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Escuela de Posgrado

"Simulation of a non-Markovian evolution using coherence"

A thesis in candidacy for the degree of Master of Science in Physics presented by

Jean Paul Marrou Osores

Advisor:
Prof. Francisco Antonio De Zela Martinez

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"Simulation of a non-Markovian evolution using coherence"

Jean Paul Marrou Osores

Presented Towards a Master's Degree in Physics

Abstract

This thesis will be oriented in the study of open quantum systems. The transition of processes that go between the Markovian and non-Markovian regime will be studied. The diagnose of non-Markovianity will be made in terms of the variation of the coherence of the state. Accordingly, an optical setup will be implemented that allows us to manipulate certain degrees of freedom, like the polarization and the optical path. Theoretically, we have found that the coherence of the system is transferred to the environment and it decreases as we move a parameter that we will take as time. This situation has been confirmed in the experiment. Then, due to the second part of the setup, which produces a non-Markovian evolution by also changing one of its parameters, we have accomplished the goal of returning the information back into the system and to measure the non-Markovianity of the process.

"Simulación de una evolución no-Markoviana usando coherencia"

Jean Paul Marrou Osores

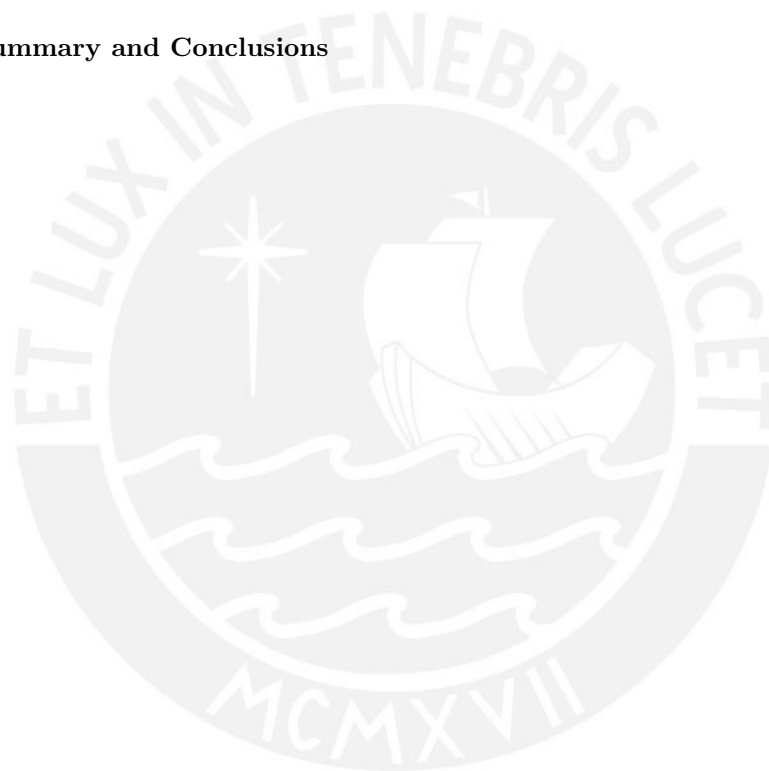Propuesto para el Grado de Magíster en Física

Resumen

El trabajo de tesis estará orientado al estudio de sistemas cuánticos abiertos. Se estudiará la transición de procesos que van del régimen Markoviano al no Markoviano en forma controlada. El diagnóstico de no Markovianidad se hará en términos de la variación de la coherencia del estado. Para ello se implementará un arreglo óptico que permita manipular varios grados de libertad, tales como polarización y camino óptico. Teóricamente, encontramos que la coherencia del sistema se transfiere al entorno y disminuye al mover uno de estos parámetros que tomaremos como el tiempo, lo que se ha podido comprobar en el experimento. Posteriormente, utilizando otro arreglo que produce una evolución no-Markoviana cambiando uno de sus parámetros también como el tiempo, se ha logrado recuperar la coherencia del sistema. De esta manera se hace posible el retorno de la información y la medición de la no-Markovianidad de dicho proceso.

Contents

Chapter 1

Qubit

1.1 Qubit and density matrix

In classical information theory a binary digit can have the value 0 or 1. On the other hand, the quantum bit is a superposition of two possible states (0 or 1) of a system. We can represent it by using Dirac notation as follows

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle, \tag{1.1}$$

where α and β are complex numbers.

In order to describe mixed states and gain generality, we can use a density matrix to represent the state

$$\hat{\rho} = |\Psi\rangle \langle\Psi| \tag{1.2}$$

$$\hat{\rho} = \alpha^2 |0\rangle \langle0| + \beta^2 |1\rangle \langle1| + \alpha\beta^* |0\rangle \langle1| + \beta\alpha^* |1\rangle \langle0| \tag{1.3}$$

And as a matrix representation we have:

$$\hat{\rho} = \begin{bmatrix} \alpha^2 & \alpha\beta^* \\ \beta\alpha^* & \beta^2 \end{bmatrix}, \tag{1.4}$$

where we can write $\alpha = ae^{i\phi_1}$ and $\beta = be^{i\phi_2}$ and we can define our state in terms of a, b and the relative phase between the states $\phi = \phi_1 - \phi_2$

$$\hat{\rho} = \begin{bmatrix} a^2 & abe^{i\phi} \\ abe^{-i\phi} & b^2 \end{bmatrix} \tag{1.5}$$

There are three properties that guarantees that the density matrix represents a physical state. We call a density matrix when it satisfy the following properties:

1. Trace=1: The sum of all probabilities should be 1.

$$\text{Tr}(\hat{\rho}) = a^2 + b^2 = 1 \tag{1.6}$$

2. Hermicity

$$\hat{\rho} = \hat{\rho}^\dagger \tag{1.7}$$

3. Positiveness: The density matrix is positive definite.

$$\hat{\boldsymbol{\rho}} \geq 0 \tag{1.8}$$

The hermicity and positiveness properties ensure the existence of real and non-negative eigenvalues of the density matrix, in other words, it ensures that the values of probabilities remain in the real domain as they are expected to be.

1.1.1 Projective Measurements

Given the the operator A, we make a measurement by applying it on the state

$$A \left| \psi \right\rangle \tag{1.9}$$

and this operator satisfies the equation

$$A \left| \phi_i \right\rangle = a_i \left| \phi_i \right\rangle, \tag{1.10}$$

where ai is the eigenvalue that corresponds to the eigenvector (state) $\left| \phi_i \right\rangle$.

In addition, it possible to make the spectral decomposition of the operator A in some orthogonal base given by $\left| \phi_i \right\rangle$

$$A = \sum_i a_i \left| \phi_i \right\rangle \left\langle \phi_i \right| \tag{1.11}$$

The probability P of obtaining as a measure the state $\left| \phi_i \right\rangle$ with eigenvalue a_i after applying the operator A on the state of the system $\left| \psi \right\rangle$ is

$$P(a_i) = \left\langle \psi \right| \left(\left| \phi_i \right\rangle \left\langle \phi_i \right| \right) \left| \psi \right\rangle \tag{1.12}$$

The mean value of the operator A is

$$< A > = \left\langle \psi \right| A \left| \psi \right\rangle \tag{1.13}$$

In density matrix notation, the probability of obtaining the measure a_i is given by

$$P(a_i) = Tr(\hat{\boldsymbol{\rho}} \left| \phi_i \right\rangle \left\langle \phi_i \right|), \tag{1.14}$$

the expected value of A is given by

$$< A > = Tr(\hat{\boldsymbol{\rho}} A), \tag{1.15}$$

and the state after performing the measure with an outcome a_i is

$$\hat{\boldsymbol{\rho}'} = \frac{\Pi_i \hat{\boldsymbol{\rho}} \Pi_i}{Tr(\Pi_i \hat{\boldsymbol{\rho}} \Pi_i)}, \tag{1.16}$$

where Π_i is the projector of the state ϕ_i and is given by

$$\Pi_i = \left| \phi_i \right\rangle \left\langle \phi_i \right|. \tag{1.17}$$

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1.1.2 Dynamics of a closed state

Unitary evolution

In the case of closed systems we can represent the time evolution of the state from 0 to t using an evolution operator U

$$U \left| \psi_0 \right\rangle = \left| \psi_t \right\rangle \tag{1.18}$$

Putting this result into the Schrödinger equation:

$$\hat{\boldsymbol{H}} \left| \psi(t) \right\rangle = i\hbar \frac{\partial \left| \psi(t) \right\rangle}{\partial t} \tag{1.19}$$

we get

$$U(t) = e^{-i\frac{\hat{\boldsymbol{H}}}{\hbar}t} \tag{1.20}$$

With $\left| \psi_t \right\rangle \left\langle \psi_t \right| = U \left| \psi_0 \right\rangle \left\langle \psi_0 \right| U^\dagger$ we get the evolution in the density matrix representation:

$$\hat{\boldsymbol{\rho}}_t = U \cdot \hat{\boldsymbol{\rho}}_0 \cdot U^\dagger \tag{1.21}$$

And its evolution is given by an equivalent equation to Schrödinger equation in matrix notation

$$\dot{\rho} = -i\left[\hat{\boldsymbol{H}}, \rho \right], \tag{1.22}$$

which is known as the Von Neumann equation.

1.2 Polarization

In the case of a laser beam, the polarization describes the direction of oscillation of the electrical field of the beam. We can have linear or circular polarization.

Polarization oscillates perpendicular to the direction of propagation in the transversal plane. Depending on our system of reference, we can see the polarization as the combination of two components of linear polarized light. Then, when they are in phase, we have linear polarized light, but when they are dephased (one delayed in terms of the other), we have circular polarization. While the oscillation of linear polarization occurs only in one direction, the direction of oscillation of circular polarized light varies over time.

Using the base of horizontal and vertical polarization we will represent a qubit and we will leave the other types of polarizations as linear combinations of these two.

