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Minimal possible counterexamples to the two-dimensional Jacobian Conjecture

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MINIMAL POSSIBLE COUNTEREXAMPLES TO THE TWO-DIMENSIONAL JACOBIAN CONJECTURE

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. . . A mis profesores, cuya paciencia y dedicación me permitieron embarcarme con buen rumbo en el mundo de las matemáticas.

Abstract

Let K be an algebraically closed field of characteristic zero. The Jacobian Conjecture (JC) in dimension two stated by Keller in [8] says that any pair of polynomials $P, Q \in L := K[x, y]$ with $[P, Q] := \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^{\times}$ (a *Jacobian pair*) defines an automorphism of L via $x \mapsto P$ and $y \mapsto Q$.

It turns out that the Newton polygons of such a pair of polynomials are closely related, and by analyzing them, much information can be obtained on conditions that a Jacobian pair must satisfy. Specifically, if there exists a Jacobian pair that does not define an automorphism (a *counterexample*) then their Newton polygons have to satisfy very restrictive geometric conditions.

Based mostly on the work in [1], we present an algorithm to give precise geometrical descriptions of possible counterexamples. This means that, assuming (P, Q) is a counterexample to the Jacobian Conjecture with $\gcd(\deg(P), \deg(Q)) = k$, we can generate the possible shapes of the Newton Polygon of P and Q and how it transforms under certain linear automorphisms. By analyzing the minimal possible counterexamples, we sketch a path to increase the lower bound of $\max(\deg(P), \deg(Q))$ to 125 for a minimal possible counterexample to the Jacobian Conjecture.

Resumen

Sea K un cuerpo algebraicamente cerrado de característica zero. La Conjetura del Jacobiano en dimensión dos postulada por Keller en [8] dice que cualquier par de polinomios $P, Q \in L \coloneqq K[x, y]$ con $[P, Q] \coloneqq \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^\times$ (un *par Jacobiano*) define un automorfismo de L via $x \mapsto P$, $y \mapsto Q$.

Resulta que los polígonos de Newton de tal par de polinomios están relacionados íntimamente, y al analizarlos, mucha información puede ser obtenida sobre condiciones que un par Jacobiano debe satisfacer. Específicamente, si existe un par Jacobiano que no define un automorfismo (un *contraejemplo*) entonces sus polígonos de Newton deben satisfacer condiciones geométricas bastante restrictivas.

Basado en gran parte en el trabajo en [1], presentamos un algoritmo para dar una descripción geométrica precisa de posibles contraejemplos. Esto significa que, asumiendo que (P, Q) es un contraejemplo a la Conjetura del Jacobiano con $\gcd(\deg(P), \deg(Q)) = k$, podemos generar las posibles formas del Polígono de Newton de P y Q y cómo se transforman bajo ciertos automorfismos lineales. Al analizar los posibles contraejemplos minimales, esbozamos un camino para incrementar la cota inferior de $\max(\deg(P), \deg(Q))$ a 125 para un posible contraejemplo minimal a la Conjetura del Jacobiano.

Contents

Introduction

Let K be a characteristic zero field and let $L := K[x,y]$ be the polynomial algebra in two indeterminates. The Jacobian Conjecture (JC) in dimension two stated by Keller in [8] says that any pair of polynomials $P, Q \in L$ with $[P,Q] := \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^\times$ defines an automorphism f of L via $f(x) := P$ and $f(y) := Q$. If this conjecture is false, then there exist $P, Q \in L$ such that $[P,Q] = K^\times$, and there exist $m, n, a, b \in \mathbb{N}$, such that $m, n > 1$ are coprime, $a < b$, the support of P is contained in the rectangle with vertices $\{(0,0), m(a,0), m(a,b), m(0,b)\}$, the support of Q is contained in the rectangle with vertices $\{(0,0), n(a,0), n(a,b), n(0,b)\}$, the point $m(a,b)$ is in the support of P and the point $n(a,b)$ is in the support of Q. Note that $\deg(P) = m(a+b)$ and $\deg(Q) = n(a+b)$.

In [7] Heitmann establishes several restrictions on these possible corners (a,b) and in [7, Theorem 2.24] he determines various of these possible corners (a,b). Moreover in [7, Theorem 2.25], for some of these corners, he finds families $\{(r+sj, t+uj) : j \in \mathbb{N}\}$ of admissible pairs (m,n). These corners were also found in [1, Remark 7.14], using more elementary methods and discrete geometry on the plane. In both articles the lists of possible corners where given without a formal proof, referring to a computer program.

In [2] we found more conditions on the points (a,b), and in this article we present an algorithm that generates the list of points satisfying all the conditions up to a fixed upper bound for $a+b$. Naturally this list is included in the one found in [1, Remark 7.14]. The algorithm also determines the families of admissible pairs (m,n), for each of these corners.

In order to exploit the simple geometric ideas of our method we also present a graphic interface of the program which includes all the filters and allows the user to grasp in detail if and why a certain corner is admissible or not.

