.02	0.3763	0.0971	0.1191	0.3763	0.5363	0.2154	0.8085	0.3824
DAX (GER)												
Normal I	0.03	0.09	0.04	0.01	0.1047	0.0971	0.6350	0.0958	0.5161	0.4403	0.0497	0.0000
Normal II	0.04	0.09	0.03	0.01	0.0228	0.0971	0.3229	0.1382	0.3871	0.4403	0.1478	0.0000
Slash I	0.03	0.08	0.06	0.01	0.1047	0.2036	0.6559	0.0583	0.1840	0.3847	0.1579	0.0000
Slash II	0.03	0.08	0.04	0.01	0.1047	0.2036	0.6350	0.0958	0.6092	0.3806	0.0502	0.0000
FTSE 100 (GBR)												
Normal I	0.03	0.07	0.04	0.01	0.1047	0.3855	0.6350	0.0958	0.2865	0.2965	0.0950	0.0000
Normal II	0.03	0.08	0.04	0.01	0.1047	0.2036	0.6350	0.0958	0.6426	0.2801	0.0950	1.1485
Slash I	0.02	0.08	0.06	0.00	0.3763	0.2036	0.6559	0.1562	0.3246	0.2801	0.1126	1.9224
Slash II	0.02	0.08	0.03	0.01	0.3763	0.2036	0.0000	0.0000	0.3246	0.2801	0.0129	0.0000
CAC 40 (FRA)												
Normal I	0.03	0.08	0.03	0.02	0.1047	0.2036	0.3229	0.1209	0.3713	0.4361	0.3241	0.9619
Normal II	0.03	0.10	0.03	0.01	0.1047	0.0421	0.3229	0.1382	0.3713	0.5355	0.3241	0.0000
Slash I	0.02	0.10	0.06	0.01	0.3763	0.0421	0.6559	0.0583	0.5570	0.5355	0.3503	0.0000
Slash II	0.04	0.09	0.03	0.01	0.0228	0.0971	0.3229	0.1382	0.4127	0.4697	0.3241	0.0000

Note.

Normal I stands for the original fixed-zero threshold THSV-SMN (normal) model.

Normal II stands for the endogenous RWMH threshold THSV-SMN (normal) model.

Slash I stands for the original fixed-zero threshold THSV-SMN (slash) model.

Slash II stands for the endogenous RWMH threshold THSV-SMN (slash) model.

Table 16: Value at Risk and Expected Shortfall metrics - ASIA

	Violation Rate				Kupiec's VaR LR test				Embrecht's et al ES measure			
	(0.01)	(0.05)	(0.95)	(0.99)	LRuc (0.01)	LRuc (0.05)	LRuc (0.95)	LRuc (0.99)	D (0.01)	D (0.05)	D (0.95)	D (0.99)
Asia												
STI (SGP)												
Normal I	0.01	0.04	0.02	0.02	1.0000	0.6350	0.1191	0.3763	0.0000	0.0246	0.0704	0.1345
Normal II	0.01	0.05	0.02	0.01	1.0000	1.0000	0.1191	0.2390	0.0000	0.0918	0.0704	0.0000
Slash I	0.01	0.05	0.02	0.01	1.0000	1.0000	0.1191	0.2390	0.0000	0.0307	0.0704	0.0000
Slash II	0.01	0.05	0.03	0.02	1.0000	1.0000	0.3229	0.1209	0.0000	0.0910	0.0225	0.1345
Nikkei 225 (JPN)												
Normal I	0.02	0.01	0.06	0.01	0.3763	0.0421	0.6559	0.0583	0.1052	0.2515	0.0857	0.0000
Normal II	0.02	0.09	0.04	0.01	0.3763	0.0971	0.6350	0.0958	0.1424	0.1880	0.0063	0.0000
Slash I	0.02	0.07	0.04	0.01	0.3763	0.3855	0.6350	0.0958	0.0818	0.1128	0.0889	0.0000
Slash II	0.01	0.06	0.03	0.00	1.0000	0.6559	0.3229	0.1562	0.0000	0.0483	0.1101	1.9302
Kospi (KOR)												
Normal I	0.03	0.08	0.02	0.00	0.1047	0.2036	0.1191	0.1562	0.4577	0.2099	0.1064	1.0923
Normal II	0.03	0.08	0.01	0.01	0.1047	0.2036	0.0261	1.0000	0.4485	0.1901	0.0267	0.1684
Slash I	0.03	0.08	0.02	0.00	0.1047	0.2036	0.1191	0.1562	0.4577	0.1901	0.1064	0.4295
Slash II	0.02	0.07	0.02	0.00	0.3763	0.3855	0.1191	0.1562	0.2311	0.1355	0.0028	0.4295
Hang Seng (HKG)												
Normal I	0.05	0.10	0.07	0.04	0.0040	0.0421	0.3855	0.0018	0.8513	0.5647	0.3538	1.5911
Normal II	0.02	0.09	0.08	0.03	0.3763	0.0971	0.2036	0.0035	0.1964	0.4816	0.5214	1.7830
Slash I	0.02	0.09	0.08	0.04	0.3763	0.0971	0.2036	0.0010	0.1964	0.5271	0.5214	1.6452
Slash II	0.03	0.10	0.08	0.02	0.1047	0.0421	0.2036	0.0118	0.4543	0.5227	0.5214	1.4844

Note.

Normal I stands for the original fixed-zero threshold THSV-SMN (normal) model.

Normal II stands for the endogenous RWMH threshold THSV-SMN (normal) model.

Slash I stands for the original fixed-zero threshold THSV-SMN (slash) model.

Slash II stands for the endogenous RWMH threshold THSV-SMN (slash) model.



Finally, in the right-tailed 1% VaR, violations ranges from 0 to 2. Regarding *LR* test, except for the F-THSV-S in IPC (MEX) and the F-THSV-N, F-THSV-S and E-THSV-S for Merval (ARG), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the E-THSV-N model ranks first for Bovespa (BRA), the F-THSV-S for IPC (MEX), the F-THSV-S for Merval (ARG) and the F-THSV-S and the E-THSV-N for Ipsa (CHL).

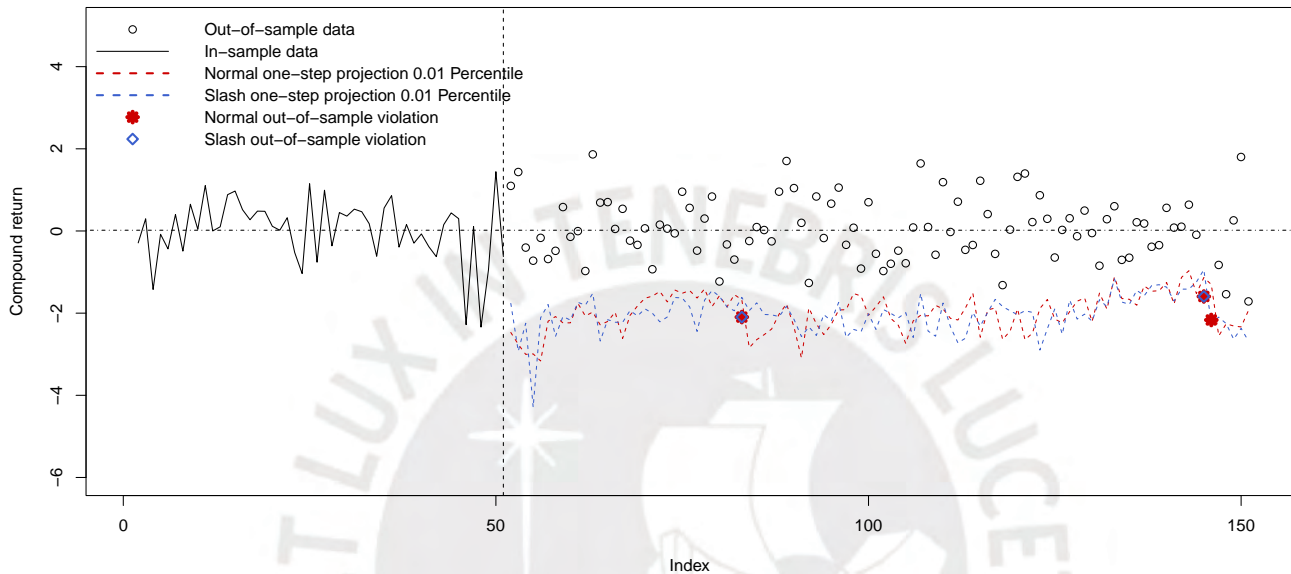
For the Euro region, in the left-tailed 1% VaR, violations ranges from 1 to 4. Regarding *LR* test, except for the E-THSV-N in DAX (GER) and the E-THSV-S for CAC 40 (FRA), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the F-THSV-S model ranks first for BFX (BEL), the F-THSV-S for DAX (GER), the F-THSV-N for FTSE 100 (GBR) and the F-THSV-N and the E-THSV-N for CAC 40 (FRA). In in the left-tailed 5% VaR, violations ranges from 6 to 10. Regarding *LR* test, except for the E-THSV-N and the F-THSV-S for CAC 40 (FRA), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the F-THSV-N model ranks first for BFX (BEL), the E-THSV-S for DAX (GER), the F-THSV-N, F-THSV-S and the E-THSV-S tie for FTSE 100 (GBR) and the F-THSV-N for CAC 40 (FRA). Next, in the right-tailed 5% VaR, violations ranges from 2 to 6. Regarding *LR* test, except for the E-THSV-S in FTSE 100 (GBR), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the E-THSV-N model ranks first for BFX (BEL), the F-THSV-N for DAX (GER), the E-THSV-S for FTSE 100 (GBR) and the F-THSV-N, E-THSV-N and the E-THSV-S for CAC 40 (FRA). Finally, in the right-tailed 1% VaR, violations ranges from 1 to 2. Regarding *LR* test, except for the E-THSV-S in FTSE 100 (GBR), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, all models tie for BFX (BEL) and DAX (GER), the F-THSV-N and the E-THSV-S also tie for FTSE 100 (GBR) and the E-THSV-N, F-THSV-S and the E-THSV-S for CAC 40 (FRA) also tie.

For the Asia region, in the left-tailed 1% VaR, violations ranges from 1 to 5. Regarding *LR* test, except for the F-THSV-N in Hang Seng (HGK), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, for STI (SGP) all models tie, the E-THSV-S for Nikkei 225 (JPN) ranks first, the E-THSV-S for Kospi (KOR) and the E-THSV-N and the F-THSV-S for Hang Seng (HGK). In in the left-tailed 5% VaR, violations ranges from 4 to 10. Regarding *LR* test, except for the F-THSV-N in the Nikkei 225 (JPN) and the F-THSV-N for Hang Seng (HGK), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the F-THSV-N model ranks first for STI (SGP), the E-THSV-S for Nikkei 225 (JPN), the E-THSV-S for Kospi (KOR) and the E-THSV-N for Hang Seng (HGK). Next, in the right-tailed 5% VaR, violations ranges from 2 to 8. Regarding *LR* test, except for the E-THSV-N in Kospi (KOR), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, the E-THSV-S model ranks first for STI (SGP), the E-THSV-N for Nikkei 225 (JPN), the E-THSV-S for Kospi (KOR) and the F-THSV-N for Hang Seng (HGK). Finally, in the right-tailed 1% VaR, violations ranges from 0 to 4. Regarding *LR* test, except for Hang Seng (HGK), in all models and markets we have no evidence to reject the null and for the ES accuracy measure *D* statistic, E-THSV-N and F-THSV-S models tie for STI (SGP), the F-THSV-N, E-THSV-N and the F-THSV-S also tie for Nikkei 225 (JPN), E-THSV-N ranks first for Kospi (KOR) and the E-THSV-S for Hang Seng (HGK) as well.