Diagonal polarization: $\left| D \right\rangle = \dfrac{\left| H \right\rangle + \left| V \right\rangle}{\sqrt{2}}$

Anti-diagonal polarization: $\left| A \right\rangle = \dfrac{\left| H \right\rangle - \left| V \right\rangle}{\sqrt{2}}$

Right circular polarization: $|R\rangle = \dfrac{|H\rangle - i\,|V\rangle}{\sqrt{2}}$

Left circular polarization: $|L\rangle = \dfrac{|H\rangle + i\,|V\rangle}{\sqrt{2}}$

In the polarization space, the density matrix is represented by the SU(2) group and is generated by the Pauli matrices. Then we can write the matrix as follows:

$$\hat{\boldsymbol{\rho}} = \frac{1}{2}(\mathbb{1} + \sum_{i=1}^{n} S_i.\sigma_i) = \sum_{i=0}^{n} S_i.\sigma_i \tag{1.23}$$

Where σ_i are the Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.24}$$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.25}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.26}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.27}$$

and S_i is the i-th component of the stokes vector, which give us information about the sate of polarization.

Each component of the Stokes vector is calculated using the measure of intensity of the light projected in each of the bases of polarizations. For example, to measure S_3 we project on vertical and horizontal polarizations to obtain

$$S_3 = \langle H|\,\hat{\boldsymbol{\rho}}\,|H\rangle - \langle V|\,\hat{\boldsymbol{\rho}}\,|V\rangle \tag{1.28}$$

All the other components are calculated in a similar way.

$$S_2 = \langle R|\,\hat{\boldsymbol{\rho}}\,|R\rangle - \langle L|\,\hat{\boldsymbol{\rho}}\,|L\rangle \tag{1.29}$$

$$S_1 = \langle D|\,\hat{\boldsymbol{\rho}}\,|D\rangle - \langle A|\,\hat{\boldsymbol{\rho}}\,|A\rangle \tag{1.30}$$

And the S_0, which represents the total intensity is given by

$$S_0 = \langle D|\,\hat{\boldsymbol{\rho}}\,|D\rangle + \langle A|\,\hat{\boldsymbol{\rho}}\,|A\rangle = \langle R|\,\hat{\boldsymbol{\rho}}\,|R\rangle + \langle L|\,\hat{\boldsymbol{\rho}}\,|L\rangle = \langle H|\,\hat{\boldsymbol{\rho}}\,|H\rangle + \langle V|\,\hat{\boldsymbol{\rho}}\,|V\rangle \tag{1.31}$$

Therefore, is sufficient to measure the stokes vector to reconstruct the density matrix of polarization.

1.2.1 Poincaré Sphere

We can represent all these bases of polarization as vectors in spherical coordinates in what is known as Poincaré sphere. In this space, the states of polarization are represented by the stokes vector S
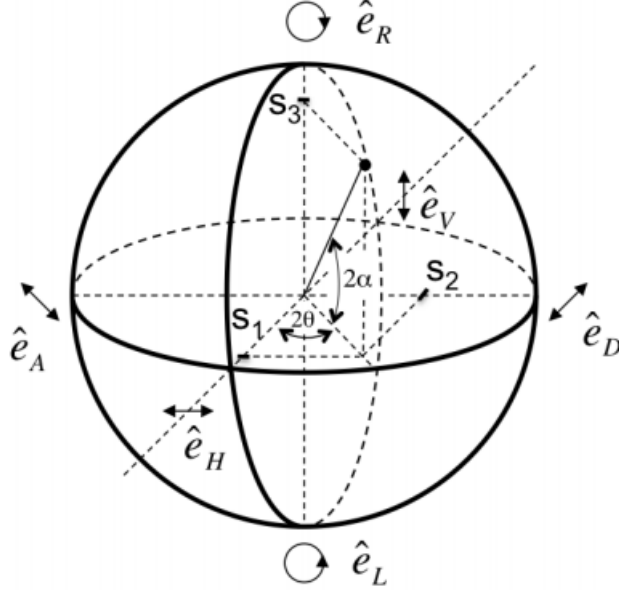

Figure 1.1: Poincaré Sphere of polarizations

In the polar plane of the sphere, we have the horizontal and vertical polarizations in one axis, one opposed to the other, and in a similar way, in a perpendicular axis we find the anti-diagonal and diagonal polarizations. On the other hand, the azimuthal plane contains elliptical polarizations while the vertical axis is composed of circular right and circular left respectively.

All the polarizations mentioned are represented as points in the surface of the sphere, while in the center of the sphere, we find the incoherent states of polarization in any basis.

This representation is useful due to the simple representation of a polarization qubit and its operations concerning phase retarders. For example, a half-wave plate performs a π rotation around and axis, which is calculated by rotating the horizontal (or vertical) axis by two times the angle of the HWP. Its operator is given by

$$U_{hwp}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \tag{1.32}$$

In a similar way, the quarter-wave plate performs a $\pi/2$ rotation around the axis.

$$U_{hwp}(\theta) = \begin{bmatrix} \cos(\theta)^2 - i\sin(\theta)^2 & (-1-i)\cos(\theta)\sin(\theta) \\ (-1-i)\cos(\theta)\sin(\theta) & -i\cos(\theta)^2 + \sin(\theta)^2 \end{bmatrix} \tag{1.33}$$

In general, any channel (an operation that preserves the properties of physical states) applied on the polarization qubit rotates the initial stokes vector and takes it to another point in the sphere, the final state. If the channel is applied over time, it may change the initial state over a path drawn over the surface.

1.2.2 Tomography

To construct the density matrix in the space of polarization we need to calculate its stokes vector according to equation (1.23). To this purpose, we need to project in each one of the six bases of polarization that we mentioned before. Then we will measure six intensities to calculate each component of the vector

$$S_1 = \frac{I_D - I_A}{S_0} \tag{1.34}$$

$$S_2 = \frac{I_R - I_L}{S_0} \tag{1.35}$$

$$S_3 = \frac{I_H - I_V}{S_0} \tag{1.36}$$

where in theory S_0 must satisfy

$$S_0 = I_D + I_A = I_R + I_L = I_H + I_V \tag{1.37}$$

In order to measure this values we make use of a QHP, which is an optical setup of a quarter-wave plate, a half-wave plate and a polarizer at the end. We measure these intensities by rotating the waveplates in specific angles

Int	qwp	hwp
H	0°	0°
V	0°	45°
R	0°	22.5°
L	90°	22.5°
D	45°	22.5°
A	45°	-22.5°

As we see, we have a set of 6 equations and 10 variables, thus technically we only need to measure 4 intensities to reconstruct the stokes vector, and hence the density matrix. However, to gain experimental accuracy, is recommended to measure all the six polarizations instead of taking advantage of the dependence between variables.

In addition, if we are in the case of a larger system composed by n polarization qubits, in order to reconstruct the total density matrix, 4^n measures will be required with the aid of multiple QHP's setups to accomplish the tomography.

1.3 Path

If we consider a beam that propagates in 2 dimensions (two degrees of freedom), its propagation path might be used as a qubit because light can be in one path, the other or in a superposition of two paths.

$$|\Psi\rangle_{path} = \alpha |0\rangle + \beta |1\rangle , \tag{1.38}$$

where, given a system of reference, $|0\rangle$ represents the state of the beam that propagates horizontally and $|1\rangle$ represents the case when light goes vertically .

1.3.1 Path operators

Mirrors

The mirrors change the path of the beam and they are represented in the operator

$$U_{mirror} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} , \tag{1.39}$$

where we see that a reflection also introduces a phase of i to the incoming beam.

Even though, theoretically, the operator is defined in this way, we see experimentally that the phase may also depend on the polarization of the incoming beam.

Beam splitters

In general a beam splitter lets one part of the photons of the beam pass through in the same direction and makes the other part of the photons reflect to the other path. In the base of the path, the operator is given by

$$U_{BS} = \begin{bmatrix} \sqrt{T} & i\sqrt{R} \\ i\sqrt{R} & \sqrt{T} \end{bmatrix} \tag{1.40}$$

where R, T are the transmission and reflection coefficients.

If we are in the case of a 50/50 beam splitter, obviously the coefficients take the values $\frac{1}{\sqrt{2}}$ and the operator will be given by

$$U_{BS} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{i}{\sqrt{2}} \\ \dfrac{i}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} \tag{1.41}$$

Polarized beam splitters

Some beam splitters may reflect the light depending on its polarization and these are called polarized beam splitters (PBS). In general, in the base of polarization and path, its operation is given by

$$U_{PBS} = |H\rangle \langle H| \otimes U_{BS_H} + |V\rangle \langle V| \otimes U_{BS_V} , \tag{1.42}$$

whereU_{BS_i} includes the coefficients of transmission and reflection depending on the incoming polarization.