At the end we list all possible corners (a,b) with $a+b<36$, and their corresponding (m,n)-families. Furthermore if (P,Q) is a counterexample to the Jacobian Conjecture that satisfy the inequality $\gcd(\deg(P), \deg(Q)) < 36$, then we give additional information on the Newton polygons of P and Q. We also provide the same information for the counterexamples that satisfy $\max\{\deg(P), \deg(Q)\} \leq 150$.

Along this thesis we will freely use the notations of [1]. This work is almost completely a transcription of the article [5], written with Jorge Alberto Guccione, Juan José Guccione and Christian Valqui.

Chapter 1

Restrictions on possible last lower corners

The first step in our strategy is to construct a set of points in $\mathbb{N}_0 \times \mathbb{N}_0$, that includes all the possible last lower corners (see [2, Definition 3.17]).

Definition 1.0.1. Let $(a, b) \in \mathbb{N} \times \mathbb{N}_0$ and $(\rho, \sigma) \in \mathfrak{V} \cap [(0, -1), (1, -1)[$ (see [1, Definition 1.5]). We say that $((a, b), (\rho, \sigma))$ is a *possible final pair* if one of the following conditions is fulfilled:

1. $b = 0$ and $(\rho, \sigma) = (0, -1)$,

2. there exists an admissible chain of length $k \in \mathbb{N}$ (see [2, Definition 3.15])

$$\mathfrak{C} = \big((C_j)_{j\in\{0,\dots,k\}}, (R_j)_{j\in\{1,\dots,k\}}, (\rho_j, \sigma_j)_{j\in\{1,\dots,k\}}\big),$$

   with $C_k = (a, b)$ and $(\rho_k, \sigma_k) = (\rho, \sigma)$.

Remark 1.0.2. Recall from [2, Definition 3.17] that if $((a, b), (\rho, \sigma))$ is a possible final pair, then (a, b) is said to be a possible last lower corner.

Remark 1.0.3. By [2, Definition 3.15(6)], if $((a, b), (\rho, \sigma))$ is a possible final pair, then $b < a$.

Remark 1.0.4. By [2, Remark 3.19], we know that if $a > 2b > 0$, then $((a, b), (1, -2))$ is a possible final pair.

Remark 1.0.5. By [2, Proposition 3.25], if (a, b) is a possible last lower corner, then $b \leq (a - b - 1)^2$, which, since $a \geq 1$ and $b < a$, is equivalent to $b \leq \frac{1}{2}\big(2a - \sqrt{4a - 3} - 1\big)$.

Proposition 1.0.6. *If $((a,b),(\rho,\sigma))$ is a possible final pair with $b>0$ and $a\leq 2b$, then $v_{\rho,\sigma}(a,b)\geq \rho$ and there exist a possible final pair $((r,s),(\rho',\sigma'))$ such that:*

1. *$r<a$, $s<b$ and $r-s<a-b$,*
2. *$v_{\rho,\sigma}(r,s)=v_{\rho,\sigma}(a,b)$,*
3. *$\overline{\vartheta}\leq \gcd(a-r,b-s)$ or $\overline{\vartheta}\mid \gcd(r,s)$, where $\overline{\vartheta}:=\frac{\rho a+\sigma b}{\gcd(\rho+\sigma,\rho a+\sigma b)}$.*

Proof. By hypothesis there exists an admissible chain

$$\mathfrak{C}=\big((C_j)_{j\in\{0,\dots,k\}},(R_j)_{j\in\{1,\dots,k\}},(\rho_j,\sigma_j)_{j\in\{1,\dots,k\}}\big)\qquad \text{with } C_k=(a,b) \text{ and } (\rho_k,\sigma_k)=(\rho,\sigma).$$

Note that $k\geq 1$ and set

$$(r,s):=C_{k-1}\qquad\text{and}\qquad(\rho',\sigma'):=\begin{cases}(\rho_{k-1},\sigma_{k-1}) & \text{if } k>1,\\ (0,-1) & \text{if } k=1.\end{cases}$$

By [2, Definition 3.15(7)] we know that $v_{\rho,\sigma}(a,b)\geq\rho$. We next prove the rest of the proposition. Item (1) follows from [2, Remark 3.16], while item (2) follows from items (4) and (5) of [2, Definition 3.15]. Moreover, by items (7) and (8) of [2, Definition 3.15], the hypothesis of [2, Proposition 3.12] are satisfied with $R=R_k$. Since $a\leq 2b$, case (1) of that proposition is impossible. Let θ and t' be as in [2, Proposition 3.12]. By [2, Remark 3.13]

$$\frac{\vartheta}{t'}=-\frac{v_{\rho,\sigma}(R)}{\rho+\sigma}=-\frac{\rho a+\sigma b}{\rho+\sigma}.$$

Hence $\overline{\vartheta}\mid\vartheta$, and so item (3) follows from items (2) and (3) of [2, Proposition 3.12]. □

Based on the previous results in Algorithm 1 we present a method for the generation of a set PLLC that includes all possible last lower corners (a,b) with $a\leq x_{max}$ for a given x_{max}. In the algorithm we use an auxiliary list PFL.