From the risk metrics statistics results we reinforce the fact that markets behave differently and a tailored-analysis should be made in order to assess the study in the market risk. Interestingly, Asia region leans towards our extended THSV-S model. From North America, Latam and Euro, results vary widely. For all that reasons, we present an additional graphical approach in order to clarify how violations occur. Figure 26 provides details in the rolling one-step forecast procedure. We use the TSX (CAN) index as an example. As previous mentioned, we have used a rolling forecast window for the last one-hundred observations in our complete data-set. In all cases, the estimation of the models have been run over the  $T - 100$  observations. The one-step rolling forecast, in addition, estimates the model another one-hundred times. Thus, for each estimation, we forecast the current  $T + 1$  observation. Then, we roll the data one step in a way that we keep the total number  $T$  observations but on each step we move forward one-step. We do so, from  $T + 1$  to  $T + 100$ . Also, for each forecast we have sampled 40,000 draws (with a 10,000 burnin period). With all that mentioned, Figure 26 contrasts the out-of-sample observations with the 0.01 percentile computed independently for each point in the endogenous THSV Normal (dotted red lines) and the endogenous

THSV Slash (dotted blue lines) and highlights the violations i.e. the true out-of-sample data-validation that is lower than the 0.01 percentile in both procedures. Those 0.01 percentiles for each point reflect the estimated left-tailed Value at Risk. In both cases we use the RWMH endogenous threshold methodology (for results over the fixed-threshold forecasts and the left and right tailed 99% and 95% VaR we refer the reader to already described Tables 13, 14, 15 and 16).

Figure 26: Out-of-sample violations in one-step projection



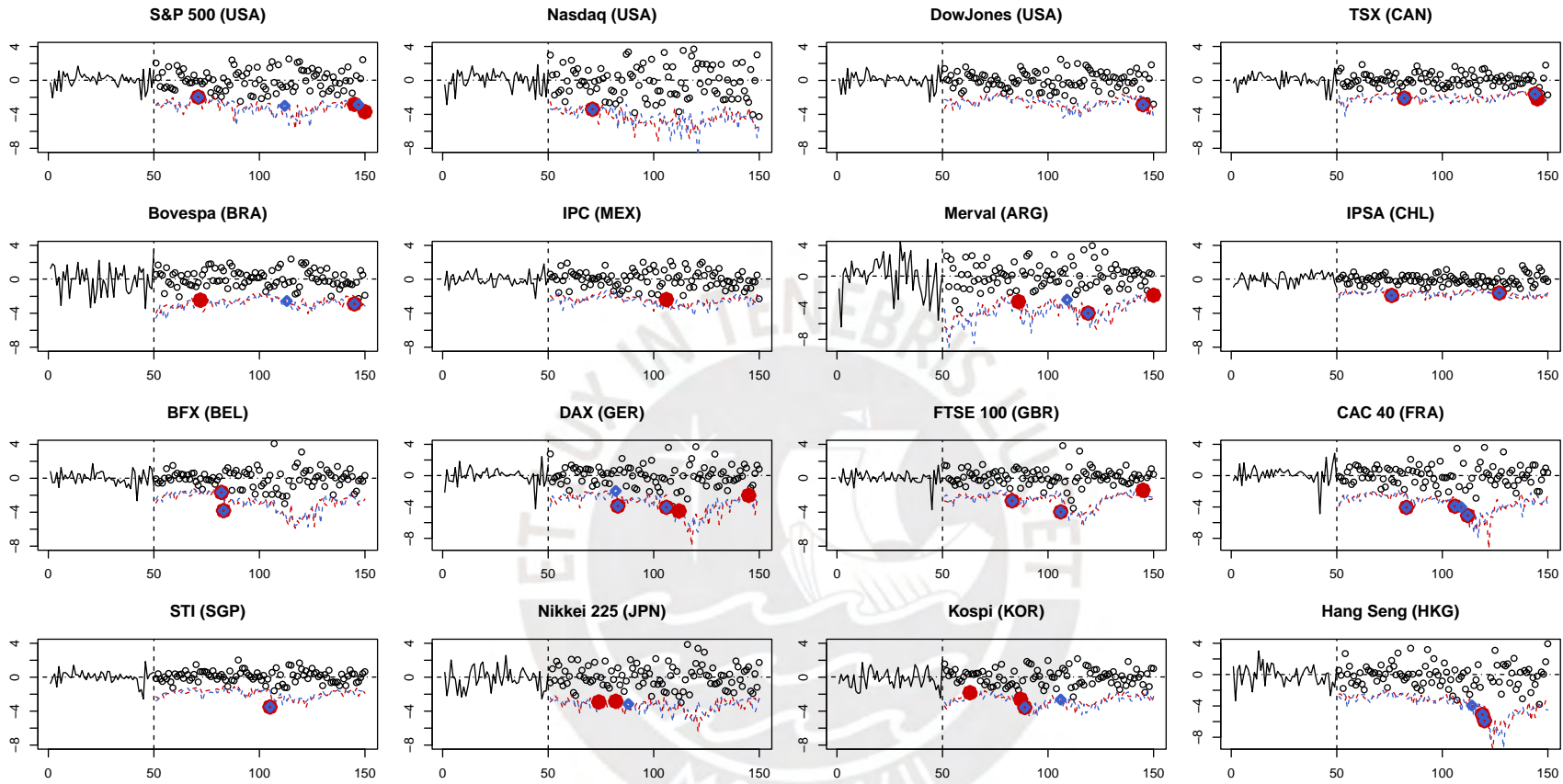
Finally, Figure 27 presents the same information, but for all the stock market indices that have been studied. We fixed the y-axis in order to highlight how one-step-ahead rolling projections differ among markets and also how the in-sample data, for where we present the last fifty observations -black lines- also differ in variability.

#### 4 Concluding remarks

Let us recall that our problem statement sought to study the impact in goodness-of-fit and prediction accuracy when we allow for an endogenous threshold extension in the THSV-SMN model developed by Abanto & Garrafa (2019). Yet, the research has a second main objective: we submit our model to a regional analysis for stock market indices across four regions in the globe. Recalling the objectives, along the document we have continually shown how markets do behave differently in terms of summary statistics, estimated posterior parameters' densities, goodness-of-fit criteria, prediction accuracy and risk metrics properties.



Figure 27: Out-of-sample violations in one-step projection – Stock Market Indices



Because of that, we conclude that our estimation results supports our hypothesis. The THSV-SMN model with endogenous threshold outperforms the fixed-zero threshold in goodness-of-fit criteria using the WAIC statistic in the majority of our sample analysis. But most importantly, results show evidence that stock market indices do have a specific behavior. Specially, evidence in the parameters' posterior distributions indicate that there exists differences within and across regions. Regarding the risk metric statistics, the use of the THSV-SMN models in order to assess value at risk computation and analysis has also shown that we ought to be prudent assessing the distribution VaR analysis: results vary widely but in the Asia region, the endogenous THSV slash model outperforms the others.

Regarding the parameters' posterior distributions, results clear show evidence on the existence of a threshold. More specifically, the persistence parameters,  $\beta_{s_t}$ , in the compound return and the location parameters,  $\alpha_{s_t}$ , in the log volatility equations do behave differently according to its regime. Future research could also aim to study the impact in the implementation of a more parsimonious model, yet we anticipate that time-efficiency gains would be marginal.

Finally, results indicate that stock market indices have also clear differences across regions. Posterior modes of the parameters vary within and across regions. Nevertheless, those specific and tailored cases can be covered by these following facts: (i) the lower variance,  $\sigma_{s_t}^2$  in the log-volatility equation (ii) the close-to-zero location,  $\alpha_{s_t}$ , and close-to-unity persistence,  $\phi_{s_t}$ , in the log-volatility equation, (iii) in our sample data analysis, the threshold,  $r$ , ranges close to zero and clearly differentiates the posterior distributions in the compound return equation,  $y_t$ , and (iv) posterior mean of the threshold,  $r$ , locates slightly above zero and evidence appear to link this threshold value to the mean of the observed compound return data. Future research and analysis may address this issue.

## 5 Discussion

In this research, we have proposed an endogenous threshold stochastic volatility model with scale mixture of normal distributions as an alternative to the standard normal -symmetric- assumption in the conditional distribution of the returns. The THSV-SMN class of models have proven to be flexible and comprehensive in the treatment of both the skewness and the kurtosis in the error distribution. Within a Bayesian framework, we have extended the MCMC sampling procedure proposed in Abanto & Garrafa (2019) in order to estimate all the parameters and latent quantities in our proposed THSV-SMN model allowing for an additional sampling step for the threshold. As a consequence of the MCMC algorithm, we have been able to produce an estimate of the latent information process which can be used in financial modeling. Also, we have applied our methods to the analysis of sixteen representative stock market indices across four regions –i.e. North America, Latam, Europe and Asia–, whose results mainly have shown that exists an important difference in the behavior within and across regions: stock market indices do behave differently and the study of each case should be addressed independently. As a consequence, an endogenous threshold has shown the capacity to improve the robustness in the majority of the cases.

Future research topic follows directly from the empirical application of our proposed endogenous threshold using the order statistics methodology. Additional extensions also can be assessed allowing for a non-zero correlation between the innovations in the compound return and the log-volatility equations.

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## Appendix: Code Quality Review – Extended THSVS Simulation

### Part I: Functions

#### a. THSV-S simulation

The function defined bellow simulates the following THSV-S model:

$$\begin{aligned}
 y_t &= \mu_{s_t} + \beta_{s_t} y_{t-1} + e^{(h_t/2)} \lambda_t^{-1/2} \epsilon_t \\
 h_{t+1} &= \alpha_{s_{t+1}} + \phi_{s_{t+1}} h_t + \sigma_{s_{t+1}} \eta_t \\
 \epsilon_t &\sim \mathcal{N}(0, 1) \\
 \eta_t &\sim \mathcal{N}(0, 1) \\
 s_t &= \begin{cases} 1 & \text{if } y_{t-1} \geq r \\ 0 & \text{if } y_{t-1} < r \end{cases}
 \end{aligned}$$

The inputs are described bellow as well:

Table 17: Inputs for THSVS

Argument	Description
$\mu_0, \mu_1$	Constants terms of (1)
$\alpha_0, \alpha_1$	Constants terms of (2)
$\beta_0, \beta_1$	AR(1) coefficients terms of $y_t$
$\phi_0, \phi_1$	AR(1) coefficients terms of $h_{t+1}$
$\sigma_0, \sigma_1$	Variance coefficients terms of $\eta_t$
$\nu$	Mixture Scale coefficient
$r$	Threshold value
$p_{11}$	Transition probability matrix main coefficient
$y_1$	Initial value for the compounded return equation
$n$	Sample size desired

We begin the sequence at  $t = 1$  and designed the vectors so they all can have the same size. That is, the vectors  $y_t, \epsilon_t, \eta_t, h_t, \lambda_t$  and  $S_t$  are defined as:

$$\begin{aligned}
 \mathbf{y}_{1:N} &= [y_1, y_2, \dots, y_{N-1}, y_N] \\
 \boldsymbol{\epsilon}_{1:N} &= [\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}, \epsilon_N] \\
 \boldsymbol{\eta}_{1:N} &= [\eta_1, \eta_2, \dots, \eta_{N-1}, \eta_N] \\
 \mathbf{h}_{1:N} &= [NA, h_2, \dots, h_{N-1}, h_N] \\
 \boldsymbol{\lambda}_{1:N} &= [NA, \lambda_2, \dots, \lambda_{N-1}, \lambda_N] \\
 \mathbf{S}_{1:N} &= [NA, S_2, \dots, S_{N-1}, S_N]
 \end{aligned}$$

Finally, the initial value for the log-volatility is defined as follows:

$$h_2 = \frac{\alpha_{S_{t=2}}}{1 - \phi_{S_{t=2}}} + \sqrt{\frac{\sigma_{S_{t=2}}^2}{1 - \phi_{S_{t=2}}^2}} \cdot \eta_2$$