In particular the operator of a PBS that lets horizontal polarization pass trough and that reflects vertical polarization is expressed as
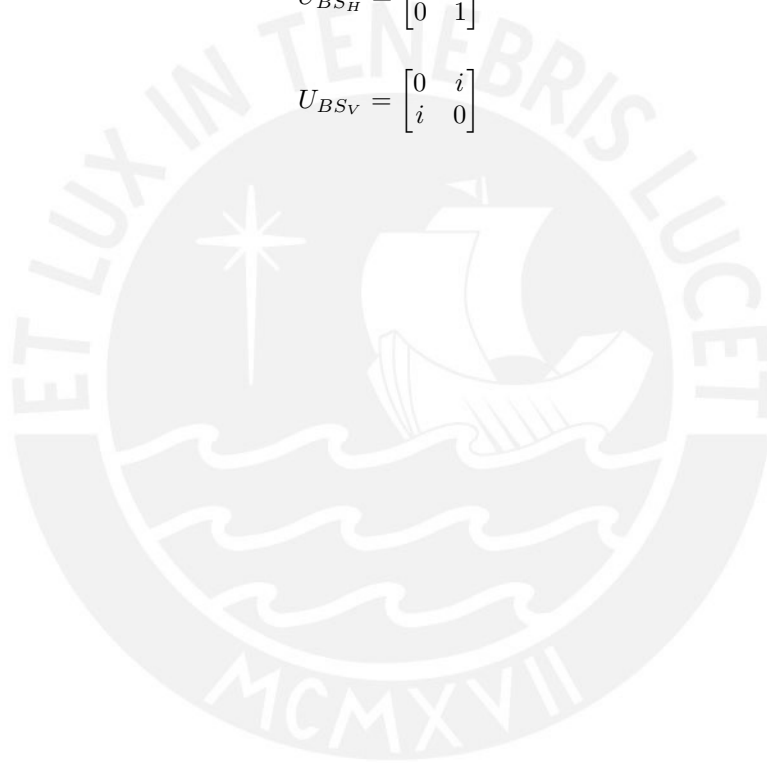
$$U_{PBS} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \tag{1.43}$$

where is evident that

$$U_{BS_H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.44}$$

and

$$U_{BS_V} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{1.45}$$

Chapter 2

Open systems

2.1 Positive Operator Valued Measure

A more general concept in the description of measurements is the Positive Operator Valued Measure. The probability of obtaining outcome a_i

$$P(a_i) = Tr(E_i \hat{\boldsymbol{\rho}}), \tag{2.1}$$

where E_i is a measure operator which may be expressed with the Kraus matrices

$$E_i = K_i^\dagger K_i \tag{2.2}$$

And the state after the measure is given by

$$|\psi_i\rangle = \frac{K_i |\psi\rangle}{\|K_i |\psi\rangle\|}. \tag{2.3}$$

Additionally, if we define the operator K_i as

$$K_i = U_i \sqrt{E_i}, \tag{2.4}$$

where U_i is an arbitrary unitary operator, the state after the measure will be now given by

$$|\psi_i\rangle = U_i \frac{\sqrt{E_i} |\psi\rangle}{\|\sqrt{E_i} |\psi\rangle\|}, \tag{2.5}$$

where is evident that the state after the measurement is not well defined due to the arbitrary evolution operator U_i. This is useful because it may consider the case of a system coupled into another one where a measurement does not determine the final state [11].

2.2 Non-unitary evolutions

In general, we may represent the system evolution using a dynamical map Λ and when it is applied to our initial state, it evolves the state from time 0 to t.

$$\hat{\boldsymbol{\rho}}_t = \Lambda_t(\hat{\boldsymbol{\rho}}_0) \tag{2.6}$$

2.2.1 Kraus representation

In open systems, we cannot use the evolution operator since the system has losses that go to the environment.

These evolutions can be represented using the Kraus representation:

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^n K_i \cdot \hat{\boldsymbol{\rho}}_0^s \cdot K_i^\dagger \tag{2.7}$$

where this matrices K_i satisfy:

$$\sum_{i=0}^n K_i^\dagger \cdot K_i = \mathbb{1} \tag{2.8}$$

When the matrices fail to satisfy this necessary condition, we cannot have a physical state evolving into another one, due to the fact that this property ensures that the trace is preserved.

2.2.2 Master equation

Another very used representation of the dynamics of an open system is through the master equation, which is an expression that includes the dynamics of the closed system plus another part that represents the losses to the environment.

In a general approach the dynamics are given by

$$\dot{\rho} = \mathcal{L}\rho, \tag{2.9}$$

where $\mathcal{L}$ is a linear map called the Linbladian.

If the Linbladian is time independent, we have the solution

$$\rho(t) = e^{\mathcal{L}t}\rho(0). \tag{2.10}$$

In the particular case of a closed system, we have the von Neumann equation

$$\mathcal{L}\rho = -i\left[\hat{\boldsymbol{H}}, \rho\right] \tag{2.11}$$

Then, we expand the exponential of eq.(2.10) at first order in Δt to get

$$e^{\mathcal{L}\Delta t} = \mathbb{1} + \Delta t \mathcal{L} \tag{2.12}$$

and we apply it on the initial state to obtain

$$e^{\mathcal{L}\Delta t}\rho(0) = \rho(0) + \Delta t \mathcal{L}\rho(0). \tag{2.13}$$

Besides, if we use the Kraus representation

$$\epsilon(\rho(0)) = \sum_i K_i \rho(0) K_i^\dagger, \tag{2.14}$$

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we can construct the kraus matrices [11]

$$K_0 = \mathbb{1} + (M - iH)\Delta t \qquad (2.15)$$

$$K_{i>0} = L_i\sqrt{\Delta t}, \qquad (2.16)$$

Additionally, from equation (2.13) we get

$$\dot{\rho} = \mathcal{L}(\rho(0)) = \frac{1}{\Delta t}(\epsilon(\rho(0)) - \rho(0)) \qquad (2.17)$$

and if we keep only terms up to first order in Δt, we get from the completeness of Kraus operators

$$\sum_{i=0}^{n} K_i^\dagger \cdot K_i = \mathbb{1} = \mathbb{1} + \Delta t(2M + \sum_{i>0} L_i^\dagger L_i) \qquad (2.18)$$

and hence

$$M = -\frac{1}{2}\sum_{i>0} L_i^\dagger L_i \qquad (2.19)$$

Finally, by substituting into eq.(2.9), we obtain the Linblad master equation

$$\dot{\rho} = \mathcal{L}(\rho(0)) = -i\left[\hat{\boldsymbol{H}}, \rho\right] + \sum_i (L_i\rho L_i^\dagger - \frac{1}{2}L_i^\dagger L_i\rho - \frac{1}{2}L_i^\dagger L_i) \qquad (2.20)$$

2.2.3 System evolution

The space of the whole system-environment state will be in the Hilbert space

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e \qquad (2.21)$$

Then, we will represent the system-environment whole initial state with the density matrix:

$$\hat{\boldsymbol{\rho}}_0^{se} = \hat{\boldsymbol{\rho}}_0^s \otimes \hat{\boldsymbol{\rho}}_0^e \qquad (2.22)$$

where we are assuming an initial separable state.

This whole state, due to being closed, evolves with an evolution operator Use and is given by

$$\hat{\boldsymbol{\rho}}_t^{se} = U_{se} \cdot \hat{\boldsymbol{\rho}}_0^{se} \cdot U_{se}^\dagger \qquad (2.23)$$

where the evolution operator is calculated from the total Hamiltonian [2]

$$H = H_s + H_e + H_{interaction} \qquad (2.24)$$

Accordingly, the system state will evolve and it can be calculated from the whole state using the partial trace in the states of the environment

$$\hat{\boldsymbol{\rho}}_t^s = \text{Tr}_e(\hat{\boldsymbol{\rho}}_t^{se}) \qquad (2.25)$$

And by using eq.(2.23), we get

$$\hat{\boldsymbol{\rho}}_t^s = \text{Tr}_e(U_{se} \cdot \hat{\boldsymbol{\rho}}_0^{se} \cdot U_{se}^\dagger) \qquad (2.26)$$

Then, by making the partial trace as in eq.(5.5) we get

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^{n} \langle i| \cdot U_{se} \cdot \hat{\boldsymbol{\rho}}_0^{se} \cdot U_{se}^\dagger \cdot |i\rangle \tag{2.27}$$

Additionally, assuming an initial state similar to eq.(2.21) with $\hat{\boldsymbol{\rho}}_0^e = |a\rangle \langle a|$ we get

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^{n} \langle i| U_{se} \cdot \hat{\boldsymbol{\rho}}_0^s \otimes |a\rangle \langle a| \cdot U_{se}^\dagger |i\rangle \tag{2.28}$$

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^{n} \langle i| U_{se} |a\rangle \hat{\boldsymbol{\rho}}_0^s \langle a| U_{se}^\dagger |i\rangle \tag{2.29}$$

where we may notice the structure of Kraus operator acting on the initial state similarly to eq.(2.7).

Finally, this Kraus operators performing the evolution in the system state will be calculated from

$$K_i = \langle i| U_{se}(t) |a\rangle \tag{2.30}$$

$$K_i^\dagger = \langle a| U_{se}^\dagger(t) |i\rangle \tag{2.31}$$

2.3 Positivity and divisibility

2.3.1 Positivity and Complete Positivity

An hermitian positive definite matrix $\hat{\boldsymbol{\rho}} > 0$ has real and positive eigenvalues.

A positive (P) map , in the case of mapping matrices, is a map that takes a positive definite matrix to another one:

$$\Lambda_t(\hat{\boldsymbol{\rho}}_0) > 0 \tag{2.32}$$

When we have multiple qubits being part of a whole system, we may have operators acting on the space of each qubit separately in the form of
$\phi_1 \otimes \phi_2 \otimes \phi_3 ... \otimes \phi_n(\hat{\boldsymbol{\rho}})$ where $H_1 \otimes H_2 \otimes H_3 ... \otimes H_n$ is the Hilbert space of the whole state $\hat{\boldsymbol{\rho}}$.

Then, a completely positive map (CP) is the one that preserves positivity of the whole system, even though the system in which the map acts is part of a larger system of qubits.

In the specific case of two qubits in the space $H_A \otimes H_B$ we may have $\mathbb{I}_A \otimes \phi(B)$ acting of the whole system state. If the map $\phi(B)$ is positive, it does not guarantee that the whole map $\mathbb{I}_A \otimes \phi(B)$ is positive as well.

For example, the transpose operation $T = |0\rangle \langle 1| + |1\rangle \langle 0|$ applied in one qubit, preserves its positivity. However, if we have 2 qubits ρ_1 and ρ_2 entangled in the state

$$\rho_{12} = (|0\rangle |0\rangle + |1\rangle |1\rangle)(\langle 0| \langle 0| + \langle 1| \langle 1|), \tag{2.33}$$

and if we perform the transpose operation T in the second qubit

$$(\mathbb{1} \otimes T)\rho_{12} = (|0\rangle |1\rangle + |1\rangle |0\rangle)(\langle 0| \langle 1| + \langle 1| \langle 0|), \tag{2.34}$$

we get as a result the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.35}$$

which is not a physical state because it has the negative eigenvalue $-1/2$. Therefore, as we see, even if the transpose operation is positive in one qubit, it does not guarantees the positivity of the whole system state, thus we say that the transpose operation is not completely positive.