Algorithm 1: GetPossibleLastLowerCorners

Input: Maximum x coordinate value $x_{max} > 0$.

Output: A list PLLC, that includes all the possible last lower corners (a, b) with $a \leq x_{max}$.

for $a \leftarrow 1$ **to** x_{max} **do**
    $b \leftarrow 0$
    while $b \leq \frac{1}{2}\left(2a - \sqrt{4a-3} - 1\right)$ **do**
        if $b = 0$ **then**
            $(\rho, \sigma)_{a,b} \leftarrow (0, -1)$, add $((a, b), (\rho, \sigma)_{a,b})$ to PFL and add (a, b) to PLLC
        else if $a > 2b > 0$ **then**
            $(\rho, \sigma)_{a,b} \leftarrow (1, -2)$, add $((a, b), (\rho, \sigma)_{a,b})$ to PFL and add (a, b) to PLLC
        else
            set $(\rho, \sigma)_{a,b} := (1, -1)$
            for $\left((r, s), (\rho, \sigma)_{r,s}\right)$ in PFL such that $r < a$, $s < b$ and $r - s < a - b$ **do**
                $N_1 \leftarrow \gcd(a - r, b - s)$
                $N_2 \leftarrow \gcd(r, s)$
                $(\rho, \sigma) \leftarrow \dfrac{1}{N_1}(b - s, r - a)$
                $g \leftarrow \gcd(\rho + \sigma, \rho a + \sigma b)$
                $\overline{\vartheta} \leftarrow \dfrac{\rho a + \sigma b}{g}$
                if $(\rho, \sigma)_{r,s} < (\rho, \sigma) < (\rho, \sigma)_{a,b}$, $v_{\rho,\sigma}(a, b) \geq \rho$ and $(\overline{\vartheta} \leq N_1$ or $\overline{\vartheta} \mid N_2)$ **then**
                    $(\rho, \sigma)_{a,b} \leftarrow (\rho, \sigma)$
            if $(\rho, \sigma)_{a,b} < (1, -1)$ **then**
                add $\left((a, b), (\rho, \sigma)_{a,b}\right)$ to PFL and add (a, b) to PLLC
        $b \leftarrow b + 1$
return PLLC.

Chapter 2

Construction of admissible complete chains up to a certain bound

Assume that the Jacobian Conjecture is false and define

$$B := \min\big\{\gcd(v_{1,1}(P), v_{1,1}(Q)) : \text{where } (P,Q) \text{ runs on the counterexamples of J.C.}\big\}. \tag{2.0.1}$$

Then, by [1, Corollary 5.21] there exists a counterexample (P,Q) and $m,n \in \mathbb{N}$ coprime such that (P,Q) is a standard (m,n)-pair and a minimal pair (that is, the greatest common divisor of $v_{11}(P)$ and $v_{11}(Q)$ is B). Let A_0 be as in Remark 2.3.4. By [1, Proposition 5.2 and Corollary 5.21(3)]

$$A_0 = \frac{1}{m}\,\mathrm{en}_{10}(P) \quad \text{and} \quad \gcd(v_{11}(P), v_{11}(Q)) = v_{11}(A_0).$$

This point A_0 corresponds to (a,b) in the introduction. In Theorem 2.3.2 below, we obtain a chain

$$(\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1}) = \big((\mathcal{A}_0, \mathcal{A}'_0), \dots, (\mathcal{A}_j, \mathcal{A}'_j), \mathcal{A}_{j+1}\big),$$

such that A_0 is the geometric realization of $\mathcal{A}_0$ (see Definition 2.1.1), and that satisfies (among others) certain geometric conditions, which are codified in Definition 2.3.1. Then, we show that this chain also satisfies certain arithmetic conditions (see the comment below Definition 2.4.2). The chains meeting the requirements of

Definitions 2.3.1 and 2.4.2 are called admissible complete chains. In Algorithm 8 we construct all the admissible complete chains that satisfy $v_{11}(A_0) \le M$ for a given positive integer bound M.

By Theorem 2.3.2 and Remark 2.4.1 we know that $\mathcal{A}_0$ is the first coordinate of $\mathcal{C}_0$ for one of the admissible complete chains $(\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$ obtained running Algorithm 8 with $M \ge B$. For example we obtain immediately that the Jacobian Conjecture is false, then $B \ge 16$, since there are no admissible complete chains with $v_{11}(A_0) < 16$ (this result was already obtained in [1]). More importantly, we will see that many of the admissible complete chains obtained in Algorithm 6 can not come from a standard (m,n)-pair as in Theorem 2.3.2.

2.1 Valid edges

In this subsection and in the next one we introduce the basic ingredients for the definition and construction of the complete chains.

For each $l \in \mathbb{N}$ we let $\mathbb{N}_{(l)}$ denote the set $\{(a,l) : a \in \mathbb{N}\}$. In the sequel we will write $a \wr l$ instead of (a,l). Moreover we will use the notation $I :=](1,-1),(1,0)]$.