With the initial values  $y_1$  and  $h_2$  we can obtain  $y_2$  and the recursion can begin.

```

1 sim_thsv <- function(mu_0, mu_1, beta_0, beta_1, alpha_0, alpha_1,
2                   phi_0, phi_1, sigma2_0, sigma2_1, nu, y_1, r, n){
3   y_t <- rep(NA,n)
4   h_t <- rep(NA,n)
5   s_t <- rep(NA, n)
6   lambda_t <- rep(NA, n)
7
8   y_t[1] <- y_1
9   s_t[2] <- ifelse(y_1 > r, 1, 0)
10  lambda_t[2] <- rbeta(1,nu,1)
11  alpha <- (1 - s_t[2])*alpha_0 + (s_t[2]*alpha_1)
12  phi <- (1 - s_t[2])*phi_0 + (s_t[2]*phi_1)
13  sigma2 <- (1 - s_t[2])*sigma2_0 + (s_t[2]*sigma2_1)
14  mu <- (1 - s_t[2])*mu_0 + (s_t[2]*mu_1)
15  beta <- (1 - s_t[2])*beta_0 + (s_t[2]*beta_1)
16  h_t[2] <- (alpha/(1-phi)) + sqrt(sigma2/(1-phi^2))*rnorm(1)
17  y_t[2] <- mu + (beta*y_t[1]) +
18            (exp(h_t[2]/2)*(lambda_t[2]^(-0.5))*rnorm(1))
19
20  for(j in 3:n){
21    s_t[j] <- ifelse(y_t[j-1] > r, 1, 0)
22    lambda_t[j] <- rbeta(1,nu,1)
23    alpha <- (1 - s_t[j])*alpha_0 + (s_t[j]*alpha_1)
24    phi <- (1 - s_t[j])*phi_0 + (s_t[j]*phi_1)
25    sigma2 <- (1 - s_t[j])*sigma2_0 + (s_t[j]*sigma2_1)
26    mu <- (1 - s_t[j])*mu_0 + (s_t[j]*mu_1)
27    beta <- (1 - s_t[j])*beta_0 + (s_t[j]*beta_1)
28    h_t[j] <- alpha + (phi*h_t[j-1]) + (sqrt(sigma2)*rnorm(1))
29    y_t[j] <- mu + (beta*y_t[j-1]) +
30            (exp(h_t[j]/2)*(lambda_t[j]^(-0.5))*rnorm(1))
31  }
32  return(THSVS <- data.table(vY = y_t, vH = h_t, vS = s_t, vL = lambda_t))
33 }

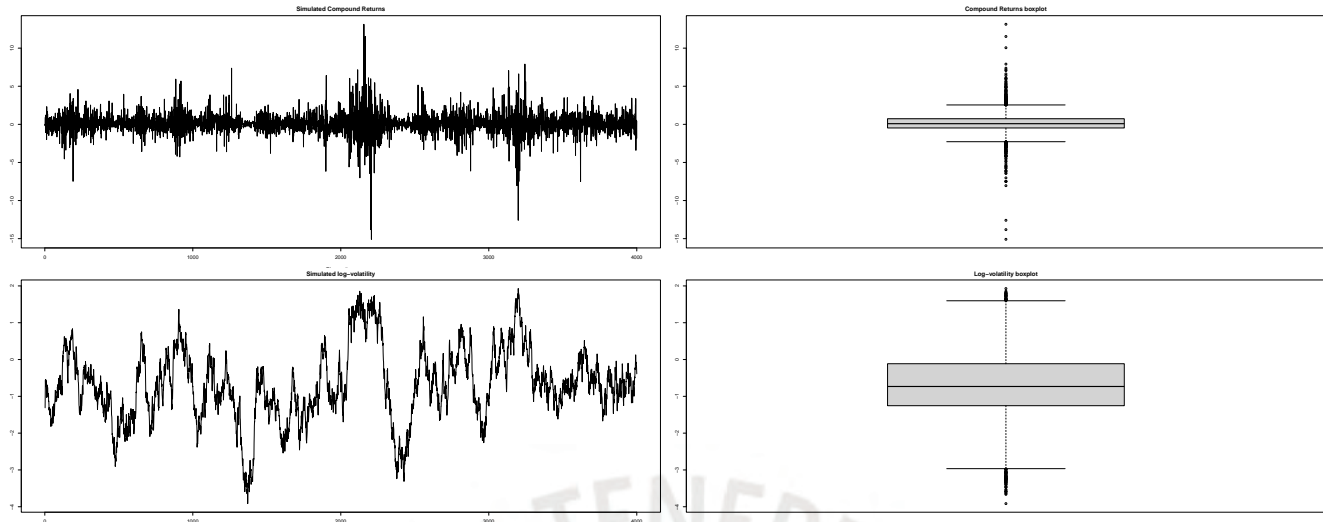
```

We have simulated  $N = 4000$  observations with the following values for the mean of our true parameters' distributions:

$$\begin{array}{ccccc}
 \mu_0 = 0.0825 & \beta_0 = -0.0680 & \alpha_0 = -0.0062 & \phi_0 = 0.9865 & \sigma_0^2 = 0.0250 \\
 \mu_1 = 0.0183 & \beta_1 = 0.0130 & \alpha_1 = -0.0154 & \phi_1 = 0.9854 & \sigma_1^2 = 0.0251 \\
 r = 0 & \nu = 1.7532 & & & 
 \end{array}$$

Figure (28) presents the simulated compound returns and the latent log-volatility:

Figure 28: Simulated data - Returns and Log-volatility



### b. randis

The randis function returns a random integer value given a vector of probabilities. This is an auxiliary function:

```

1 randis <- function(weight){
2   ns <- length(weight)
3   ff <- as.integer(rep(0, ns+1))
4   f2 <- as.integer(rep(0, 1))
5   f1 <- 0
6   gg <- cumsum(c(f1, weight))
7   uu <- runif(1)
8
9   for(j in 1:(ns+1)){
10    ff[j] <- ifelse( gg[j] < uu, 1, 0)
11    f2 <- sum(ff)
12  }
13  return(f2)

```

### c. updvZ

We use the 'updvZ function in the process of sampling the log-volatilities. We have defined:

$$z_t = \ln(y_t - \mu_{s_t} - \beta_{s_t} y_{t-1})^2$$

So, the vector  $\mathbf{z}_t$  has the following structure compared with  $\mathbf{y}_t$ :

$$\mathbf{z}_{1:N} = [\text{NA}, z_2, \dots, z_{N-1}, z_N]$$

$$\mathbf{y}_{1:N} = [y_1, y_2, \dots, y_{N-1}, y_N]$$

```

1 updvZ <- function(N, vY, vMu, vBeta){
2   updvZ <- rep(0, N)
3   updvZ[1] <- NA
4   updvZ[N] <- NA
5   for(t in 2:N){

```

```

6   updvZ[t] <- log( (vY[t] - vMu[t] - (vBeta[t]*vY[t-1]) )^2 )
7   }
8   return(updvZ)
9   }

```

#### d. updmix

Recall that  $z_t$ , after the log-square transformation, can be re-written as:

$$z_t = -\ln \lambda_t + h_t + \ln \epsilon_t^2$$

where  $\ln \epsilon_t^2 \sim \ln \chi_1^2$ . Now, note that we have a non-Gaussian state space model. Following Kim et al. (1998) and Omori et al. (2007), we approximate  $\ln \epsilon_t^2$  by a finite mixture of normal distributions. The `updmix` function returns a vector of indices from where we are going call both the mean and variance from the 10 distributions of normal densities that gives an approximation of the  $\ln \chi_1^2$  density:

$$f(\ln \epsilon_t^2) = \sum_{i=1}^P q_i f(\ln \epsilon_t^2 | k_t = i)$$

where  $k_t$  is a discrete mixing variable,  $\ln \epsilon_t^2 | k_t = i \sim \mathcal{N}(\theta_i, \varpi_i^2)$  and  $q_i = \Pr(k_t = i)$ :

```

1  updmix <- function(N, vY, vH, vL, pmix, mumix, varmix){
2  index <- rep(NA,N)
3  gmix <- length(pmix)
4
5  for(jt in 2:N){
6  qf <- rep(0, gmix)
7  pf <- rep(0, gmix)
8  for(k in 1:gmix){
9  qf[k] <- pmix[k] * dnorm( vZ[jt] + log(vL[jt]) -
10                          vH[jt] - mumix[k], sd = sqrt(varmix[k]) )
11  }
12  pf <- qf / sum(qf)
13  index[jt] <- randis(pf)
14  }
15  return(index)
16  }

```

#### e. updvS

The `updvS` function computes the vector  $\mathbf{S}_t$ . Recall that the THSV-SMN requires:

$$S_t = \begin{cases} 1 & \text{if } y_{t-1} > r \\ 0 & \text{if } y_{t-1} \leq r \end{cases}$$

So, the structure of the  $\mathbf{S}_t$  vector takes the form:

$$\mathbf{S}_{1:N} = [\text{NA}, S_2, \dots, S_{N-1}, S_N]$$

The  $r$  input will give us the flexibility to change the conditions and compute  $\mathbf{S}_t$  with different characteristics.

```

1  updvS <- function(N, r, vY){
2    vS <- rep(0, N)
3    vS[1] <- NA
4    for (jt in 2:N ){
5      vS[jt] <- ifelse( vY[jt-1] > r, 1, 0)
6    }
7    return(vS)
8  }

```

#### f. updsigma

The updsigma function computes the full conditional of  $\sigma_{S_t}^2$  for  $i = 0, 1$ . We have set as prior  $\sigma_{S_t}^2 \sim \mathcal{IG}(5, 0.5)$  and the full conditional is given by:

$$\sigma_{S_t}^2 \sim \mathcal{IG}\left(\frac{n_i^*}{2}, \frac{S_i^*}{2}\right)$$

where

$$n_i^* = n_i + \sum_{t=0}^N \mathbb{I}(S_{t+1} = i)$$

$$S_i^* = S_i + \sum_{t=0}^N (h_{t+1} - H'_{t+1} \varphi_{S_t})^2 \quad \forall S_t = 0, 1$$

with  $H'_{t+1} = [1 \quad h_t]$  and  $\varphi_{S_t} = [\alpha_{S_t} \quad \phi_{S_t}]'$ :

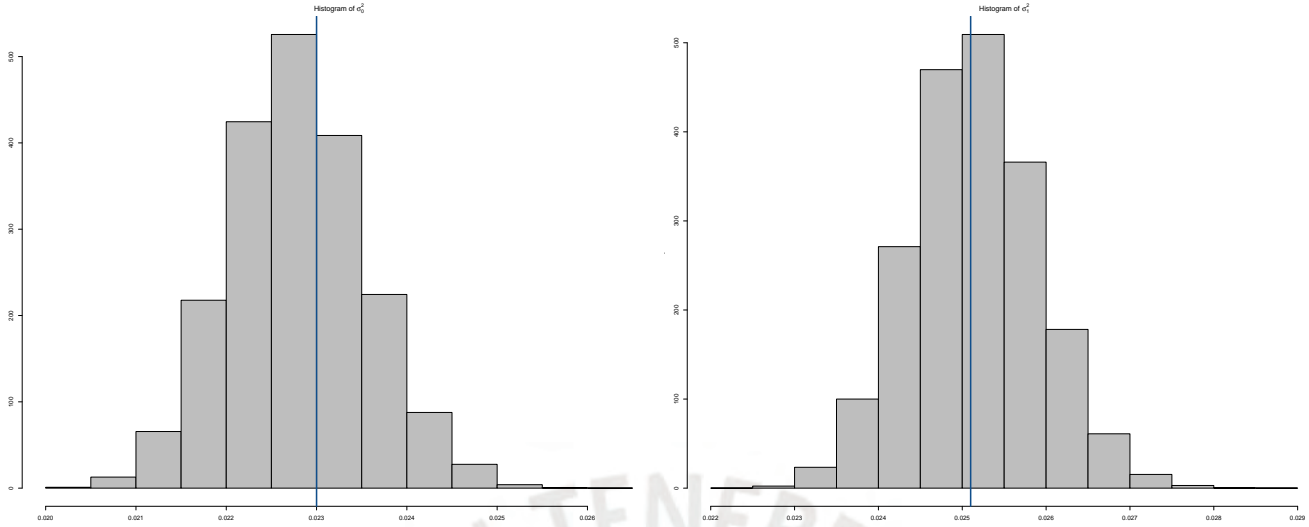
```

1  updsigma <- function(N, vH, vAlpha, vPhi, g_S0, g_S1, g_n0, g_n1, vS){
2    sigma2upd <- rep(0,2)
3    S00 <- g_S0
4    S11 <- g_S1
5    n0 <- g_n0 + length(na.omit(vS[vS==0]))
6    n1 <- g_n1 + length(na.omit(vS[vS==1]))
7
8    for (j in 3:N ){
9      if(vS[j] == 0)
10       S00 <- S00 + (( vH[j] - vAlpha[j] - (vPhi[j] * vH[j-1]) )^2)
11      else S00 <- S00 + 0
12
13      if(vS[j] == 1)
14       S11 <- S11 + (( vH[j] - vAlpha[j] - (vPhi[j] * vH[j-1]) )^2)
15      else S11 <- S11 + 0
16    }
17    sigma2upd[1] <- 1 / rgamma(1, shape = n0/2, rate = S00/2)
18    sigma2upd[2] <- 1 / rgamma(1, shape = n1/2, rate = S11/2)
19    return(sigma2upd)
20  }

```

Figure (29) presents the posterior 20,000 draws of  $\sigma_{S_t}^2$  where the blue vertical lines are the true simulated means:

Figure 29: Posterior draws of  $\sigma_{s_t}^2$



g. updalphaphi

The updalphaphi function computes the full conditional of  $\varphi_{s_t}$  for  $i = 0, 1$ . We have set as prior

$$\varphi_{s_t} \sim \mathcal{N}_2[|\phi_{s_t}| < 1](\bar{\varphi}_{s_t}, S_\varphi)$$

and the full conditional is given by:

$$\varphi_{s_t} \sim \mathcal{N}_2[|\phi_{s_t}| < 1](\bar{\varphi}_{s_t}^*, \Sigma_\varphi)$$

where  $\mathcal{N}_2[|\phi_{s_t}| < 1]$  denotes the bi-variate truncated normal distribution and

$$\bar{\varphi}_{s_t}^* = \Sigma_\varphi \left[ S_\varphi^{-1} \bar{\varphi}_{s_t} + \sum_{t=0}^N \frac{h_{t+1} H_{t+1}}{\sigma_{s_t}^2} \right]$$

$$\Sigma_\varphi = \left[ S_\varphi^{-1} + \sum_{t=0}^N \frac{H_{t+1} H_{t+1}'}{\sigma_{s_t}^2} \right]^{-1} \quad \forall S_t = 0, 1$$

```

1 updalphaphi <- function(vH, N, gSvp0, gSvp1, gvp0, gvp1, vSigma2, vS){
2   vphi0 <- rep(0,2)
3   vphi1 <- rep(0,2)
4   s00 <- matrix(0,2,2)
5   s01 <- matrix(0,2,2)
6   m00 <- rep(0,2)
7   m01 <- rep(0,2)
8  .mvp0 <- rep(0,2)
9  .mvp1 <- rep(0,2)
10  Ht <- matrix(0,N,2)
11  Ht[,1] <- 1
12  Ht[,2] <- vH
13
14  for (j in 3:N){
15
16    if(vS[j] == 0)

```

```

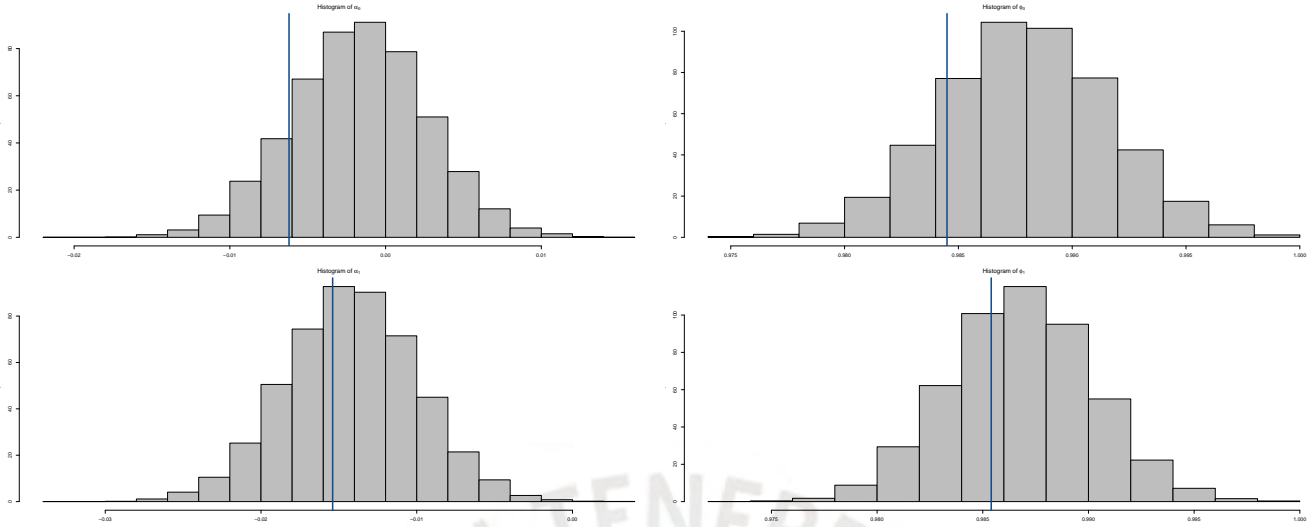
17     s00 <- s00 + (( Ht[j-1,] % * % t(Ht[j-1,]) ) / vSigma2[j])
18     else s00 <- s00 + matrix(0,2,2)
19
20     if(vS[j] == 0)
21         m00 <- m00 + (( vH[j]*Ht[j-1,] ) / vSigma2[j])
22     else m00 <- m00 + 0
23 }
24
25 s00 <- s00 + solve(gSvp0)
26 m00 <- m00 + (solve(gSvp0) % * % gvp0)
27 invs00 <- solve(s00)
28.mvp0 <- invs00 % * % m00
29 B0 <- chol(invs00)
30 d0 <- 2
31
32 while (d0 > 0){
33     vphi0 <-.mvp0 + (t(B0) % * % rnorm(2))
34     if ( (vphi0[2] > -1) & (vphi0[2] < 1) )
35         {d0 = 0}
36 }
37
38 for (j in 3:N){
39
40     if(vS[j] == 1)
41         s01 <- s01 + (( Ht[j-1,] % * % t(Ht[j-1,]) ) / vSigma2[j])
42     else s01 <- s01 + matrix(0,2,2)
43
44     if(vS[j] == 1)
45         m01 <- m01 + ( vH[j]*Ht[j-1,] ) / vSigma2[j]
46     else m01 <- m01 + 0
47 }
48
49 s01 <- s01 + solve(gSvp1)
50 m01 <- m01 + (solve(gSvp1) % * % gvp1)
51 invs01 <- solve(s01)
52.mvp1 <- invs01 % * % m01
53 B1 <- chol(invs01)
54 d1 <- 2
55
56 while (d1 > 0){
57     vphi1 <-.mvp1 + (t(B1) % * % rnorm(2))
58     if ( (vphi1[2] > -1) & (vphi1[2] < 1) )
59         {d1 = 0}
60 }
61
62 vphi <- cbind(t(vphi0), t(vphi1))
63 colnames(vphi) <- c("alpha0", "phi0", "alpha1", "phi1")
64 return(vphi)
65 }

```

Figure (30) presents the posterior 20,000 draws of  $\varphi_s$ , where the blue vertical lines are the true simulated means (first row:  $\alpha_0, \phi_0$ ; second row:  $\alpha_1, \phi_1$ ):



Figure 30: Posterior draws of  $\varphi_{s_t} = (\alpha_{s_t}, \phi_{s_t})$



#### h. updmubeta

The updmubeta function computes the full conditional of  $\psi_{s_t}$  for  $i = 0, 1$ . We have set as prior

$$\psi_{s_t} \sim \mathcal{N}_{2[|\beta_{s_t}| < 1]}(\bar{\psi}_{s_t}, S_{\psi})$$

and the full conditional is given by:

$$\psi_{s_t} \sim \mathcal{N}_{2[|\beta_{s_t}| < 1]}(\bar{\psi}_{s_t}^*, \Sigma_{\psi})$$

where  $\mathcal{N}_{2[|\beta_{s_t}| < 1]}$  denotes the bi-variate truncated normal distribution and

$$\bar{\psi}_{s_t}^* = \Sigma_{\psi} \left[ S_{\psi}^{-1} \bar{\psi}_{s_t} + \sum_{t=0}^N \lambda_t \exp(-h_t) r_t y_t \right]$$

$$\Sigma_{\psi} = \left[ S_{\psi}^{-1} + \sum_{t=0}^N \lambda_t \exp(-h_t) r_t r_t' \right] \quad \forall S_t = 0, 1$$

where  $r_t' = [1 \quad y_{t-1}]$

```

1  updmubeta <- function(vY, vH, vL, N, gSps0, gSps1, gps0, gps1, vS){
2  vpsi0 <- rep(0,2)
3  vpsi1 <- rep(0,2)
4  S00 <- matrix(0,2,2)
5  S01 <- matrix(0,2,2)
6  m00 <- rep(0,2)
7  m01 <- rep(0,2)
8  mps0 <- rep(0,2)
9  mps1 <- rep(0,2)
10
11  Mt <- matrix(0,N,2)
12  Mt[,1] <- 1
13  Mt[,2] <- vY
14
15  for (j in 2:N){
16
17    if(vS[j] == 0)

```

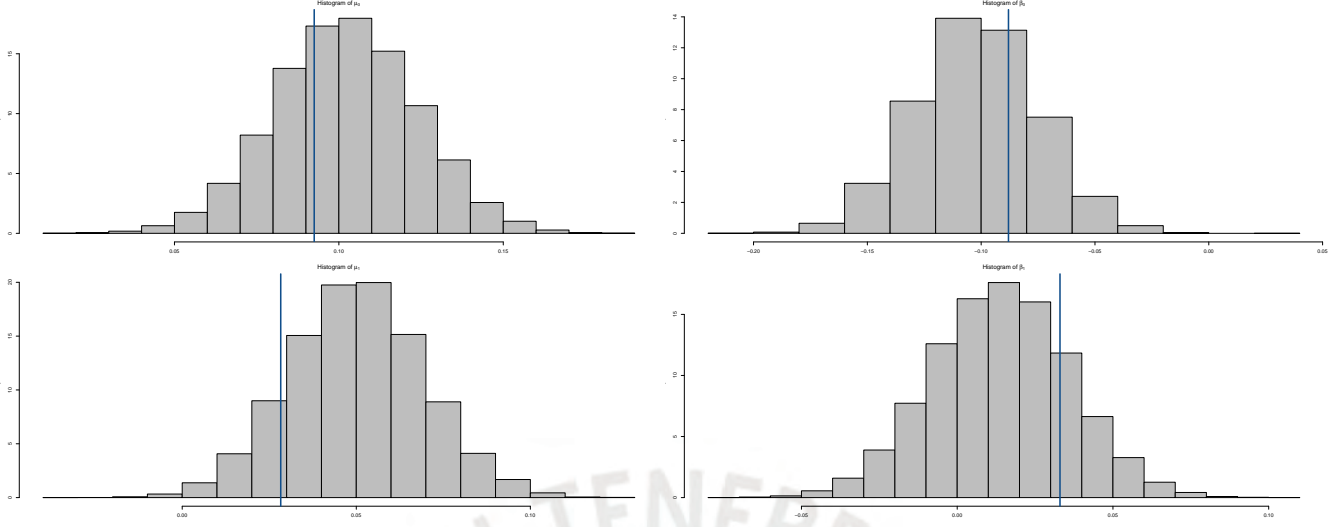
```

18     S00 <- S00 + ( vL[j]*exp(-vH[j]) * (Mt[j-1,] % * % t(Mt[j-1,]) ) )
19     else S00 <- S00 + matrix(0,2,2)
20
21     if(vS[j] == 0)
22         m00 <- m00 + (Mt[j-1,] * vL[j] * exp(-vH[j]) * vY[j])
23     else m00 <- m00 + 0
24 }
25
26 S00 <- S00 + solve(gSps0)
27 m00 <- m00 + (solve(gSps0) % * % gps0)
28 invS00 <- solve(S00)
29 mps0 <- invS00 % * % m00
30 B0 <- chol(invS00)
31 d0 <- 2
32
33 while (d0 > 0){
34     vpsi0 <- mps0 + (t(B0) % * % rnorm(2))
35     if ( (vpsi0[2] > -1) & (vpsi0[2] < 1) )
36         {d0 = 0}
37 }
38
39 for (j in 2:N){
40
41     if(vS[j] == 1)
42         S01 <- S01 + ( vL[j]*exp(-vH[j]) * (Mt[j-1,] % * % t(Mt[j-1,]) ) )
43     else S01 <- S01 + matrix(0,2,2)
44
45     if(vS[j] == 1)
46         m01 <- m01 + (Mt[j-1,] * vL[j] * exp(-vH[j]) * vY[j])
47     else m01 <- m01 + 0
48 }
49
50 S01 <- S01 + solve(gSps1)
51 m01 <- m01 + (solve(gSps1) % * % gps1)
52 invS01 <- solve(S01)
53 mps1 <- invS01 % * % m01
54 B1 <- chol(invS01)
55 d1 <- 2
56
57 while (d1 > 0){
58     vpsi1 <- mps1 + (t(B1) % * % rnorm(2))
59     if ( (vpsi1[2] > -1) & (vpsi1[2] < 1) )
60         {d1 = 0}
61 }
62
63 vpsi <- cbind(t(vpsi0), t(vpsi1))
64 colnames(vpsi) <- c("mu0", "beta0", "mu1", "beta1")
65 return(vpsi)
66 }