In addition, if the whole map Λ_t is positive for n qubits, we will say that it is n-positive; and in the case where n is the total number of qubits of the system, we will say that the map is CP.

Thus, in the specific case of two qubits where $n = 2$ includes the whole system, we shall say, for example, that the map $\Lambda_t = \mathbb{I}_A \otimes \phi(B)$ is CP.

A CP map ensures the mapping of physical states into physical states, in other words, a CP maps preserves the state hermicity, preserves the positivity and preserves the property of having a trace equal to 1. We are evolving, then, a density matrix into another one.

Furthermore, it is said that if a map has a Kraus decomposition, it is a Completely Positive map [9].

2.3.2 Map divisibility

An important property is the map divisibility. It describes the possibility of splitting the map in steps (inside the time interval that it acts):

For example, this map that takes the state from 0 to t is divisible in two maps, one acting from 0 to s and another one acting from s to t.

$$\hat{\boldsymbol{\rho}}_t = \Lambda_{t,s}\Lambda_{s,0}(\hat{\boldsymbol{\rho}}_0), \tag{2.36}$$

where $t > s > 0$

The division of the map

$$\Lambda_{t,0} = \Lambda_{t,s}\Lambda_{s,0} \tag{2.37}$$

is possible if the inverse map Λ_s^{-1} exists in order to get

$$\Lambda_{t,s} = \Lambda_t \Lambda_s^{-1} \tag{2.38}$$

where $\Lambda_{t,0} = \Lambda_t$

A map ϕ that is CP and divisible is called a CP-divisible map.

Additionally, divisible process lead us to master equations of the form

$$\dot{\rho} = \mathcal{K}(\rho(0)) = -i\left[\hat{\boldsymbol{H}}_s(t), \rho\right] + \sum_i \gamma_i(t)(A_i(t)\rho A_i^\dagger(t) - \frac{1}{2}A_i^\dagger(t)A_i(t)\rho - \frac{1}{2}A_i^\dagger(t)A_i(t))$$
(2.39)

where the rates $\gamma(t)$ must be positive at all times to ensure CP-divisibility [14]

2.4 Amplitude Damping Channel

Consider a system s in an environment e both represented as qubits:

- If the system is in the ground state nothing happens

$$|0_s\rangle|0_e\rangle \implies |0_s\rangle|0_e\rangle$$
(2.40)

- If the system is in an excited state, theres a probability $P(t)$ of giving this information to the environment

$$|1_s\rangle|0_e\rangle \implies \sqrt{1-P}|1_s\rangle|0_e\rangle + \sqrt{P}|0_s\rangle|1_e\rangle$$
(2.41)

Assuming that the probability may change over time $P = P(t)$, we can construct the evolution operator of the ADC in the total state space

$$U_{adc}(t) = |0_s\rangle\langle 0_s|\otimes|0_e\rangle\langle 0_e| + \sqrt{1-P(t)}|1_s\rangle\langle 1_s|\otimes|0_e\rangle\langle 0_e| + \sqrt{P(t)}|0_s\rangle\langle 1_s|\otimes|1_e\rangle\langle 0_e|$$
(2.42)

Then, if we consider only the system state, the evolution of the Amplitude Damping Channel is described by the the dynamical map:

$$\phi_{ADC}(\hat{\boldsymbol{\rho}}_0^s) = \hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^n K_i \hat{\boldsymbol{\rho}}_0^s K_i^\dagger$$
(2.43)

Furthermore, assuming an initial separable state $\hat{\boldsymbol{\rho}}_0^s \otimes |0_e\rangle\langle 0_e|$, we get

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^n \langle i_s| U_{adc} \cdot \hat{\boldsymbol{\rho}}_0^s \otimes |0_e\rangle\langle 0_e| \cdot U_{adc}^\dagger |i_s\rangle$$
(2.44)

$$\hat{\boldsymbol{\rho}}_t^s = \sum_{i=0}^n \langle i_s| U_{adc} |0_e\rangle \hat{\boldsymbol{\rho}}_0^s \langle 0_e| U_{adc}^\dagger |i_s\rangle$$
(2.45)

where

$$K_i(t) = \langle i_s| U_{adc}(t) |0_e\rangle$$
(2.46)

$$K_i^\dagger(t) = \langle 0_{se}| U_{adc}^\dagger(t) |i_s\rangle$$
(2.47)

In this way, we obtain the Kraus operators K_1, K_2 of the ADC, which do not depend on the initial system state $\hat{\boldsymbol{\rho}}_0^s$

$$K_1(t) = |0_s\rangle\langle 0_s| + \sqrt{1-P(t)}|0_s\rangle\langle 0_s|$$
(2.48)

$$K_2(t) = \sqrt{P(t)} \left| 0_s \right\rangle \left\langle 1_s \right| \tag{2.49}$$

and in matrix form

$$K_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-P(t)} \end{bmatrix} \tag{2.50}$$

$$K_2(t) = \begin{bmatrix} 0 & \sqrt{P(t)} \\ 0 & 0 \end{bmatrix} \tag{2.51}$$

In general, we consider the initial system state $\hat{\boldsymbol{\rho}}_0^s$

$$\hat{\boldsymbol{\rho}}_0^s = \frac{1}{2} \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \tag{2.52}$$

which evolves into

$$\hat{\boldsymbol{\rho}}_t^s = \frac{1}{2} \begin{bmatrix} \rho_{00} + P(t)\rho_{11} & \sqrt{1-P(t)}\rho_{01} \\ \sqrt{1-P(t)}\rho_{10} & (1-P(t))\rho_{11} \end{bmatrix} \tag{2.53}$$

In particular, if we consider the initial state $\hat{\boldsymbol{\rho}}_0^s$ as coherent state, the final state $\hat{\boldsymbol{\rho}}_t^s$ in matrix form is given by

$$\hat{\boldsymbol{\rho}}_t^s = \frac{1}{2} \begin{bmatrix} 1+P(t) & \sqrt{1-P(t)} \\ \sqrt{1-P(t)} & 1-P(t) \end{bmatrix} \tag{2.54}$$

In addition, if we apply the amplitude damping channel repeatedly in succession in order to perform a discrete evolution

$$\phi_{ADC}^{t_n}(...\phi_{ADC}^{t_2}(\phi_{ADC}^{t_1}(\hat{\boldsymbol{\rho}}_0^s))), \tag{2.55}$$

we get the final state

$$\hat{\boldsymbol{\rho}}_t^s = \frac{1}{2} \begin{bmatrix} \rho_{00} + (1-(1-P(t))^n)\rho_{11} & (1-P(t))^{n/2}\rho_{01} \\ (1-P(t))^{n/2}\rho_{10} & (1-P(t))^n\rho_{11} \end{bmatrix} \tag{2.56}$$

evidently, if we begin with a coherent state, we will get

$$\hat{\boldsymbol{\rho}}_t^s = \frac{1}{2} \begin{bmatrix} 1+(1-(1-P(t))^n) & (1-P(t))^{n/2} \\ (1-P(t))^{n/2} & (1-P(t))^n \end{bmatrix} \tag{2.57}$$

Chapter 3

Qubit coherence

3.1 Coherent and incoherent states

The maximal coherent state is

$$|\Psi_c\rangle = \sum_{i=0}^{n} c_i \, |i\rangle \tag{3.1}$$

and we define the incoherent state by

$$\hat{\boldsymbol{\delta}} = \sum_{i=0}^{n} \delta_i \, |i\rangle \, \langle i| \tag{3.2}$$

As we can see from the definition, we can only represent an incoherent state using a density matrix since its an statistical mixture of other states.

In matrix notation an incoherent state is represented with a diagonal matrix, while coherent states have values outside the diagonal.

3.2 Incoherent maps

We will say that the state $\hat{\boldsymbol{\rho}}$ belongs to the set of incoherent states I ($\hat{\boldsymbol{\rho}} \in I$) and we denote it by $\hat{\boldsymbol{\rho}}_I$

An incoherent map Λ_I is the one that takes an incoherent state $\hat{\boldsymbol{\rho}}_I$ to another one, in other words, this map does not create coherence.

$$\Lambda_I(\hat{\boldsymbol{\rho}}_I) \in I \tag{3.3}$$

In Kraus representation:

$$\sum_{i=0}^{n} K_i \cdot \hat{\boldsymbol{\rho}}_I \cdot K_i^\dagger \in I \tag{3.4}$$

From now on we will recognize an incoherent completely positive trace preserving map as ICPTP

3.3 Coherence measure

State coherence, from a mathematical point of view, is related with the values outside the diagonal of the density matrix, which are called coherences while the elements in the diagonal are called populations.
When we have, for example, an

3.3.1 Measure requirements

In order to quantify coherence, many measure definitions were made. Then, Baumgatz et al. [1] established some requirements that should be satisfied by the measure in order to be considered as a proper coherence quantifier.