Definition 2.1.1. A *corner* is a pair $(a \wr l, b)$ with $a \wr l \in \mathbb{N}_{(l)}$ and $b \in \mathbb{N}_0$. For $l=1$ we will write (a,b) instead of $(a \wr 1, b)$. The *geometric realization* of a corner $\mathcal{A} = (a \wr l, b)$ is the point $A := \left(\frac{a}{l}, b\right) \in \frac{1}{l}\mathbb{N} \times \mathbb{N}_0$.

Let $l \in \mathbb{N}$. In the rest of this section given $\mathcal{A}, \mathcal{A}' \in \mathbb{N}_{(l)} \times \mathbb{N}_0$ with $\mathcal{A} \neq \mathcal{A}'$, we write

$$\mathcal{A} = (a \wr l, b), \quad \mathcal{A}' = (a' \wr l, b'), \quad (\rho,\sigma) := \operatorname{dir}(A - A') \quad \text{and} \quad \operatorname{gap}(\rho, l) := \frac{\rho}{\gcd(\rho, l)}.$$

Definition 2.1.2. Set $d := \gcd(a,b)$, $\overline{a} := \frac{a}{d}$ and $\overline{b} := \frac{b}{d}$, The pair $(\mathcal{A}, \mathcal{A}')$ is called a *valid edge* if

1. $(\rho,\sigma) \in I$,
2. $v_{1,-1}(A') \neq 0$, $v_{1,-1}(A) < 0$ and $v_{1,-1}(A) < v_{1,-1}(A')$,
3. there exist $\mathrm{enF} \in \mathbb{N}_{(l)} \times \mathbb{N}$ and $\mu \in \mathbb{N}$, with $\mu \le l(bl-a) + 1/\overline{b}$ and $d \nmid \mu$, such that

$$\mathrm{enF} = \frac{\mu}{d}\mathcal{A} := \mu(\overline{a} \wr l, \overline{b}), \quad v_{\rho,\sigma}(\mathrm{enF}) = \rho + \sigma \quad \text{and} \quad \text{if } l = 1, \text{ then } \mu < d.$$

4. If $l = 1$ and $v_{1,-1}(A') > 0$, then A' is a possible last lower corner.

The valid edge $(\mathcal{A}, \mathcal{A}')$ is called *simple* if $v_{01}(\text{enF}) - 1 = \text{gap}(\rho, l)$ and $(\text{gap}(\rho, l) > 1$ or $v_{01}(A') > 0)$.

Remark 2.1.3. By item (1) the last inequality in item (2) is equivalent to $v_{01}(A - A') > 0$. Moreover $d > 1$ since $d \nmid \mu$. We can also replace condition (3) by

(3') $\exists\ \mu \in \mathbb{N}$, such that $\frac{\mu}{d} = \frac{\rho+\sigma}{v_{\rho,\sigma}(A)}$, $\mu \leq l(bl - a) + 1/\bar{b}$, $d \nmid \mu$ and if $l = 1$, then $\mu < d$.

Moreover, such a μ univocally determines enF via the equality $\text{enF} = \frac{\mu}{d}\mathcal{A}$. Write $\text{enF} = (f_1 \wr l, f_2)$. Since $v_{\rho,\sigma}(\text{enF}) = \rho + \sigma$ and $f_2 \geq 1$,

$$(\rho, \sigma) = \frac{1}{\gcd(f_1 - l, f_2 l - l)}(f_2 l - l, l - f_1).$$

This equality implies $f_2 > 1$, because by condition (1) we have $\rho > 0$. Thus, by [2, Remark 3.9] we know that

$$\text{gap}(\rho, l) = \frac{f_2 - 1}{\gcd(f_1 - l, f_2 - 1)}.$$

Consequently $v_{01}(\text{enF}) - 1 = \text{gap}(\rho, l)$ if and only if $\gcd(f_1 - l, f_2 - 1) = 1$.

Notation 2.1.4. Fixed $l \in \mathbb{N}$ and given $A = \left(\frac{a}{l}, b\right) \in \frac{1}{l}\mathbb{N} \times \mathbb{N}_0$ we set $\mathcal{A} := (a \wr l, b) \in \mathbb{N}_{(l)} \times \mathbb{N}_0$.

In Algorithm 2 we obtain a list StartingEdges consisting of all valid edges $(\mathcal{A}, \mathcal{A}')$ starting with a given $A \in \mathbb{N} \times \mathbb{N}$ such that $v_{1,-1}(A) < 0$. We use freely the results of Remark 2.1.3. Before running this algorithm with input a corner $\mathcal{A} = (a, b)$ it is necessary to run Algorithm 1 with input greater than or equal to a, in order to obtain a list PLLC.

Algorithm 2: GetStartingEdges

Input: A corner $A = (a, b) \in \mathbb{N} \times \mathbb{N}$ with $a < b$, and a list PLLC.