```

Figure (31) presents the posterior 20,000 draws of  $\psi_{s_t}$  where the blue vertical lines are the true simulated means (first row:  $\mu_0, \beta_0$ ; second row:  $\mu_1, \beta_1$ ):

Figure 31: Posterior draws of  $\psi_{S_t} = (\mu_{S_t}, \beta_{S_t})$



i. updvH

The updvH function simulates the states for the log-volatility  $h_t$ . We use the simulation smoothing method proposed by McCausland et al. (2011) to sample from the states in the system defined by the following equations:

$$z_t = -\ln \lambda_t + h_t + \xi_t, \quad \xi_t | k_t = i \sim \mathcal{N}(\vartheta_i, \varpi_i^2)$$

$$h_{t+1} = \alpha_{S_{t+1}} + \phi_{S_{t+1}} h_t + \sigma_{S_{t+1}} \eta_t$$

Conditional to  $k_t$  and  $S_t$ , the system follows a linear Gaussian state space model. This method has been already described in Section 2.4. Nevertheless, the simulations using the MMP procedure for the  $j$ -th iteration are also described below:

*MMP algorithm*

1. Simulate  $h_T^{(j)} \sim \mathcal{N}(m_T^{(j)}, \Sigma_T^{(j)})$
2. For  $t = T - 1, \dots, 1$ , simulate  $h_t^{(j)} \sim \mathcal{N}(m_t^{(j)} - \Sigma_t^{(j)} \Omega_{t,t+1}^{(j)} h_{t+1}^{(j)}, \Sigma_t^{(j)})$

where

$$\Sigma_1 = \frac{1}{\Omega_{11}}$$

$$\Sigma_t = (\Omega_{t,t} - \Omega_{t-1,t}^2 \Sigma_{t-1})^{-1}$$

$$m_1 = \Sigma_1 c_1$$

$$m_t = \Sigma_t (c_t - \Omega_{t-1,t} m_{t-1})$$

$\Omega$ , a  $T \times T$  symmetric matrix, on its general form is defined as follows:

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & 0 & \dots & 0 \\ \Omega_{12} & \Omega_{22} & \Omega_{23} & \dots & 0 \\ 0 & \Omega_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \Omega_{T-1,T-1} & \Omega_{T-1,T} \\ 0 & 0 & \dots & \Omega_{T-1,T} & \Omega_{T,T} \end{pmatrix}$$

where

$$\begin{aligned}\Omega_{11} &= A_{1,1} + [(1 - S_1)\phi_0 + S_1\phi_1] \times A_{1,2} + \frac{1}{P_1} \\ \Omega_{tt} &= A_{t,1} + [(1 - S_t)\phi_0 + S_t\phi_1] \times A_{t,2} + A_{t-1,2} \\ \Omega_{TT} &= \frac{1}{\varpi_{k_T}^2} + A_{T-1,2} \\ \Omega_{t,t+1} &= -[(1 - S_t)\phi_0 + S_t\phi_1] \times A_{t,2}\end{aligned}$$

We also have defined  $A_t$  as:

$$A_t = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ \vdots & \vdots \\ A_{t,1} & A_{t,2} \end{bmatrix}$$

where

$$\begin{aligned}A_{1,t} &= \frac{1}{\varpi_{k_t}^2} \\ A_{2,t} &= \frac{1}{[(1 - S_t)\sigma_0^2 + S_t\sigma_1^2]}\end{aligned}$$

and

$$\begin{aligned}c_1 &= A_{1,1} \times [z_1 + \log(\lambda_1) + \vartheta_{k_1}] - [(1 - S_1)\phi_0 + S_1\phi_1] \times A_{1,2} \times [(1 - S_1)\alpha_0 + S_1\alpha_1] + \frac{a_1}{P_1} \\ c_t &= A_{1,1} \times [z_t + \log(\lambda_t) + \vartheta_{k_t}] - [(1 - S_t)\phi_0 + S_t\phi_1] \times A_{t,2} \times [(1 - S_t)\alpha_0 + S_t\alpha_1] \\ &\quad + A_{t-1,2} \times [(1 - S_{t-1})\alpha_0 + S_{t-1}\alpha_1] \\ c_T &= \frac{1}{\varpi_{k_T}^2} [z_T + \log(\lambda_T) + \vartheta_{k_T}] + A_{T-1,2} \times [(1 - S_{T-1})\alpha_0 + S_{T-1}\alpha_1]\end{aligned}$$

with

$$\begin{aligned}a_1 &= \frac{\alpha_{S_{i=1}}}{1 - \phi_{S_{i=1}}} \\ P_1 &= \frac{\sigma_{S_{i=1}}^2}{1 - \phi_{S_{i=1}}^2}\end{aligned}$$

```

1 updvH <- function(N, vZ, vL, vSigma2, vAlpha, vPhi, kmix, mumix, varmix){
2   P1      <- vSigma2[2] / (1 - vPhi[2]^2)
3   a1      <- vAlpha[2] / (1 - vPhi[2])
4   states  <- rep(NA, N)
5   Omegat  <- rep(0, N)
6   Omegattplus1 <- rep(0, N)
7   Sigmat  <- rep(0, N)
8   mt      <- rep(0, N)
9   At      <- matrix(NA, N, 2)
10  ct      <- rep(NA, N)
11
12  for (j in 2:N){
13    At[j,1] <- 1 / varmix[kmix[j]]
14    At[j,2] <- 1 / vSigma2[j]
15  }
16
17  Omegat[2] <- At[2,1] + At[2,2]*((vPhi[2])^2) + (1/P1)

```

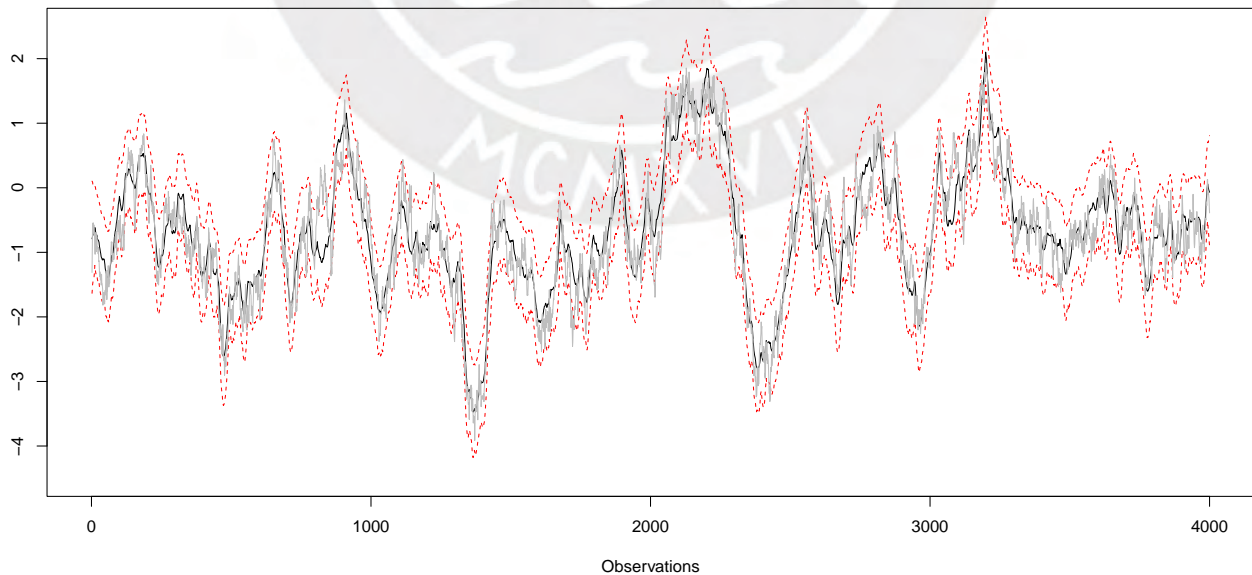
```

18 ct[2]      <- At[2,1]*(vZ[2] + log(vL[2]) - mumix[kmix[2]]) -
19   (vPhi[2]*At[2,2]*vAlpha[2]) + (a1/P1)
20 Sigmat[2] <- 1/Omegat[2]
21 mt[2]     <- Sigmat[2]*ct[2]
22 Omegattplus1[2] <- -vPhi[2]*At[2,2]
23
24 for(j in 3:(N-1) ){
25   Omegat[j]      <- At[j,1] + (At[j,2]*(vPhi[j]^2)) + At[j-1,2]
26   ct[j]         <- At[j,1]*(vZ[j] + log(vL[j]) - mumix[kmix[j]]) -
27   (vPhi[j]*At[j,2]*vAlpha[j]) + (At[j-1,2]*vAlpha[j-1])
28   Omegattplus1[j] <- -vPhi[j]*At[j,2]
29   Sigmat[j]      <- 1 / (Omegat[j] - (Omegattplus1[j-1]^2)*Sigmat[j-1])
30   mt[j]         <- Sigmat[j]*(ct[j] - Omegattplus1[j-1]*mt[j-1])
31 }
32
33 Omegat[N] <- 1/varmix[kmix[N]] + At[N-1,2]
34 ct[N]    <- (1/varmix[kmix[N]])*(vZ[N] + log(vL[N]) - mumix[kmix[N]])
35   + (At[N-1,2]*vAlpha[N-1])
36 Sigmat[N] <- 1/(Omegat[N] - ((Omegattplus1[N-1]^2)*Sigmat[N-1]))
37 mt[N]    <- Sigmat[N]*(ct[N] - (Omegattplus1[N-1]*mt[N-1]))
38 states[N] <- mt[N] + (sqrt(Sigmat[N])*rnorm(1))
39
40 for(j in (N-1):2){
41   states[j] <- mt[j] - (Sigmat[j]*Omegattplus1[j]*states[j+1]) +
42   (sqrt(Sigmat[j])*rnorm(1))
43 }
44 return(states)
45 }

```

Figure (32) presents the posterior 20,000 draws mean (black line), the 95% CI (i.e. Confidence Intervals in red dotted lines) and the true simulated chain (gray line) for each point of the chain  $h_t$  :

Figure 32: Log-volatility - Posterior mean and 95% CI vs. True data



j. upd\_r

The `upd_r` function computes the full conditional of  $r_t$ . We have set the true mean as  $r = 0$  and the full conditional of  $r_t$ , has been already defined in section 2.4:

```

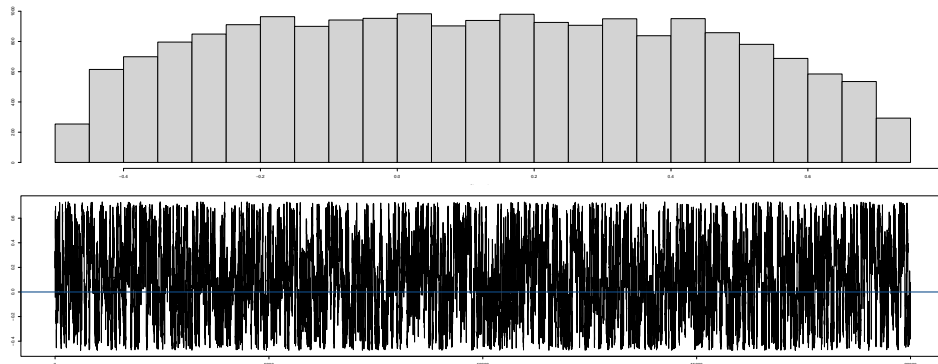
1 target <- function(N, vY, vH, vL, mu_0, mu_1, beta_0, beta_1, alpha_0,
2                   alpha_1, phi_0, phi_1, sigma2_0, sigma2_1, r){
3   likel <- rep(NA, N)
4   likel_vY <- rep(NA, N)
5   likel_vH <- rep(NA, N)
6   Smix     <- rep(NA, N)
7
8   for (jt in 2:N){
9     Smix[jt] <- ifelse( vY[jt-1] > r, 1, 0)
10  }
11  vMu   <- mu_1*Smix + ( rep(1,N) - Smix ) * mu_0
12  vBeta <- beta_1*Smix + ( rep(1,N) - Smix ) * beta_0
13  vAlpha <- alpha_1*Smix + ( rep(1,N) - Smix ) * alpha_0
14  vPhi  <- phi_1*Smix + ( rep(1,N) - Smix ) * phi_0
15  vSigma2 <- sigma2_1*Smix + ( rep(1,N) - Smix ) * sigma2_0
16
17  for(i in 3:N){
18    likel_vY   <- (1/(exp(vH[i]/2)*(vL[i]^(-0.5))))*
19                exp(-0.5*(1/(exp(vH[i])*vL[i]^(-1))*
20                (vY[i]-vMu[i]-vBeta[i]*vY[i-1])^2)
21    likel_vH[i] <- (1/vSigma2[i])*exp(-0.5*(1/vSigma2[i])*
22                (vH[i]-vAlpha[i]-vPhi[i]*vH[i-1])^2)
23    likel[i]    <- likel_vY[i] * likel_vH[i]
24  }
25
26  likel2 <- log(likel)
27  sum_llikel <- sum(na.omit(likel2))
28  return(sum_llikel)
29 }
30
31 proposed <- function(x){
32   rnorm(1, mean=x, sd=0.15)
33 }
34
35 upd_r <- function(r){
36   dd <- 2
37   while (dd > 0){
38     r_proposed <- proposed(r)
39     if ( r_proposed > q1 & r_proposed < q3)
40       {dd = 0}
41   }
42   alpha <- min(1, exp(target(N, vY, vH, vL, vpsi, vphi, sigma2, r_proposed) -
43               target(N, vY, vH, vL, vpsi, vphi, sigma2, r)))
44   accept <- runif(1) < alpha
45   r     <- ifelse(accept, r_proposed, r)
46   return(r)
47 }