- The coherence measure of an incoherent state must be zero :

$$C(\hat{\boldsymbol{\delta}}) = 0 \tag{3.5}$$

- If an incoherent map is applied to the state, the coherence measure must not increase:

$$C(\rho) \geq C(\Lambda_{ICTP}(\rho)) \tag{3.6}$$

3.3.2 L-norm measure

The more intuitive of all these measures that satisfy the requirements is the l-norm measure, which is also known as linear coherence C_l :

$$C_l = \sum_{i=0}^{n} |\rho_{i,j}| \tag{3.7}$$

In the case of a qubit ρ_1 which is represented by a 2×2 matrix the coherence measure may take a values

$$0 \leq C_l(\rho_1) \leq 1 \tag{3.8}$$

But if n-qubits are part of a total system ρ_n we may have a $2^n \times 2^n$ matrix representing the total system and in this case its measure varies between

$$0 \leq C_l(\rho) \leq n^2 - 1 \tag{3.9}$$

3.3.3 Entropy measure

Another very used measure is the entropy measure, which is given by

$$C_s = S(\rho_{diag}) - S(\rho) \tag{3.10}$$

where ρ_{diag} is the density matrix with all its non-diagonal elements replaced by zeros and S represents the von Neumann entropy which is defined by

$$S = -\sum_{i=0}^{n} \lambda_i \log_2(\lambda_i) \tag{3.11}$$

where λ are the eigenvalues of the density matrix ρ

As an analogy to the l-norm measure, if n-qubits are part of a total system ρ_n we may have a $2^n \times 2^n$ matrix representing the total system, but in this case its measure varies between

$$0 \leq C_s(\rho) \leq n \tag{3.12}$$

3.3.4 ADC and coherence

An example of an ICPTP is the Amplitude damping channel. It is easy to verify that system coherence cannot grow when we apply the ADC on an incoherent state

$$\phi_{adc}(\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \frac{1}{2} \begin{bmatrix} 1 + P(t) & 0 \\ 0 & 1 - P(t) \end{bmatrix} \tag{3.13}$$

Furthermore, when the Kraus matrices related to a map ϕ does not have more than one non-zero term per column, we say that the map is incoherent. This is a sufficient condition to ensure that an evolution will not create coherence when an incoherent state is given as an initial state.

As we can see, the Kraus matrices given by eq. (2.20) and (2.21) satisfy this condition and it is another way to verify the incoherence of the ADC.

Then, if we consider again an initial coherent state $\hat{\boldsymbol{\rho}}_0^s$ ($C(\hat{\boldsymbol{\rho}}_0^s) = 1$) going into the ADC and we measure the coherence of the final state $\hat{\boldsymbol{\rho}}_t^s$ using the norm measure we get

$$C_l(\frac{1}{2} \begin{bmatrix} 1 + P(t) & \sqrt{1 - P(t)} \\ \sqrt{1 - P(t)} & 1 - P(t) \end{bmatrix}) = \sqrt{1 - P(t)} \tag{3.14}$$

Is evident that the measure satisfy the condition

$$C_l(\hat{\boldsymbol{\rho}}_t^s) \leq C_l(\hat{\boldsymbol{\rho}}_0^s) \tag{3.15}$$

$$\sqrt{1 - P(t)} \leq 1 \tag{3.16}$$

due to the fact that $0 \leq P(t) \leq 1$

Chapter 4

Non-Markovianity

4.1 Markovian processes

4.1.1 Classical Markovian process

Consider the case where we have an stochastic process that involves having different values of the variable $x(t)$ at different times $t \geq 0$, and that this process is described by a hierarchy of joint probability distributions

$$P_n = P_n(x_{n+1}, t_{n+1}; x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; ...; x_1, t_1) \tag{4.1}$$

which satisfies the condition

$$\sum_m P_n(x_n, t_n; ...; x_m, t_m; ...; x_1, t_1) = P_{n-1}(x_n, t_n; ...; x_1, t_1) \tag{4.2}$$

The conditional probability of having the next value x_{n+1} at t_{n+1}, given all the previous ones, is defined by

$$P(x_{n+1}, t_{n+1} \mid x_n, t_n; ...; x_1, t_1) = \frac{P(x_{n+1}, t_{n+1}; x_n, t_n; ...; x_1, t_1)}{P(x_n, t_n; ...; x_1, t_1)} \tag{4.3}$$

We say a process is Markovian if the next state in the chain of states depends only in the present one. Then the conditional probability of having the next state x_{n+1} at t_{n+1}, given all the previous ones, is given by

$$P(x_{n+1}, t_{n+1} \mid x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; ...; x_1, t_1) = P(x_{n+1}, t_{n+1} \mid x_{n-1}, t_{n-1}) \tag{4.4}$$

Since the future state depends only in the present state we say that a Markovian processes lacks of memory.

It is possible, then, to reconstruct the n-point joint probability from the one-point initial probability and from knowing the conditional transition probability T

$$P_n(x_{n+1}, t_{n+1}; ...; x_1, t_1) = \prod_1^{n-1} T(x_{i+1}, t_{i+1} \mid x_i, t_i) P_1(x_1, t_1) \tag{4.5}$$

where

$$T(x_{i+1}, t_{i+1} \mid x_i, t_i) = P_{1|1}(x_{i+1}, t_{i+1} \mid x_i, t_i) \tag{4.6}$$

and these transition probabilities have to satisfy the Chapman-Kolmogorov equation

$$T(x, t \mid y, s) = \sum_z T(x, t \mid z, \tau) T(z, \tau \mid y, s) \tag{4.7}$$

This latter condition ensures that eq.(4.2) is satisfied and also guarantees the markovianity of the process [8]. It is possible to obtain an equivalent differential equation in the form

$$\frac{d}{dt} T(x, t \mid y, s) = \sum_z W_{xz}(t) T(z, t \mid y, s) - W_{zx}(t) T(x, t \mid y, s) \tag{4.8}$$

where $W_{zx}(t) \geq 0$ is the rate (conditional probability per unit of time) that a transition to the state z happens while the system is at state x at time t.

Furthermore, the Chapman-Kolmogorov equation also holds for the one point probability [4]

$$\frac{d}{dt} P_1(x, t) = \sum_z W_{xz}(t) P_1(z, t) - W_{zx}(t) P_1(x, t), \tag{4.9}$$

which is known as a Pauli master equation for a classical Markov process, where again $W_{zx}(t) \geq 0$.(CITAR)

4.1.2   Quantum Markovian process

The concept of markovianity in the quantum regime is not as well defined like in the classical scheme [15]. This becomes evident when we try to make an analogy between the joint probability and a set of projective measurements [4]. For example, consider the system-environment state ρ_{se} and the operator $X = \sum_i x_i \ket{\phi_i} \bra{\phi_i}$. If we define the super-operator that corresponds to the outcome x_i

$$M_i \rho_{se} = \ket{\phi_i} \bra{\phi_i} \rho_{se} \ket{\phi_i} \bra{\phi_i} \tag{4.10}$$

and the unitary evolution super-operator

$$\mathbb{U} = U_t \rho_{se} U_t, \tag{4.11}$$

it is possible to measure the value of quantity X at each step in time prior to the following unitary evolution, in other words

$$P_n(x_n, t_n; x_{n-1}, t_{n-1}; ...; x_1, t_1) = Tr(M_{x_n} \mathbb{U}_{t_n} ... M_{x_1} \mathbb{U}_{t_1} \rho_{se}) \tag{4.12}$$

However, in this formulation is easy to see that the successive measures will collapse the system states and the evolution will be conditioned to the election of the measure M_i.

$$\rho'_{se} = \frac{M_i \rho_{se}}{Tr(M_i \rho_{se})} = \ket{\phi_i} \bra{\phi_i} \otimes \rho_e^{x_i} \tag{4.13}$$

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where we denote the environment state $\rho_e^{x_i}$ to visualize that it has been affected by the measurement M_i.

Thus, due to this fact, the Markovian description of the evolution should not depend on the way we measure. The usual approach is to analyse the Markovianity in the interaction between the environment and the system, which is depicted in the total state evolution ρ_{se} itself.

In the case of a non-Markovian processes the future states depend also on the past ones, then we say this type of evolutions involve memory effects.

When we talk of a system interacting with an environment we say that it losses information and in the case of non-Markovian evolutions, we have a backflow of information returning to the system.

4.2   CP-divisibility and Markovianity

The relation between CP-divisibility and Markovianity starts with the concept of classical Markovianity and the Chapman-Kolmogorov equation. As we have seen, this latter equation is obtained from the Markov condition.

In spite of making a relation between CP-divisibility and quantum Markovianity, we consider the matrix

$$(\Lambda_{t,s})_{x,y} = T(x,t|y,s) \tag{4.14}$$

which is an stochastic matrix whose entries are the conditional probabilities $T(x,t|y,s)$

Then, as we have seen the conditional probabilities satisfy the Chapman-Kolmogorov equation (4.7), and if we rewrite the equation with the matrices $\Lambda_{t,s})_{x,y}$ we get

$$\Lambda_{t,s} = \Lambda_{t,r}\Lambda_{r,s}, \tag{4.15}$$

where $t > r > s > 0$, and as we see this equation is very similar to the equation that defines the divisibility of the map (2.36)

Due to this fact and due to the idea that the characterization of Markovianity has to be made with the analysis of the evolution of the system-environment whole state, which in other words means that the Markovianity property will be proven by the map behaviour, we say that a divisible map involves a Markovian evolution.

Accordingly, the equation $\Lambda_{t,s} = \Lambda_{t,r}\Lambda_{r,s}$ is taken as a quantum couterpart to the classic Chapman-Kolmogorov equation [12].

CP-divisibility, then, implies Markovianity, conversely in the case of indivisibility of the map, we are in the case of a non-Markovian process.

Furthermore, there Rivas, Huelga and Plenio made a measure of non-Markovianity based on the property of the divisibility of the map [12].

4.3   non-Markovianity and distinguishability

One approach to the subject is the the idea of having Alice sending information to Bob through a quantum channel. Let's say that Alice prepares two states ρ_1, ρ_2 and that she can distinguish both states using the trace distance measure [7], which is given by

$$D(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2|| \tag{4.16}$$

where the norm of an operator $||A|| = Tr|A|$

The modulus of the operator is given by $|A| = \sqrt{A^\dagger A}$ and in the case of an hermitian operator like a density matrix, the trace of the modulus is given by

$$Tr|A| = \sum_{i=1}^{n}|a_i| \tag{4.17}$$

where a_i are the eigenvalues of the operator.

For example, the distinguishability between $\rho_1 = |H\rangle\langle H|$ and $\rho_1 = |V\rangle\langle V|$ where in matrix form $\rho_1 - \rho_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, is given by $\frac{1}{2}(|1| + |-1|)=1$.