Output: A list StartingEdges, consisting of all valid edges $(\mathcal{A}, \mathcal{A}')$.

$d \leftarrow \gcd(a, b)$
for $\mu = 1$ **to** $d - 1$ **do**
  enF $\leftarrow \frac{\mu}{d}(a, b)$
  $(\rho, \sigma) \leftarrow \operatorname{dir}(\text{enF} - (1, 1))$
  for $i = 1$ **to** $\left\lfloor \frac{b}{\rho} \right\rfloor$ **do**
    $A' \leftarrow (a, b) - i(-\sigma, \rho)$
    if $v_{1,-1}(A') < 0$ **or** $(\, v_{1,-1}(A') > 0$ **and** $A' \in \text{PLLC})$ **then**
      add $(\mathcal{A}, \mathcal{A}')$ **to** StartingEdges
RETURN StartingEdges

In the following proposition we show among other things how a regular corner of an (m, n)-pair (P, Q) gives rise to a valid edge.

Proposition 2.1.5. *Let $l \geq 1$ and let (P, Q) be an (m, n)-pair in $L^{(l)}$. Assume that if $l = 1$, then (P, Q) is a standard (m, n)-pair in L (see* [1, Definition 4.3]*). Let $(A, (\rho, \sigma))$ be a regular corner of (P, Q) (see* [1, Definition 5.5]*) and let $A' := \frac{1}{m} \operatorname{st}_{\rho,\sigma}(P)$. Write*

$$\ell_{\rho,\sigma}(P) = x^{m\frac{a'}{l}} y^{mb'} p(z) \quad \text{with } z := x^{-\frac{\sigma}{\rho}} y,\ p \in K[z] \text{ and } p(0) \neq 0.$$

The following facts hold:

1. *If $l = 1$, then the regular corner $(A, (\rho, \sigma))$ is of type II.*

2. *If $(A, (\rho, \sigma))$ is of type II (see the comments above* [1, Definition 5.9]*), then $(\mathcal{A}, \mathcal{A}')$ is a valid edge.*

3. *If $\lambda \in K^{\times}$ is a root of p, then*

$$\frac{m_\lambda}{m} \leq \frac{v_{01}(A - A')}{\operatorname{gap}(\rho, l)}, \quad \text{where } m_\lambda \text{ denotes the multiplicity of } \lambda.$$

If moreover $(\mathcal{A}, \mathcal{A}')$ is simple, then $\frac{m_\lambda}{m} = \frac{v_{01}(A - A')}{\operatorname{gap}(\rho, l)}$.

4. If $(A,(\rho,\sigma))$ is of type II.b), then there exists a root $\lambda \in K^\times$ of p such that

$$b' < \frac{\rho a+\sigma b l}{l(\rho+\sigma)} \le \frac{m_\lambda}{m}, \tag{2.1.2}$$

where m_λ denotes the multiplicity of λ in p.

Proof. 1) By [1, Remark 5.10 and Propositions 5.22 and 6.1].

2) First note that by [1, Remark 1.8] we have $A \in \frac{1}{l}\mathbb{N} \times \mathbb{N}_{(0)}$. We now check that the pair $(\mathcal{A},\mathcal{A}')$ satisfies conditions (1)–(4) of Definition 2.1.2. The fact that $(\rho,\sigma) \in I$ and the inequality $v_{1,-1}(A) < 0$ follow from [1, Definition 5.5]). Moreover, $v_{1,-1}(A') \neq 0$ by [1, Corollary 5.7(1) and Theorem 2.6(4)], while $v_{1,-1}(A) < v_{1,-1}(A')$ by Remark 2.1.3, because $v_{01}(A') < v_{01}(A)$. So conditions (1) and (2) are true. Let μ and F be as in [1, Proposition 5.14] and set $\mathrm{enF} := \mathrm{en}_{\rho,\sigma}(F)$. All the assertions in condition (3), with the exception of the last one, follow from the definition of μ and items (3) and (4) of that proposition. Assume now $l=1$ (which by hypothesis implies that $P,Q \in L$). By [12, Theorem 10.2.1 and Proposition 10.2.6] there exists $k \in \mathbb{N}$ such that $(km,0) \in \mathrm{Supp}(P)$. So

$$v_{\rho,\sigma}(A) = \frac{1}{m}v_{\rho,\sigma}(P) \ge \frac{1}{m}v_{\rho,\sigma}(km,0) = k\rho \ge \rho \ge \rho+\sigma = v_{\rho,\sigma}(\mathrm{enF}).$$

Since $\mu v_{\rho,\sigma}(A) = d v_{\rho,\sigma}(\mathrm{enF})$ and $d \nmid \mu$, this implies that $\mu < d$. We finally prove item (4). Since $(A,(\rho,\sigma))$ is of type II and $v_{1,-1}(A') > 0$, it is of type II.b). Consequently if $l=1$ it follows from [1, Remark 6.3] that $(A,A',(\rho,\sigma))$ is the starting triple of (P,Q) (see [1, Definition 6.2]), and so condition (4) is true by [2, Remark 3.23], because by hypothesis $P,Q \in L$.