```

Figure (33) presents the posterior 20,000 draws histogram:



Figure 33: RW-MH  $r_t$



k. updnv

The updnv function extract draws from the posterior distribution of  $\nu$ . The prior and the full conditional posterior distribution for  $\nu$  are:

$$\nu \sim \mathcal{G}(a_\nu, b_\nu)$$

$$\nu^* \sim \mathcal{G}_{(\nu>1)}\left(T + a_\nu, b_\nu - \sum_{t=1}^T \log \lambda_t\right)$$

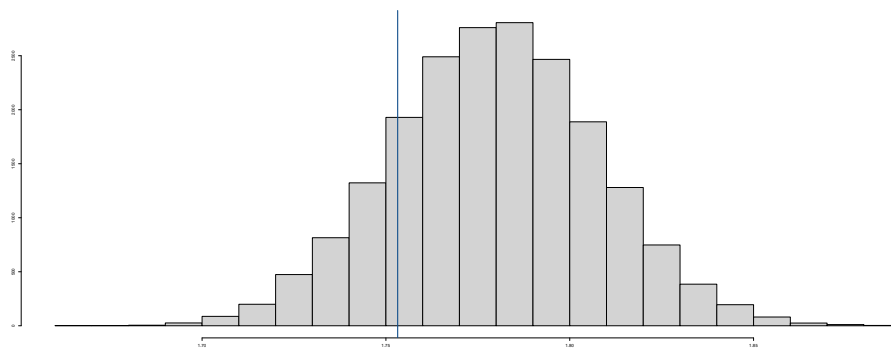
```

1 updnv <- function(vL, N){
2   Sauv <- sum(log(vL), na.rm = T)
3   TT <- N - 1
4   a <- TT + gl0
5   b <- gl1 - Sauv
6
7   dd <- 2
8   while (dd > 0){
9     nu <- rgamma(1, shape = a, rate = b)
10    if ( nu > 1 )
11      {dd = 0}
12    }
13    return(nu)
14  }

```

Figure (34) presents the posterior 20,000 draws histogram of  $\nu$ :

Figure 34:  $\nu$  posterior draws



## l. updvL

In order to simulate  $\lambda_t$  we rely on our updvL function. For that, we have computed an auxiliary function `igaright`:

```

1  igaright <- function(a, b, U){
2    yaux = exp(-b*U) * runif(1)
3    xaux = min(b, -log(yaux))
4    igr = exp( (1/a)*log(runif(1)) + log(xaux))
5    igr = igr/b
6    return(igr)
7  }

```

For  $\lambda_t$ , following Abanto & Garrafa (2019) we have both the prior and the posterior distribution below:

$$\lambda_t \sim \mathcal{Be}(\nu, 1)$$

$$\lambda_t | y_t, y_{t-1}, h_t, \Theta \sim \mathcal{G}_{(\lambda_t < 1)}\left(\nu + \frac{1}{2}, \frac{1}{2} [y_t - r'_t \phi_{s_t}]^2 e^{-h_t}\right)$$

We use the `igaright` function in order to sample the  $\lambda_t$  chain more efficiently. Now we present the updvL function:

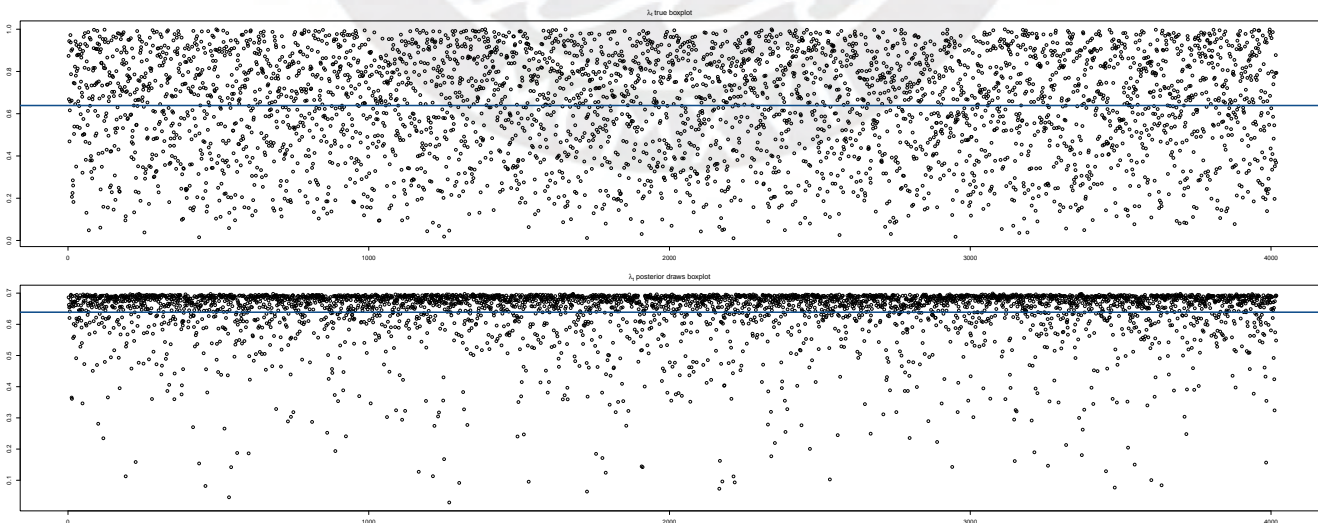
```

1  updvL <- function(vH, vY, vMu, vBeta, nu, vLprev){
2    vL <- rep(NA, N)
3    aa = nu + 0.5
4
5    for(j in 2:N){
6      bb <- 0.5*exp(-vH[j]) * ((vY[j] - vMu[j] - vBeta[j]*vY[j-1])^2)
7      vL[j] <- igaright(aa, bb, vLprev[j])
8    }
9    return(vL)
10 }

```

Figure (35) presents the posterior 20,000 draws histogram of  $\lambda_t$ :

Figure 35:  $\lambda_t$  posterior draws vs true chain boxplots



## m. upd\_forecast

The `upd_forecast` function has been also used to compute the one-step ahead forecast of  $y_t$ .

```

1  upd_forecast <- function(vphi, vpsi, sigma2, nu, r, vY, vH){
2  Sn1 <- ifelse(vY[N] > r, 1, 0)
3  alpha <- (1 - Sn1)*vphi[1,1] + Sn1*vphi[1,3]
4  phi <- (1 - Sn1)*vphi[1,2] + Sn1*vphi[1,4]
5  sigma2 <- (1 - Sn1)*sigma2[2] + Sn1*sigma2[1]
6  mu <- (1 - Sn1)*vpsi[1,1] + Sn1*vpsi[1,3]
7  beta <- (1 - Sn1)*vpsi[1,2] + Sn1*vpsi[1,4]
8
9  Hn1 <- alpha + phi + vH[N] + sqrt(sigma2)*rnorm(1)
10 Ln1 <- rbeta(1, nu, 1)
11 Yn1 <- mu + beta*vY[N] + exp(Hn1/2)*((Ln1)^(-1/2))*rnorm(1)
12 return(Yn1)
13 }

```

#### n. deviance

The deviance function has been used to compute the Watanabe-Akaike information criterion proposed by Watanabe (2010, 2013).

```

1  deviance <- function(N, vY, vH, vL, vMu, vBeta){
2  llikel <- 0
3
4  for (tt in 2:N){
5    llikel <- llikel + dnorm(vY[tt] - vMu[tt] - (vBeta[tt] * vY[tt-1]),
6                          , sd = sqrt(1/vL[tt])*exp(0.5*vH[tt]), log=T)
7  }
8  dev <- -2*llikel
9  return(dev)
10 }

```

#### o. devWAIC

The `devWAIC` function has been also used to compute the Watanabe-Akaike information criterion (WAIC).

```

1  devWAIC <- function(N, vY, vH, vL, vMu, vBeta){
2  dev1 <- rep(NA, N)
3  dev2 <- rep(NA, N)
4
5  for (tt in 2:N){
6    dev1[tt] <- dnorm(vY[tt] - vMu[tt] - (vBeta[tt]*vY[tt-1]),
7                    sd = sqrt(1/vL[tt])*exp(0.5*vH[tt]))
8    dev2[tt] <- dnorm(vY[tt] - vMu[tt] - (vBeta[tt]*vY[tt-1]),
9                    sd = sqrt(1/vL[tt])*exp(0.5*vH[tt]), log=T)
10 }
11 waic <- cbind(dev1, dev2)
12 return(waic)
13 }

```

## Part II: Main

The following code corresponds to the main structure to estimate the original THSV-SMN in the specific case where the mixture results is a Slash distribution. This includes the endogenous threshold extension:

```

1 ##### THSV - SMN Slash - Random Walk Metropoli Hasting Threshold #####
2 ##### Written by:          Ronaldo Robles (PUCP - PER)
3 ##### Main reference and C++ code:      Ph. D Carlos Abanto (UFRJ - BRA)
4
5 # Calling functions #####
6
7 set.seed(20051997)
8 source("THSV-S - Threshold - Functions.R")
9
10 # Priors #####
11
12 #sigma2
13 sigma2 <- rep(0,2)
14 sigma2[1] <- mean(1/rgamma(10000, shape = 5, rate = 0.5))
15 sigma2[2] <- mean(1/rgamma(10000, shape = 5, rate = 0.5))
16 g_n0 <- 10.0
17 g_n1 <- 10.0
18 g_S0 <- 0.1
19 g_S1 <- 0.1
20
21 #varpsi
22 mu <- rep(0,2)
23 mu[1] <- 0.05
24 mu[2] <- 0.05
25 beta <- rep(0,2)
26 beta[1] <- 0.05
27 beta[2] <- 0.05
28 gps0 <- rep(0,2)
29 gps1 <- rep(0,2)
30 gSps0 <- 100*diag(2)
31 gSps1 <- 100*diag(2)
32
33 #varphi
34 alpha <- rep(0,2)
35 alpha[1] <- 0.05
36 alpha[2] <- 0.05
37 phi <- rep(0.98,2)
38 phi[1] <- 0.98
39 phi[2] <- 0.98
40 gSvp0 <- 100*diag(2)
41 gSvp1 <- 100*diag(2)
42 gvp0 <- rep(0,2)
43 gvp0[2] <- 0.98
44 gvp1 <- rep(0,2)
45 gvp1[2] <- 0.98
46
47 r <- 0
48 gl0 <- 0.08
49 gl1 <- 0.04
50 nu <- mean(rgamma(10000, shape = gl0, rate = gl1))
51
52 # Data #####
53 Proyect <- getwd()
54 Data <- read.table("Global/Data/LATAM/MERVAL.txt")$V1
55 vY <- Data[1:(length(Data)-100)]
56
57 N <- length(vY)
58 q1 <- summary(vY)[2]
59 q3 <- summary(vY)[5]
60 pmix <- c(0.00609, 0.04775, 0.13057, 0.20674, 0.22715, 0.18842, 0.12047,
61          0.05591, 0.01575, 0.00115) #probs