The trace distance of two orthogonal states is, then, equal to $D(\rho, \rho_\perp) = 1$ and in the case of the same state it takes the value $D(\rho, \rho) = 0$. Thus, we have that the trace distance is bounded

$$0 \leq D(\rho_1, \rho_2) \leq 1 \tag{4.18}$$

Taking again the situation of Alice and Bob, we shall say that Alice prepares two states with equal probability and sends them to Bob trough a Channel, thus Bob has a chance of measuring them correctly, which maximally is given by [6]

$$P_{max} = \frac{1}{2}(1 + D(\rho_1, \rho_2)) \tag{4.19}$$

To illustrate a system-environment interaction, we say that Alice sends the states to Bob through a noisy channel that affects the states and thus affects also the distinguishability. We shall say that the probability P_{max} that Bob makes get the correct measure will decrease, and hence we may say that the system information was lost due to the interaction with the channel and this lost information went to the environment. Here naturally we are calling information to the distinguishability of the quantum states prepared by Alice.

In addition, it is neccesary to recall that the trace distance measure has the property of never increase if a completely positive trace preserving map is applied to the states [13].

$$D(\lambda(\rho_1), \lambda(\rho_2)) \leq D(\rho_1, \rho_2) \tag{4.20}$$

In the particular case of a unitary evolution, is easy to verify that

$$D(U(\rho_1), U(\rho_2)) = D(\rho_1, \rho_2) \tag{4.21}$$

Therefore, when we apply a completely positive trace preserving dynamical map on the initial states we have the inequality

$$D(\phi_t(\rho_1), \phi_t(\rho_2)) = D(\rho_1(t), \rho_2(t)) \leq D(\rho_1(0), \rho_2(0)), \tag{4.22}$$

which shows that in the Markovian scheme the distinguishability must decrease monotonically, because information always goes into the environment and never returns to the system.

On the other hand, if we verify that the trace distance measure $D(\rho_1(t), \rho_2(t))$ increases, at least in a time interval, we are in the presence of non-Markovianity and we shall say that, at least in a brief period of time, the information stored in the environment flew back into the system.

Thus we take the distinguishability increase over time as a non-Markovianity witness. Then, in order to quantify this return of information, Breuer et al [3] defined the measure

$$N(\phi) = max_{\rho_s^{1,2}} \int_{\sigma \geq 0} \sigma(t) dt \tag{4.23}$$

where

$$\sigma(t) = \frac{dD(\rho_1(t), \rho_2(t))}{dt} \tag{4.24}$$

This functional $N(\phi)$ characterizes non-Markovianity of the map ϕ and takes the maximal value of all the possibles pairs of initial states of the system considering only the intervals in time where the distinguishability is positive. It represents then, the total maximal flow of information from the environment into the system that the map ϕ is able to return [4].

It is proven that this definition of the measure can be simplified taking into account that the maximal of the functional is achieved by using optimal pair of states, which are in most cases orthogonal states of the system space [16] and they naturally have $D(\rho_1, \rho_2) = 1$.

Furthermore, Breuer et al.[4] define the system information by

$$I_s = D(\rho_1^s(t), \rho_2^s(t)) \tag{4.25}$$

and the external information by

$$I_{ext} = D(\rho_1^{se}(t), \rho_2^{se}(t)) - D(\rho_1^s(t), \rho_2^s(t)) \tag{4.26}$$

it is easy to notice that the sum of external and internal information

$$I_{ext} + I_s = D(\rho_1^{se}(t), \rho_2^{se}(t)) \tag{4.27}$$

depends only on the information of the whole systems states

However, we know that the system-environment (se) is closed and evolves under a unitary operator, so using eq. (4.22) we have

$$D(\rho_1^{se}(t), \rho_2^{se}(t)) = D(\rho_1^{se}(0), \rho_2^{se}(0)) \tag{4.28}$$

Finally we say that the sum, which represents the total information, becomes a constant

$$I_{ext} + I_s = D(\rho_1^{se}(0), \rho_2^{se}(0)) = const. \tag{4.29}$$

4.4 Coherence and non-Markovianity witness

The use of coherence as information has been subject of study recently to model non-Markovian evolutions [5, 10] . Under ICPTP maps, the coherence should decrease monotonically, since we are in the Markovian regime, where the information (coherence) does not return. Then, if a dynamical ICPTP map ϕ_t is applied to the state we will get

$$C(\phi_t(\rho_s(t))) \leq C(\rho_s(t))) \tag{4.30}$$

When this does not happen and $C(\phi_t(\rho_s(t)))$ does not decrease monotonically we are in presence of a back-flow of information (Non-Markovian process), which is made evident by the non-Markovianity witness:

$$\frac{dC}{dt} \geq 0 \tag{4.31}$$

As an analogy of the trace distance measure we define

$$N(\phi) = \int_{\sigma \geq 0} \sigma(t) dt \tag{4.32}$$

where

$$\sigma(t) = \frac{dC(t)}{dt} \tag{4.33}$$

and as we can see comparing to the trace distance measure, the maximal of the functional $N(\phi)$ is no longer needed when we work with coherence and hence it simplifies the calculations.

Additionally, by making an analogy with the trace distance measure, we define the system's information as

$$I_{sist} = C(\rho_{pol}(t)) \tag{4.34}$$

and also the external information

$$I_{ext} = C(\rho_{se}(t)) - C(\rho_{pol}(t)) \tag{4.35}$$

and now we sum to calculate the total information of the s-e

$$I_{total} = I_{sist} + I_{ext} = C(\rho_{se}(t)) \neq const \tag{4.36}$$

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And as we can see, this result differs to the trace distance measure case, where the s-e information is constant due to the fact that it is a closed system, nonetheless, when we work with coherence, the unitary evolutions taking place in the system-environment naturally create coherence.

Besides, we also can re-define the external information

$$I_{ext} = I_{env} + I_{ent} \tag{4.37}$$

where I_{env} is the information calculated by making the partial trace over the system and the term I_{ent} is the information due to the entanglement between system and environment.

We can calculate then I_{ent} by

$$I_{ent} = I_{total} - (I_{sist} + I_{env}) \tag{4.38}$$

As we can see in the graph of coherence over time, at the beginning of the evolution, the state is separable so $I_{ent} = 0$ and the total information I_{total} equals the sum of the terms $I_{sist} + I_{env}$. But then, as the evolution continue, the state is no longer separable and the entanglement term begins to contribute to the total information. That occurs until the state is separable again, just at the time when the amplitude damping channel takes all the coherence into the environment.

Chapter 5

Non-Markovian evolution simulation

To simulate a Non-Markovian evolution, we will use as information the coherence of the state of a system. This coherence will flow between the system and the environment.

In the case of our experiment, we will produce an initial separable state represented in the space of polarization (system) and path (environment). It is given by

$$\mathcal{H} = \mathcal{H}_{polarization} \otimes \mathcal{H}_{path} \tag{5.1}$$

The mentioned separable state will interact with a channel composed of two sections. The first, an amplitude damping channel, will produce the decoherence and the second part will be used to return the information back into the system.

Then, the whole evolution will be described and we will characterize the flow of information with the aid of the linear coherence measure, which we will use as a witness of non-Markovianity. In both sections of the evolution, the change of coherence over time in the system and environment will be shown to have a better understanding of the dynamics of information between qubits. Finally, the non-Markovianity of the evolution will be quantified using eq.(4.33)
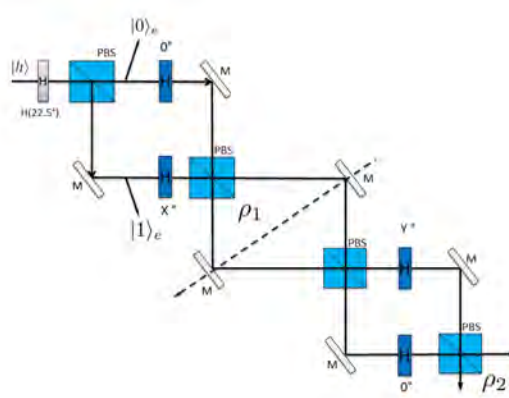
5.1 Experimental setup

Figure 5.1: The setup has a Markovian (ADC) and a non-Markovian sections.

5.2 Preparing the initial state

In order to simulate a system-environment interaction, two qubits were used for the experiment: the system is represented as a qubit of polarization (p) and the environment as the path(c).

The initial state of polarization will be the diagonal polarization state going in the path $|0\rangle$

$$|\psi(0)\rangle_{se} = (\frac{|h\rangle + |v\rangle}{\sqrt{2}}) \otimes |0\rangle \tag{5.2}$$

As we see this is a separable state. Equivalently, in density matrix notation we have

$$|\psi_0\rangle \langle \psi_0| = \hat{\boldsymbol{\rho}}_0^{pc} = \hat{\boldsymbol{\rho}}_0^{p} \otimes \hat{\boldsymbol{\rho}}_0^{c} \tag{5.3}$$

where $\hat{\boldsymbol{\rho}}_0^{pc}$ in matrix form in the base of polarization and path is:

$$\hat{\boldsymbol{\rho}}_o^{pc} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

This state is prepared using a 633 nm laser beam, which is a source of visible linear polarized light. Then, with the help of a polarizer we filter all the other polarizations that we don't need in order to stay with vertical polarization. Finally using a half-wave plate we will rotate this vertical polarization into a diagonal one.

5.3 Experimental procedure

Our initial state, as we can see, consists of a coherent state coupled to an incoherent environment qubit. The evolution made by the amplitude damping

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channel has the objective of taking this coherence, now taken as information of the system, into the environment so in the end we have the state

$$|\psi_f\rangle = |H\rangle \, (\frac{|0\rangle + |1\rangle}{\sqrt{2}}). \tag{5.5}$$

Here we can see an incoherent state of polarization coupled with a coherent environment state, namely all the information (coherence) went from system to environment. What happened in between one state and the other cannot be represented as a separable state because at those times of the evolution the polarization is entangled with the path qubit, hence it is not separable.

The second part of the evolution consists of a setup identical to the amplitude damping channel, but it does not make the same operation due to the possibility of having as an initial state consisting of polarization in both paths going into the interferometer. This difference with the first channel creates a way by which the information may return, that is to say it is not an amplitude damping channel and it does not perform an incoherent operation. It has the potential of returning the coherence and mapping an incoherent state into a coherent one using the coherence available in the environment qubit (path). In other words, it makes possible the backflow of information.