3) Let F be as in [1, Theorem 2.6] and write

$$F = x^{\frac{u}{l}} y^v f(z) \quad \text{with } z := x^{-\frac{\sigma}{\rho}}y,\ f \in K[z] \text{ and } f(0) \neq 0.$$

By [2, Remark 3.9] there exist $\overline{p},\overline{f} \in K[z]$ such that

$$p(z) = \overline{p}(z^k) \quad \text{and} \quad f(z) = \overline{f}(z^k), \qquad \text{where } k := \mathrm{gap}(\rho,l).$$

So,

$$t := \deg \overline{p} = \frac{\deg p}{k} = \frac{v_{01}(\mathrm{en}_{\rho,\sigma}(P) - \mathrm{st}_{\rho,\sigma}(P))}{k} = m\frac{v_{01}(A-A')}{k}.$$

By [2, Remark 3.8] we have $m_\lambda \le \deg \overline{p}$, which yields $\frac{m_\lambda}{m} \le \frac{v_{01}(A-A')}{\mathrm{gap}(\rho,l)}$. Assume now that $(\mathcal{A}, \mathcal{A}')$ is simple. Since $k = v_{01}(\mathrm{en}_{\rho,\sigma}(F)) - 1$, we have

$$k + 1 = v_{01}(\mathrm{en}_{\rho,\sigma}(F)) = v_{01}(F) = v + \deg(f) = v + k\deg(\overline{f}),$$

which implies $\deg(\overline{f}) = v = 1$ or $k = 1$, $v = 0$ and $\deg(\overline{f}) = 2$. But if $v = 0$, then by [1, Theorem 2.6(2)]

$$\Big(\frac{u}{l}, 0\Big) = \mathrm{st}_{\rho,\sigma}(F) \sim A',$$

which is impossible since $v_{01}(A') > 0$, since $k = 0$ and $(\mathcal{A}, \mathcal{A}')$ is simple. Hence, $\deg(\overline{f}) = 1$ and so, by [1, Proposition 2.11(3)] we have $\overline{p}(z^k) = (z^k - c)^t$ for some constant $c \in K^{\times}$. Consequently, by [2, Remark 3.8], every linear factor of p has multiplicity t. Thus $m_\lambda = t = m\frac{v_{01}(A-A')}{\mathrm{gap}(\rho,l)}$, as desired.

4) By [1, Proposition 5.16] there exists $\lambda \in K^{\times}$ such that the second inequality in (2.1.2) is true. Since $\rho > 0$ and $\frac{a'}{l} - b' > 0$, we have

$$\Big(\rho\frac{a'}{l} + \sigma b'\Big) - (\rho + \sigma)b' = \rho\Big(\frac{a'}{l} - b'\Big) > 0.$$

Since $\rho + \sigma > 0$ and $v_{\rho,\sigma}(A) = v_{\rho,\sigma}(A')$, this implies the first inequality in (2.1.2). $\square$

Remark 2.1.6. Let $l \geq 1$ and let (P, Q) be an (m, n)-pair in $L^{(l)}$. Let $(A, (\rho, \sigma))$ be a regular corner of (P, Q) and let $A' := \frac{1}{m}\mathrm{st}_{\rho,\sigma}(P)$. Write

$$\ell_{\rho,\sigma}(P) = x^{m\frac{a'}{l}} y^{mb'} p(z) \quad \text{with } z := x^{-\frac{\sigma}{\rho}} y,\ p \in K[z] \text{ and } p(0) \neq 0,$$

If $(A, (\rho, \sigma))$ is of type I, then all the roots of p are simple. In fact if $p(z) = (z-\lambda)^2\tilde{p}(z)$, then

$$\begin{aligned}[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] &= [x^{m\frac{a'}{l}} y^{mb'}(z-\lambda)^2\tilde{p}(z), \ell_{\rho,\sigma}(Q)] \\ &= 2(z-\lambda)x^{m\frac{a'}{l}} y^{mb'}\tilde{p}(z)[(z-\lambda), \ell_{\rho,\sigma}(Q)] + (z-\lambda)^2[x^{m\frac{a'}{l}} y^{mb'}\tilde{p}(z), \ell_{\rho,\sigma}(Q)],\end{aligned}$$

which contradicts the fact that $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \in K^{\times}$.