```

```

62 mumix <- c(1.92677, 1.34744, 0.73504, 0.02266, -0.85173, -1.97278, -3.46788,
63          -5.55246, -8.68384, -14.65000) #means
64 varmix <- c(0.11265, 0.17788, 0.26768, 0.40611, 0.62699, 0.98583, 1.57469,
65          2.54498, 4.16591, 7.33342) #variances
66
67 # Ordering data #####
68
69 #Order <- sort(vY, index.return= TRUE)
70 #iy <- Order$ix
71 #vYsort <- Order$x
72 vS <- updvS(N, r, vY)
73
74 # Parameters (given S = 1 o S = 0) #####
75
76 vMu <- mu[2]*vS + ( rep(1,N) - vS ) * mu[1]
77 vBeta <- beta[2]*vS + ( rep(1,N) - vS ) * beta[1]
78 vAlpha <- alpha[2]*vS + ( rep(1,N) - vS ) * alpha[1]
79 vPhi <- phi[2]*vS + ( rep(1,N) - vS ) * phi[1]
80 vSigma2 <- sigma2[2]*vS + ( rep(1,N) - vS ) * sigma2[1]
81
82 # Number of simulations #####
83
84 burnin = 10000
85 finit = 40000
86 vH <- rep(0,N)
87 vL <- rbeta(N, nu, 1)
88
89 # Gibbs Sampling #####
90
91 vZ <- updvZ(N, vY, vMu, vBeta)
92 kmix <- updmix(N, vY, vH, vL, pmix, mumix, varmix)
93 vH <- updvH(N, vZ, vL, vSigma2, vAlpha, vPhi, kmix, mumix, varmix)
94 nu <- updnu(vL, N)
95 vL <- updvL(vH, vY, vMu, vBeta, nu, vL)
96
97 for (jt in 1:burnin){
98   #sigma
99   sigma2 <- updsigma(N, vH, vAlpha, vPhi, g_S0, g_S1, g_n0, g_n1, vS)
100  vSigma2 <- sigma2[2] * vS + ( rep(1,N) - vS ) * sigma2[1]
101
102  #alpha and phi
103  vphi <- updalphaphi(vH, N, gSvp0, gSvp1, gvp0, gvp1, vSigma2, vS)
104  vAlpha <- vphi[1,3] * vS + ( rep(1,N) - vS ) * vphi[1,1]
105  vPhi <- vphi[1,4] * vS + ( rep(1,N) - vS ) * vphi[1,2]
106
107  #mu y beta
108  vpsi <- updmubeta(vY, vH, vL, N, gSps0, gSps1, gps0, gps1, vS)
109  vMu <- vpsi[1,3] * vS + ( rep(1,N) - vS ) * vpsi[1,1]
110  vBeta <- vpsi[1,4] * vS + ( rep(1,N) - vS ) * vpsi[1,2]
111
112  vZ <- updvZ(N, vY, vMu, vBeta)
113  kmix <- updmix(N, vY, vH, vL, pmix, mumix, varmix)
114  vH <- updvH(N, vZ, vL, vSigma2, vAlpha, vPhi, kmix, mumix, varmix)
115  nu <- updnu(vL, N)
116  vL <- updvL(vH, vY, vMu, vBeta, nu, vL)
117  r <- upd_r(r)
118
119  vS <- updvS(N, r, vY)
120  vSigma2 <- sigma2[2] * vS + ( rep(1,N) - vS ) * sigma2[1]
121  vAlpha <- vphi[1,3] * vS + ( rep(1,N) - vS ) * vphi[1,1]
122  vPhi <- vphi[1,4] * vS + ( rep(1,N) - vS ) * vphi[1,2]

```

```

123 vMu      <- vpsi[1,3] * vS + ( rep(1,N) - vS ) * vpsi[1,1]
124 vBeta   <- vpsi[1,4] * vS + ( rep(1,N) - vS ) * vpsi[1,2]
125 print(jt)
126 }
127
128 # Containers
129
130 vHfile   <- matrix(NA,finit, N)
131 vLfile   <- matrix(NA,finit, N)
132 nusimf   <- rep(NA, finit)
133
134 sigma2simf <- matrix(NA,finit,2)
135 colnames(sigma2simf) <- c("sigma0", "sigma1")
136
137 vphisimf <- matrix(NA,finit,4)
138 colnames(vphisimf) <- c("alpha0", "alpha1", "phi0", "phi1")
139
140 vpsisimf <- matrix(NA,finit,4)
141 colnames(vpsisimf) <- c("mu0", "mu1", "beta0", "beta1")
142
143 r_simf    <- rep(NA, finit)
144 vY_Tplus1_simf <- rep(NA, finit)
145 devsimf   <- rep(NA, finit)
146 waic      <- matrix(0, N, 2)
147 dev       <- 0
148
149 for (jt in (burnin+1):(burnin+finit) ){
150   #sigma
151   sigma2 <- updsigma(N, vH, vAlpha, vPhi, g_S0, g_S1, g_n0, g_n1, vS)
152   vSigma2 <- sigma2[2] * vS + ( rep(1,N) - vS ) * sigma2[1]
153
154   #alpha and phi
155   vphi <- updalphaphi(vH, N, gSvp0, gSvp1, gvp0, gvp1, vSigma2, vS)
156   vAlpha <- vphi[1,3] * vS + ( rep(1,N) - vS ) * vphi[1,1]
157   vPhi <- vphi[1,4] * vS + ( rep(1,N) - vS ) * vphi[1,2]
158
159   #mu y beta
160   vpsi <- updmubeta(vY, vH, vL, N, gSps0, gSps1, gps0, gps1, vS)
161   vMu <- vpsi[1,3] * vS + ( rep(1,N) - vS ) * vpsi[1,1]
162   vBeta <- vpsi[1,4] * vS + ( rep(1,N) - vS ) * vpsi[1,2]
163
164   vZ <- updvZ(N, vY, vMu, vBeta)
165   kmix <- updmix(N, vY, vH, vL, pmix, mumix, varmix)
166   vH <- updvH(N, vZ, vL, vSigma2, vAlpha, vPhi, kmix, mumix, varmix)
167   nu <- updnu(vL, N)
168   vL <- updvL(vH, vY, vMu, vBeta, nu, vL)
169   r <- upd_r(r)
170
171   vS <- updvS(N, r, vY)
172   vSigma2 <- sigma2[2] * vS + ( rep(1,N) - vS ) * sigma2[1]
173   vAlpha <- vphi[1,3] * vS + ( rep(1,N) - vS ) * vphi[1,1]
174   vPhi <- vphi[1,4] * vS + ( rep(1,N) - vS ) * vphi[1,2]
175   vMu <- vpsi[1,3] * vS + ( rep(1,N) - vS ) * vpsi[1,1]
176   vBeta <- vpsi[1,4] * vS + ( rep(1,N) - vS ) * vpsi[1,2]
177   dev <- dev + deviance(N, vY, vH, vL, vMu, vBeta)
178
179   rr = jt%%20
180   ss = (jt-burnin)/1
181   if (rr == 0){
182     sigma2simf[ss,] <- t(sigma2)
183     vphisimf[ss,] <- c(vphi[1,1], vphi[1,3], vphi[1,2], vphi[1,4])

```



```

184     vpsisimf[ss,]    <- c(vpsi[1,1], vpsi[1,3], vpsi[1,2], vpsi[1,4])
185     vHfile[ss,]     <- t(vH)
186     vLfile[ss,]     <- t(vL)
187     nusimf[ss]      <- nu
188     r_simf[ss]      <- r
189     devsimf[ss]     <- dev
190     waic <- waic + devWAIC(N, vY, vH, vL, vMu, vBeta)
191     vY_Tplus1_simf[jt] <- upd_forecast(vphi, vpsi, sigma2, nu, r, vY, vH)
192   }
193   else {rr = 1}
194   print(jt)
195 }
196
197 dev1      <- waic[,1]
198 dev2      <- waic[,2]
199 vHmean    <- rep(NA,N)
200 vHmode    <- rep(NA,N)
201 vHmode025 <- rep(NA,N)
202 vHmode097 <- rep(NA,N)
203
204 for (i in 2:N) {
205   vHmean[i]    <- mean(vHfile[,i], na.rm = T)
206   vHmode[i]    <- mlv(na.omit(vHfile[,i]), method = "meanshift")
207   vHmode025[i] <- quantile(vHfile[,i], probs = 0.025)
208   vHmode097[i] <- quantile(vHfile[,i], probs = 0.975)
209   print(i)
210 }
211
212 vHsimf <- cbind(vHmean, vHmode, vHmode025, vHmode097)
213
214 vLmean <- rep(NA,N)
215 vLmode <- rep(NA,N)
216 vLmode025 <- rep(NA,N)
217 vLmode097 <- rep(NA,N)
218
219 for (i in 2:N) {
220   vLmean[i]    <- mean(vLfile[,i], na.rm = T)
221   vLmode[i]    <- mlv(na.omit(vLfile[,i]), method = "meanshift")
222   vLmode025[i] <- quantile(vLfile[,i], probs = 0.025)
223   vLmode097[i] <- quantile(vLfile[,i], probs = 0.975)
224   print(i)
225 }
226 vLsimf <- cbind(vLmean, vLmode, vLmode025, vLmode097)
227
228 for(i in 1:length(devsimf)){
229   if(is.infinite(devsimf[i])){
230     devsimf[i] <- NA
231   }
232 }
233
234 dbar      <- mean(na.omit(devsimf))
235
236 for(i in 1:length(dev1)){
237   if(is.infinite(dev1[i])){
238     dev1[i] <- NA
239   }
240 }
241 for(i in 1:length(dev2)){
242   if(is.infinite(dev2[i])){
243     dev2[i] <- NA
244   }

```

```

245 }
246
247 dev1mean <- mean(na.omit(dev1))
248 dev2mean <- mean(na.omit(dev2))
249
250 # Parametros finales ####
251
252 sigma2mean <- rep(0,2)
253 sigma2mean[1] <- mean(sigma2simf[,1], na.rm = T)
254 sigma2mean[2] <- mean(sigma2simf[,2], na.rm = T)
255 sigma2mode <- rep(0,2)
256 sigma2mode[1] <- mlv(sigma2simf[,1], method = "meanshift")
257 sigma2mode[2] <- mlv(sigma2simf[,2], method = "meanshift")
258
259 alphamean <- rep(0,2)
260 alphamean[1] <- mean(vphisimf[,1], na.rm = T)
261 alphamean[2] <- mean(vphisimf[,2], na.rm = T)
262 alphamode <- rep(0,2)
263 alphamode[1] <- mlv(vphisimf[,1], method = "meanshift")
264 alphamode[2] <- mlv(vphisimf[,2], method = "meanshift")
265
266 phimean <- rep(0,2)
267 phimean[1] <- mean(vphisimf[,3], na.rm = T)
268 phimean[2] <- mean(vphisimf[,4], na.rm = T)
269 phimode <- rep(0,2)
270 phimode[1] <- mlv(vphisimf[,3], method = "meanshift")
271 phimode[2] <- mlv(vphisimf[,4], method = "meanshift")
272
273 numean <- rep(0,2)
274 numean[1] <- mean(vpsisimf[,1], na.rm = T)
275 numean[2] <- mean(vpsisimf[,2], na.rm = T)
276 mumode <- rep(0,2)
277 mumode[1] <- mlv(vpsisimf[,1], method = "meanshift")
278 mumode[2] <- mlv(vpsisimf[,2], method = "meanshift")
279
280 betamean <- rep(0,2)
281 betamean[1] <- mean(vpsisimf[,3])
282 betamean[2] <- mean(vpsisimf[,4])
283 betamode <- rep(0,2)
284 betamode[1] <- mlv(vpsisimf[,3], method = "meanshift")
285 betamode[2] <- mlv(vpsisimf[,4], method = "meanshift")
286
287 r_mean <- mean(r_simf)
288 r_mode <- mlv(r_simf, method="meanshift")
289
290 numean <- mean(nusimf)
291 numode <- mlv(nusimf, method = "meanshift")
292
293 vS <- updvS(N, r_mean, vY)
294 vSigma2 <- numean[2] * vS + ( rep(1,N) - vS ) * sigma2[1]
295 vAlpha <- alphamean[2] * vS + ( rep(1,N) - vS ) * alphamean[1]
296 vPhi <- phimean[2] * vS + ( rep(1,N) - vS ) * phimean[1]
297 vMu <- numean[2] * vS + ( rep(1,N) - vS ) * numean[1]
298 vBeta <- betamean[2] * vS + ( rep(1,N) - vS ) * betamean[1]
299
300 # WAIC Statistic ####
301
302 # With the mean
303 dv <- deviance(N, vY, vHmean, vLmean, vMu, vBeta)
304 pD <- dbar - dv
305 DIC <- dbar + pD