5.4 Evolving the sate in the ADC

First, the initial state is going to interact with an amplitude damping channel. This channel is constructed with two polarizing beam splitters and a half-wave plate. To simulate an evolution over time, the angle of the HWP is said to represent the time as it changes in discrete steps from 0 to $\pi/4$ Then, we will have the following unitary evolution acting on the whole system-environment state

$$U_{adc} = U_{mirror}.U_{pbs}.U_{mirror}.U_{hwp}.U_{pbs} \tag{5.6}$$

So the evolution in the system-environment closed state is

$$|\psi_t\rangle = U_{adc}|\psi_0\rangle \tag{5.7}$$

In density matrix form:

$$\hat{\boldsymbol{\rho}}_t^{pc} = U_{adc} \cdot \hat{\boldsymbol{\rho}}_0^{pc} \cdot U_{adc}^\dagger, \tag{5.8}$$

Explicitly in matrix form in the base of polarization and path U_{adc} is composed by

$$U_{pbs} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \tag{5.9}$$

$$U_{mirror} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{5.10}$$

$$U_{hwp} = \begin{bmatrix} \cos(2x) & \sin(2x) \\ \sin(2x) & -\cos(2x) \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \cos(0) & \sin(0) \\ \sin(0) & -\cos(0) \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.11}$$

As we can see, in the HWP matrix, there are two HWP acting one on each path. The one which simulates time that changes over time and another just at 0 degrees to compensate the difference in the path.

Accordingly, U_{adc} makes the following transformation

- If the system is in the horizontal polarization state, the state remains the same

$$|H\rangle |0\rangle \implies |H\rangle |0\rangle \tag{5.12}$$

- If the system is in an vertical polarization state, theres a chance of giving information to the environment

$$|V\rangle |0\rangle \implies \sqrt{P} |V\rangle |0\rangle + \sqrt{1-P} |H\rangle |1\rangle \tag{5.13}$$

where in this specific case the probability P is in terms of the HWP angle x

$$P(x) = \sin^2(2x) \tag{5.14}$$

Taking $x = t$ we apply the channel and calculate the total evolved state :

$$\hat{\boldsymbol{\rho}}_{pc}(t) = \begin{bmatrix} \frac{1}{2} & -i\cos(t)\sin(t) & \frac{1}{2}\cos(2t) & 0 \\ \cos(t)\sin(t) & 2\cos^2(t)\sin^2(t) & \frac{i}{4}\sin(4t) & 0 \\ \frac{1}{2}\cos(2t) & -\frac{i}{4}\sin(4t) & \frac{1}{2}\cos^2(2t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.15}$$

Now we can calculate the system (p) state by tracing partially in the environment to obtain

$$\hat{\boldsymbol{\rho}}_p(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\sin^2(2t) & \frac{1}{2}\cos(2t) \\ \frac{1}{2}\cos(2t) & \frac{1}{2}\cos^2(2t) \end{bmatrix} \tag{5.16}$$

We can calculate likewise the environment (c) state by tracing partially in the system to get

$$\hat{\boldsymbol{\rho}}_c(t) = \begin{bmatrix} \frac{1}{4}(3 + \cos(4t)) & -i\cos(t)\sin(t) \\ i\cos(t)\sin(t) & 2\cos^2(t)\sin^2(t) \end{bmatrix} \tag{5.17}$$

5.5 Non-Markovian evolution

5.5.1 Initial state

The initial state of the second part of the setup is the final state of the first part, which is at time $t = \frac{\pi}{4}$ and is the time when the maximal decoherence is achieved.

$$\hat{\boldsymbol{\rho}}_{pc}(\pi/4) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} & 0 & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.18}$$

Then we say that the initial state of the second part is at $t = 0$

$$\hat{\boldsymbol{\rho}}_{pc}^2(0) = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} & 0 & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.19)$$

5.5.2 Evolving the state

The initial state is going to interact with a setup that is very similar to the first one. This channel is constructed with two polarizing beam splitters and a half-wave plate, but this time the HWP will act on the other path. To simulate an evolution over time again, the angle of the HWP is said to represent the time t as it changes in discrete steps from 0 to $\pi/4$ until the coherence is recovered.

In a similar way to the first part, we have the following unitary evolution acting on the complete system-environment state

$$U_{nc} = U_{pbs}.U_{mirror}.U_{hwp2}.U_{pbs} \qquad (5.20)$$

where this time the HWP operator is given by

$$U_{hwp2} = \begin{bmatrix} \cos(2x) & \sin(2x) \\ \sin(2x) & -\cos(2x) \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \cos(0) & \sin(0) \\ \sin(0) & -\cos(0) \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (5.21)$$

Therefore, in the end we get the final state

$$\hat{\boldsymbol{\rho}}_{pc}^2(t) = \begin{bmatrix} \frac{1}{2} & \frac{i}{2}\cos(2t) & \frac{1}{2}\sin(2t) & 0 \\ \frac{-i}{2}\cos(2t) & \frac{1}{2}\cos^2(2t) & \frac{-i}{4}\sin(4t) & 0 \\ \frac{1}{2}\sin(2t) & \frac{i}{4}\sin(4t) & 2\cos^2(t)\sin^2(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.22)$$

Now we can calculate the system (p) state after the second part of the setup by tracing partially in the environment to obtain

$$\hat{\boldsymbol{\rho}}_p^2(t) = \begin{bmatrix} \frac{1}{4}(3 + \cos(4t)) & -\cos(t)\sin(t) \\ \cos(t)\sin(t) & 2\cos^2(t)\sin^2(t) \end{bmatrix} \qquad (5.23)$$

Similarly we calculate the environment (c) state by tracing partially in the system again to get

$$\hat{\boldsymbol{\rho}}_c^2(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\sin^2(2t) & i\frac{1}{2}\cos(2t) \\ -i\frac{1}{2}\cos(2t) & \frac{1}{2}\cos^2(2t) \end{bmatrix} \qquad (5.24)$$

5.6 Calculating the Coherence

We have now, the evolved states (in terms of t) of the first and second section of the setup, where in each one we have the state of polarization, path and the

total system.

Then, we calculate the linear coherence of each one to get

$$C(\hat{\boldsymbol{\rho}}_p^1(t)) = |\cos(2t)| \tag{5.25}$$

$$C(\hat{\boldsymbol{\rho}}_c^1(t)) = |\sin(2t)| \tag{5.26}$$

$$C(\hat{\boldsymbol{\rho}}_p^2(t)) = |\sin(2t)| \tag{5.27}$$

$$C(\hat{\boldsymbol{\rho}}_c^2(t)) = |\cos(2t)| \tag{5.28}$$

$$C(\hat{\boldsymbol{\rho}}_{pc}^1(t)) = |\cos(2t)| + |\sin(2t)| + \frac{1}{2}|\sin(4t)| \tag{5.29}$$

$$C(\hat{\boldsymbol{\rho}}_{pc}^2(t)) = |\cos(2t)| + |\sin(2t)| + \frac{1}{2}|\sin(4t)| \tag{5.30}$$

Furthermore, we have plotted the evolution of these coherences over time in Fig. 5.1, where as we can see there is a gap between the coherence of the whole system $C(\rho_{se})$ and the sum of coherences of the system and environment separately $C(\rho_s) + C(\rho_e)$, which were calculated by the partial trace. Thus, this lead us to think that this gap occurs due to the entanglement between system and environment, which encloses information in form of coherence as we have noticed in eq.(4.37).
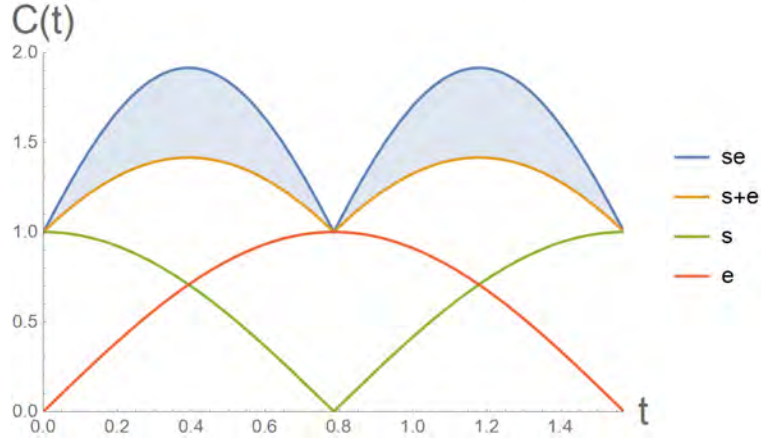

Figure 5.2: Coherence of system, environment, whole system-environment and system plus environment in both sections of the setup

5.6.1 Calculating the non-Markovianity

From the system coherence of the non-Markovian section of the setup (eq. 5.27) we can get the change of coherence over time

$$\frac{d}{dt}C(\hat{\boldsymbol{\rho}}_p^2(t)) = \frac{2\cos(2t)\sin(2t)}{\sqrt{\sin(2t)^2}} \tag{5.31}$$

Then, we integrate this in the interval where it is positive to get the theoretical measure

$$\int_0^{\pi/4} \frac{d}{dt} C(\hat{\boldsymbol{\rho}}_p^2(t)) dt = 1. \tag{5.32}$$

5.6.2 Kraus Matrices

In both parts of the evolution the following Kraus matrices are mapping the system and environment states respectively.