Remark 2.1.7. Let $l \geq 1$ and let (P, Q) be an (m, n)-pair in $L^{(l)}$. Let $(A, (\rho, \sigma))$ be a regular corner of (P, Q) and let $A' := \frac{1}{m}\mathrm{st}_{\rho,\sigma}(P)$. Write

$$\ell_{\rho,\sigma}(P) = x^{\frac{k}{l}}\mathfrak{p}(z) \quad \text{where } z := x^{-\frac{\sigma}{\rho}} y \text{ and } \mathfrak{p}(z) \in K[z].$$

Let $\lambda \in K^{\times}$ be a root of $\mathfrak{p}$ of multiplicity m_λ and let $\gamma := \frac{m_\lambda}{m}$ (note that $\deg(\mathfrak{p}) = mb$

and that since $\mathfrak{p} = (x^{-\sigma/\rho}y)^{b'}p$, the multiplicity of λ as a root of p is also m_λ). By Proposition 2.1.5(3)

$$\gamma \le \frac{b-b'}{\operatorname{gap}(\rho,l)} \le b.$$

Hence, if $b=\gamma$, then $b'=0$, $\operatorname{gap}(\rho,l)=1$ and $\mathfrak{p}(z)=\mu(z-\lambda)^{mb}$, and consequently $(A,(\rho,\sigma))$ is not of type II. Since $mb>1$ it follows from Remark 2.1.6 that it is not of type I either, and so it is necessarily of type III. In line 7 of Algorithm 3 we set gmax $:= \min\left\{\frac{b-b'}{\operatorname{gap}(\rho,l)}, b-1\right\}$ in order to avoid the regular corners of type III. We can ignore these corners, since they do not appear in a complete chain of an (m,n)-pair (see Proposition 2.3.2). Note that from $b'=0$ and $\operatorname{gap}(\rho,l)=1$ it follows that $(\mathcal{A},\mathcal{A}')$ is not simple.

2.2 The children of a valid edge

Let (P,Q) be an (m,n)-pair in $L^{(l)}$, let $(A,(\rho,\sigma))$ be a regular corner of type II of (P,Q) and let $A' := \frac{1}{m}\operatorname{st}_{\rho,\sigma}(P)$. If $(A,(\rho,\sigma))$ is of type II.b), then applying [1, Propositions 5.16 and 5.18(4)], we obtain a regular corner $(A_1,(\rho',\sigma'))$ of an (m,n)-pair (P_1,Q_1). In the sequel we will call $\mathcal{A}_1$ the corner generated by $(\mathcal{A},\mathcal{A}')$. If moreover $(A_1,(\rho',\sigma'))$ is of type II, then we say that $(\mathcal{A}_1,\mathcal{A}_1')$, where $A_1' := \frac{1}{m}\operatorname{st}_{\rho',\sigma'}(P_1)$, is a child of $(\mathcal{A},\mathcal{A}')$. On the other hand, if $(A,(\rho,\sigma))$ is of type II.a), then we set $A_1 := A'$ and $A_1' := \frac{1}{m}\operatorname{st}_{\rho_1,\sigma_1}(P)$, where $(\rho_1,\sigma_1) := \operatorname{Pred}_P(\rho,\sigma)$ (which is well defined by [1, Proposition 4.6(5)]). As before, in this case we also call $\mathcal{A}_1$ the corner generated by $(\mathcal{A},\mathcal{A}')$ and we say that $(\mathcal{A}_1,\mathcal{A}_1')$ is a child of (A,A').

For a general valid edge $(\mathcal{A},\mathcal{A}')$ we will construct all its possible children $(\mathcal{A}_1,\mathcal{A}_1')$ (see Definition 2.2.8) in two steps:

- **GenerateCorners** $(\mathcal{A},\mathcal{A}')$**:** We find the corners $\mathcal{A}_1$ generated by a valid edge $(\mathcal{A},\mathcal{A}')$ (see Definition 2.2.5).

- **GetCornerChildren** $((\mathcal{A},\mathcal{A}'),\mathcal{A}_1)$**:** Given a corner $\mathcal{A}_1$ generated by a valid edge $(\mathcal{A},\mathcal{A}')$, we determine all possible $\mathcal{A}_1'$, such that $(\mathcal{A}_1,\mathcal{A}_1')$ is a child of $(\mathcal{A},\mathcal{A}')$.

In the rest of this subsection $(\mathcal{A},\mathcal{A}')$ denotes a valid edge.

Definition 2.2.1. We set $\gamma_{\max} \coloneqq \min\bigl(\frac{b-b'}{\operatorname{gap}(\rho,l)}, b-1\bigr)$ and we define the *set of multiplicities*

$$\Gamma = \Gamma(\mathcal{A},\mathcal{A}') \coloneqq \begin{cases} \{\gamma_{\max}\} & \text{if } (\mathcal{A},\mathcal{A}') \text{ is simple} \\ \{b',\dots,\gamma_{\max}\} & \text{if } (\mathcal{A},\mathcal{A}') \text{ is not simple.} \end{cases}$$

Remark 2.2.2. Note that from the equality

$$\gamma_{\max} = \min\bigl(\gcd(a-a',b-b'), b-1\bigr)$$

(see [2, equality (3.9)]) it follows that $\gamma_{\max} \in \mathbb{N}$. Moreover if $\gamma_{\max} < \frac{b-b'}{\operatorname{gap}(\rho,l)}$, then $\operatorname{gap}(\rho,l) = 1$ and $b' = 0$, which, as we saw in Remark 2.1.7, excludes the case $(\mathcal{A},\mathcal{A}')$ simple.