```

```

306 lppd <- sum (log(na.omit(dev1mean)))
307 pwaic <- 2 * ( lppd - sum(na.omit(dev2mean)))
308 WAIC <- -2 * ( lppd - pwaic )
309 Statistics <- cbind(dbar, pD, DIC, lppd, pwaic, WAIC)
310 colnames(Statistics) <- c("dBar", "pD", "DIC", "lppd", "pWAIC", "WAIC")
311
312 save.image("Workspace - Merval.RData")

```

## Part III: VaR

For VaR and ES statistics, we have to use an abrasive and computationally intensive procedure. This is the recursion we use for each  $T + 1$  estimation. We have computed the likelihood ratio test Kupiec (1995) for the VaR theoretic percentiles and the measure of Embrechts et al (2004) for the Expected Shortfall prediction accuracy. We compare four models i.e the THSV - Normal with a fixed-zero threshold, the THSV - Normal with a RWMH endogenous threshold and the THSV - Slash, also with a fixed-zero and a RWMH threshold:

```

1 # VaR Estimation
2 # Written by: Ronaldo Robles
3 # Based on R code from Phd. Carlos Abanto
4
5 ##### Running the model with the One-step ahead forecast
6 vY <- read.table("WD/SP500.txt")$V1
7 N <- length(vY) - 100
8
9
10 # Initial point
11 l<- 1
12 vY <- vY[1:N]
13 source("THSV-N - Endo threshold - Light.R")
14 vY[(N+1)] <- mean(vYforesimf)
15 rm(list=setdiff(ls(), c("vY", "N")))
16
17 # Recursion
18 for(l in 2:100){
19   vY <- vY[2:(N+1)]
20   source("THSV-N - Endo threshold - Light.R")
21   vY[(N+1)] <- mean(vYforesimf)
22   rm(list=setdiff(ls(), c("vY", "N")))
23 }
24
25
26 vY_forecast_means <- matrix(NA, 100, 5)
27 # Load recursive data and store means and quantiles
28 for(k in 1:100){
29   vY_nn <- read.table(paste("Results THSV-N Endo/vY_forecasts_", k, ".txt"))$V1
30   vY_forecast_means[k,1] <- mean(vY_nn)
31   vY_forecast_means[k,2] <- quantile(vY_nn, probs = 0.05)
32   vY_forecast_means[k,3] <- quantile(vY_nn, probs = 0.95)
33   vY_forecast_means[k,4] <- quantile(vY_nn, probs = 0.01)
34   vY_forecast_means[k,5] <- quantile(vY_nn, probs = 0.99)
35   print(k)
36 }
37
38 # Load real out of sample data
39 SP500 <- read_csv("WD/SP500.txt",
40                 col_names = FALSE)$X1
41 N <- length(SP500)
42 SP500 <- SP500[(N-100+1):N]
43

```

```

44 # LRuc Kupiec (1995) likelihood ratio test
45
46 mn                <- 100
47
48 test005           <- rep(NA, mn)
49 test095           <- rep(NA, mn)
50 test001           <- rep(NA, mn)
51 test099           <- rep(NA, mn)
52
53 test005           <- ifelse(market < vY_forecast_means[,2],1,0)
54 test095           <- ifelse(market > vY_forecast_means[,3],1,0)
55 test001           <- ifelse(market < vY_forecast_means[,4],1,0)
56 test099           <- ifelse(market > vY_forecast_means[,5],1,0)
57
58 xnhat005          <- sum(test005)
59 alphahat005       <- xnhat005/mn
60
61 xnhat095          <- sum(test095)
62 alphahat095       <- xnhat095/mn
63
64 xnhat001          <- sum(test001)
65 alphahat001       <- xnhat001/mn
66
67 xnhat099          <- sum(test099)
68 alphahat099       <- xnhat099/mn
69
70 # LRuc 0.01
71
72 s00               <- log( (alphahat001^xnhat001) * ((1-alphahat001)^(mn-xnhat001)) )
73 s01               <- log( (0.01^xnhat001) * ( (1 - 0.01)^(mn-xnhat001)) )
74 LRuc001          <- 2*( s00 - s01 )
75
76 pchisq(q=LRuc001, df=1, lower.tail = F)
77
78 # LRuc 0.05
79
80 s00               <- log( (alphahat005^xnhat005) * ((1-alphahat005)^(mn-xnhat005)) )
81 s01               <- log( (0.05^xnhat005) * ( (1 - 0.05)^(mn-xnhat005)) )
82 LRuc005          <- 2*( s00 - s01 )
83
84 # p-value
85 pchisq(q=LRuc005, df=1, lower.tail = F)
86
87 # LRuc 0.95
88
89 s00               <- log( (alphahat095^xnhat095) * ((1-alphahat095)^(mn-xnhat095)) )
90 s01               <- log( (0.05^xnhat095) * ( (1 - 0.05)^(mn-xnhat095)) )
91 LRuc095          <- 2*( s00 - s01 )
92
93 pchisq(q=LRuc095, df=1, lower.tail = F)
94
95 # LRuc 0.99
96
97 s00               <- log( (alphahat095^xnhat099) * ((1-alphahat099)^(mn-xnhat099)) )
98 s01               <- log( (0.01^xnhat099) * ( (1 - 0.01)^(mn-xnhat099)) )
99 LRuc099          <- 2*( s00 - s01 )
100
101 pchisq(q=LRuc099, df=1, lower.tail = F)
102
103 # Measure of Embrechts et al (2004) for Expected Shortfall
104

```

```

105 # D 0.05
106 shortfall05n <- mean(SP500[test005==1])
107
108 deltan <- SP500 - shortfall05n
109 deltan05 <- quantile(deltan,0.05)
110
111 Sn005 <- ifelse(deltan < deltan05, 1, 0)
112 ss005 <- sum(Sn005)
113 D1 <- sum(deltan[test005==1]) / xnhat005 # standard back-testing for ES estimates
114 D2 <- sum(deltan[Sn005==1]) / ss005 # correction for VaR dependence
115 D005 <- ( abs(D1) + abs(D2) ) / 2
116
117 # D 0.095
118 shortfall95n <- mean(SP500[test095==1])
119
120 deltan <- SP500 - shortfall95n
121 deltan95 <- quantile(deltan,0.95)
122
123 Sn095 <- ifelse(deltan > deltan95, 1, 0)
124 ss095 <- sum(Sn095)
125 D1 <- sum(deltan[test095==1]) / xnhat095 # standard back-testing for ES estimates
126 D2 <- sum(deltan[Sn095==1]) / ss095 # correction for VaR dependence
127 D095 <- ( abs(D1) + abs(D2) ) / 2
128
129
130 # D 0.01
131 shortfall01n <- mean(SP500[test001==1])
132
133 deltan <- SP500 - shortfall01n
134 deltan01 <- quantile(deltan,0.01)
135
136 Sn001 <- ifelse(deltan < deltan01, 1, 0)
137 ss001 <- sum(Sn001)
138 D1 <- sum(deltan[test001==1]) / xnhat001 # standard back-testing for ES estimates
139 D2 <- sum(deltan[Sn001==1]) / ss001 # correction for VaR dependence
140 D001 <- ( abs(D1) + abs(D2) ) / 2
141
142 # D 0.099
143 shortfall99n <- mean(SP500[test099==1])
144
145 deltan <- SP500 - shortfall99n
146 deltan99 <- quantile(deltan,0.99)
147
148 Sn099 <- ifelse(deltan > deltan99, 1, 0)
149 ss099 <- sum(Sn099)
150 D1 <- sum(deltan[test099==1]) / xnhat099 # standard back-testing for ES estimates
151 D2 <- sum(deltan[Sn099==1]) / ss099 # correction for VaR dependence
152 D099 <- ( abs(D1) + abs(D2) ) / 2

```

## Appendix: Order Statistics - Mathematical framework

### Endogenous threshold with order statistics

In this appendix we present our mathematical framework for our proposed methodology based on work from Chib (1998). For a fixed two regime model, with data following the specific behavior of financial assets returns, the posterior density of our state  $k_t$  will be defined as follows:

$$p(k_t | \mathbf{y}_{1:T}, K^{t+1}, \Theta, P) \propto p(k_t | Y_t, \Theta, P) \cdot p(k_{t+1} | k_t, P)$$

where

$$p(k_t = q | Y_t, \Theta, P) = \frac{p(k_t = q | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_q)}{\sum_{r=q-1}^q p(k_t = r | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_r)} \quad (8)$$

with

$$p(k_t = q | Y_{t-1}, \Theta, P) = \sum_{r=q-1}^q p_{rq} \cdot p(k_{t-1} = r | Y_{t-1}, \Theta, P) \quad (9)$$

In our unique threshold specification,  $q = 0, 1$ . First, we expand (8):

$$p(k_t = q | Y_t, \Theta, P) = \frac{p(k_t = q | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_q)}{p(k_t = q-1 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_{q-1}) + p(k_t = q | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_q)}$$

Given the values of  $q$ , we analyze the expression for both cases:

- $q = 0$

$$\begin{aligned} p(k_t = 0 | Y_t, \Theta, P) &= \frac{p(k_t = 0 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_0)}{p(k_t = -1 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_{-1}) + p(k_t = 0 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_0)} \\ &= \frac{p(k_t = 0 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_0)}{0 + p(k_t = 0 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_0)} \\ &= 1 \end{aligned}$$

- $q = 1$

$$p(k_t = 1 | Y_t, \Theta, P) = \frac{p(k_t = 1 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_1)}{p(k_t = 0 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_0) + p(k_t = 1 | Y_{t-1}, \Theta, P) \cdot f(y_t | Y_{t-1}, \Theta_1)} \quad (10)$$

The former expression is not trivial. Nevertheless, we can use (9) with  $q = 1$ .

$$p(k_t = 1 | Y_{t-1}, \Theta, P) = p_{01} \cdot p(k_{t-1} = 0 | Y_{t-1}, \Theta, P) + p_{11} \cdot p(k_{t-1} = 1 | Y_{t-1}, \Theta, P)$$

If we simplify the notation, we can present:

$$p(k_t = 1) = (1 - p_{00}) \cdot p(k_{t-1} = 0) + p(k_{t-1} = 1) \quad (11)$$

After that, we replace (11) in (10):

$$p(k_t = 1 | Y_t) = \frac{[(1 - p_{00}) \cdot p(k_{t-1} = 0) + p(k_{t-1} = 1)] \cdot f(y_t | \Theta_1)}{p(k_t = 0) \cdot f(y_t | \Theta_0) + [(1 - p_{00}) \cdot p(k_{t-1} = 0) + p(k_{t-1} = 1)] \cdot f(y_t | \Theta_1)}$$



We can also obtain the first addend in the denominator from (9) with  $q = 0$ :

$$\begin{aligned} p(k_t = 0) &= p_{-10} \cdot p(k_{t-1} = -1) + p_{00} \cdot p(k_{t-1} = 0) \\ &= p_{00} \cdot p(k_{t-1} = 0) \end{aligned}$$

and finally we have:

$$p(k_t = 1 | Y_t) = \frac{[(1 - p_{00}) \cdot p(k_{t-1} = 0 | Y_{t-1}) + p(k_{t-1} = 1 | Y_{t-1})] \cdot f(y_t | \Theta_1)}{p_{00} \cdot p(k_{t-1} = 0 | Y_{t-1}) \cdot f(y_t | \Theta_0) + [(1 - p_{00}) \cdot p(k_{t-1} = 0 | Y_{t-1}) + p(k_{t-1} = 1 | Y_{t-1})] \cdot f(y_t | \Theta_1)} \quad (12)$$

Regarding  $p_{00}$ , we have defined the transition probability matrix  $P$  as:

$$P = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 0 & 1 \end{pmatrix}$$

Following Chib (1998), the prior for  $p_{00}$  will be:

$$p_{00}^{(0)} \sim \text{Beta}(a, b)$$

with mean  $a/(a + b)$  and duration  $d = a^b \cdot (1 - a)^{(1-b)}$ . Both  $a$  and  $b$  are positive and satisfy  $a \gg b$ . The full conditional of  $p_{00}$  is equivalent to:

$$p_{00} = \text{Beta}(a + n_{ii}, b + 1)$$

where  $n_{ii}$  is the number on transitions from state  $i$  to  $i$ , and only when  $i = 0$  (i.e the total number of zeros in the vector state  $K_t$  minus 1).