- ADC matrices that evolve the polarization state:

$$K_1^{pol1} = \begin{bmatrix} -1 & 0 \\ 0 & -\cos(2t) \end{bmatrix}, K_2^{pol1} = \begin{bmatrix} 0 & -i\sin(2t) \\ 0 & 0 \end{bmatrix} \tag{5.33}$$

- Matrices that return the coherence of the polarization state:

$$K_1^{pol2} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ \sin(2t) & \cos(2t) \end{bmatrix}, K_2^{pol2} = \frac{i}{\sqrt{2}} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ 0 & 1 \end{bmatrix} \tag{5.34}$$

- Matrices of the path in the first evolution:

$$K_1^{path1} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ \sin(2t) & \cos(2t) \end{bmatrix}, K_2^{path1} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ i\sin(2t) & \cos(2t) \end{bmatrix} \tag{5.35}$$

- Matrices of the path in the second part of the setup:

$$K_1^{path2} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ i\cos(2t) & 0 \end{bmatrix}, K_2^{path2} = \frac{i}{\sqrt{2}} \begin{bmatrix} -\sin(2t) & 0 \\ 0 & 0 \end{bmatrix} \tag{5.36}$$

where all these Kraus operators, even the ones involving the whole state and the partial systems, satisfy the eq.(2.8) and hence this ensures that these maps are indeed completely positives.

5.7 Results

We have measured the coherence in the whole evolution by taking data in each of the setup's sections. The first set of data corresponds to the interval where the HWP(x) of the ADC varies between $0 < x < \pi/4$. In a similar way, the second part of the setup give us the final part of the data, that corresponds to the HWP(y) of the non-Markovian section, where $0 < y < \pi/4$. In order to measure coherence, we have made topographies to reconstruct the polarization matrices and we have measured the intensity in each path in steps during the evolution to calculate its coherence.

We have measured the coherence of path and polarization over time(that depends on the angles) in both sections of the setup to obtain
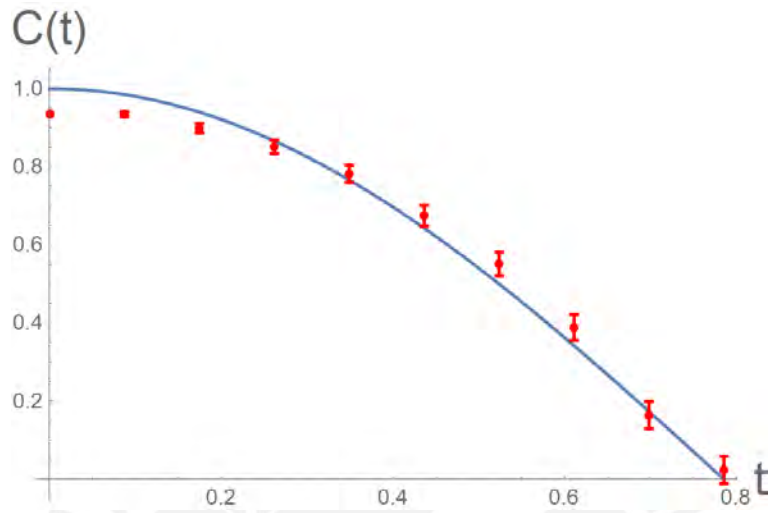

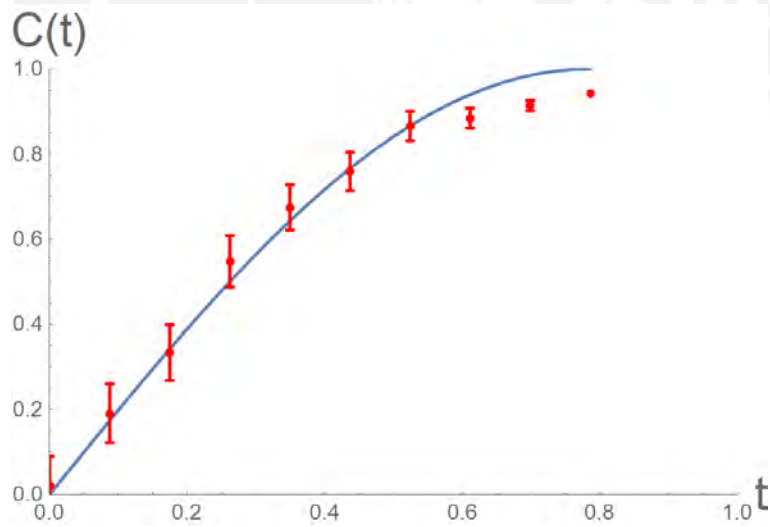Figure 5.3: Polarization (system) coherence in the Markovian section

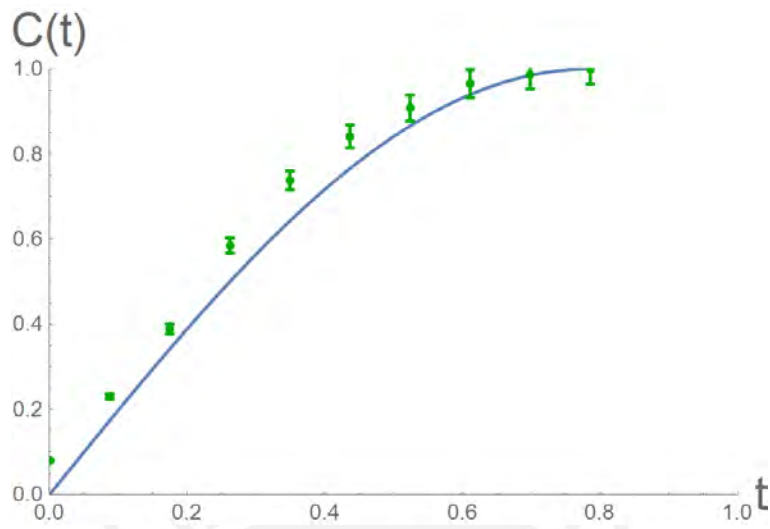Figure 5.4: Polarization coherence in the non-Markovian section

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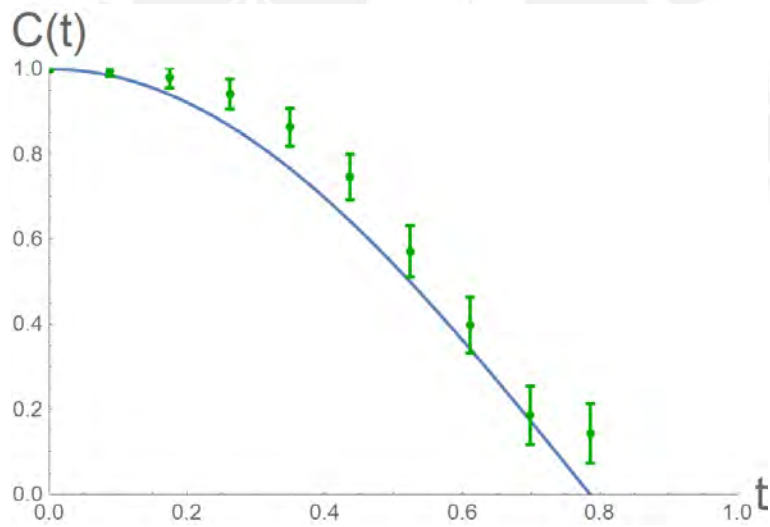Figure 5.5: Path (environment) coherence in the Markovian section

Figure 5.6: Path (environment) coherence in the non-Markovian section

Then, we have made a fit of the second part of the evolution (Fig. 5.2), where the polarization coherence returns. The differentiation of that fit function lead us to Fig. 5.6, where an experimental and theoretical value of $\frac{d}{dt}C$ is plotted.
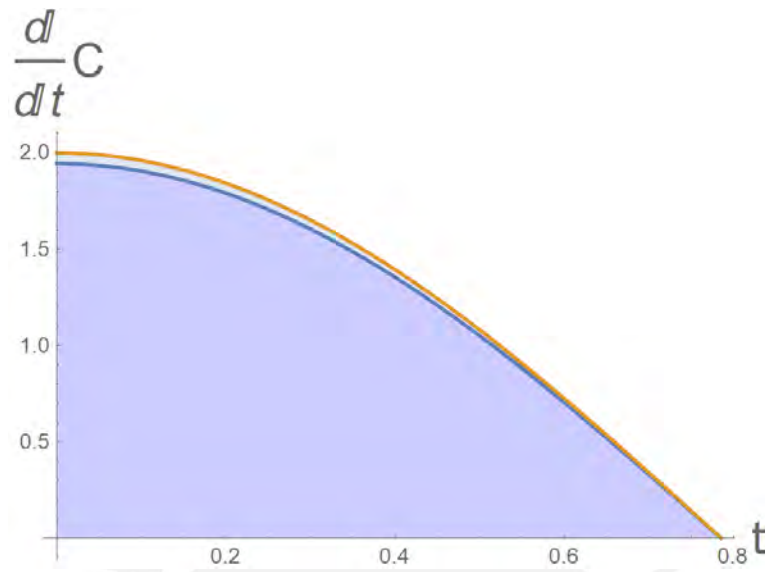
Figure 5.7: Polarization coherence in the first part. The area beneath the blue curve (experimental) is the amount of non-Markovianity of the process, the lighter area represents the difference with the area beneath the red line (theoretical).

Finally when we integrate the experimental function to obtain a measure of non-Markovianity of 0.972 ± 0.034 .

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Chapter 6

Summary and Conclusions

- We have been able to simulate a non-Markovian evolution using the coherence of the system as information. First, the coherence of the system was lost into the environment due to an amplitude damping channel that produces decoherence. Then, it was recovered by applying a non-Markovian CP map on the incoherent state of the system. In other words, in order to be in the presence of a backflow of information, in the second section of the setup, we have mapped an incoherent polarization state with a coherent one by taking the coherence available in the qubit of the optical path, which represents the environment.

- According to the theoretical results, we have noticed that the entanglement between polarization and optical path encloses information in form of coherence. This information could be measured if we use a tomography technique that allow us to measure the 4×4 matrix of the whole state.

- We have been successful at quantifying the non-Markovianity of the evolution by measuring a value of 0.972, which has a relative error of 2.81% when compared to the theoretical result.

- Even though the simulation was successful, the idea of information based on the concept of coherence differs to other approaches like the distinguishability of the states, because the coherence varies under unitary transformations. Therefore, this lead us to think that there should be certain requirements that a physical property must fulfil in order to be interpreted as information.

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