Remark 2.2.3. The previous definition is motivated by the properties established in Proposition 2.1.5(3) for the case of (m,n)-pairs.

For each γ such that $b' \le \gamma \le \gamma_{\max}$, we let $\mathcal{A}_{(\gamma)}$ denote $\bigl(a_1 \wr l_1, b_1\bigr)$, where

$$l_1 \coloneqq \operatorname{lcm}(l,\rho), \quad b_1 \coloneqq \gamma \quad \text{and} \quad a_1 \coloneqq \frac{al_1}{l} + (\gamma - b)\frac{-\sigma l_1}{\rho}.$$

Note that $v_{\rho,\sigma}(A_{(\gamma)}) = v_{\rho,\sigma}(A)$. So $A_{(\gamma)}$ is in the line determined by A and A'.

Definition 2.2.4. We say that $\mathcal{A}_{(\gamma)}$ is *admissible* if

1. $v_{1,-1}(A_{(\gamma)}) < 0$,
2. $l_1 - \frac{a_1}{b_1} > 1$ or $\gcd(a_1,b_1) > 1$.

Definition 2.2.5. Let $\mathcal{A},\mathcal{A}' \in \mathbb{N}_{(l)} \times \mathbb{N}_0$ be such that $(\mathcal{A},\mathcal{A}')$ is a valid edge. We say that an element $\mathcal{A}_1 \in \mathbb{N}_{(l_1)} \times \mathbb{N}$ is a *corner generated* by $(\mathcal{A},\mathcal{A}')$, if either $\mathcal{A}_1 = \mathcal{A}'$ and $v_{1,-1}(A') < 0$, or $v_{1,-1}(A') > 0$ and there exists $\gamma \in \Gamma(\mathcal{A},\mathcal{A}')$ such that $\mathcal{A}_{(\gamma)}$ is admissible and $\mathcal{A}_1 = \mathcal{A}_{(\gamma)}$ (which implies $\mathcal{A}_1 \neq \mathcal{A}'$).

Proposition 2.2.6. *Assume that* $(\mathcal{A},\mathcal{A}')$ *is simple. Let*

$$l_1 \coloneqq \operatorname{lcm}(l,\rho), \quad a_1 \coloneqq \frac{al_1}{l} + (\gamma_{\max} - b)\frac{-\sigma l_1}{\rho} \quad \textit{and} \quad b_1 \coloneqq \gamma_{\max}.$$

If $v_{1,-1}(A') < 0$*, then* $v_{1,-1}(A_1) > 0$*, where* $A_1 \coloneqq \bigl(\frac{a_1}{l_1}, b_1\bigr)$.

Proof. By Definition 2.1.2 and Remark 2.2.2 we know that

$$f_2 = \operatorname{gap}(\rho, l) + 1 \qquad \text{and} \qquad \operatorname{gmax} = \frac{b - b'}{\operatorname{gap}(\rho, l)}. \tag{2.2.3}$$

Let μ and d be as in Definition 2.1.2. By Definition 2.1.2 and item (3') of Remark 2.1.3 we have

$$f_2 = \frac{\mu}{d} b \qquad \text{and} \qquad \frac{\mu}{d} = \frac{(\rho + \sigma)l}{\rho a + \sigma bl}. \tag{2.2.4}$$

Moreover combining $v_{\rho,\sigma}(A) = v_{\rho,\sigma}(A')$ with the fact that $v_{1,-1}(A') > 0$, we obtain

$$b' < \frac{a'}{l} = -b'\frac{\sigma}{\rho} + \frac{a}{l} + b\frac{\sigma}{\rho}.$$

Hence

$$b'\left(\frac{\rho + \sigma}{\rho}\right) < \frac{\rho a + \sigma lb}{l\rho},$$

which, by the second equality in (2.2.4), implies

$$b' < \frac{\rho a + \sigma lb}{l(\rho + \sigma)} = \frac{d}{\mu}.$$

But then, by the first equalities in (2.2.3) and (2.2.4),

$$b = \frac{d}{\mu} f_2 = \frac{d}{\mu}(\operatorname{gap}(\rho, l) + 1) > \frac{d}{\mu} \operatorname{gap}(\rho, l) + b',$$

and so, by the second equality in (2.2.3),

$$\operatorname{gmax} = \frac{b - b'}{\operatorname{gap}(\rho, l)} > \frac{d}{\mu}.$$

Consequently,

$$v_{1,-1}(A_1) = \frac{a\rho + b\sigma l}{\rho l} - \operatorname{gmax} \frac{\rho + \sigma}{\rho} < \frac{a\rho + b\sigma l}{\rho l} - \frac{d}{\mu}\frac{\rho + \sigma}{\rho} = 0,$$

where the last equality follows from the second equality in (2.2.4). □

In Algorithm 3 we obtain a list GeneratedCorners consisting of all the corners generated by a valid edge $(\mathcal{A}, \mathcal{A}')$.

Algorithm 3: GetGeneratedCorners

Input: A valid edge $(\mathcal{A}, \mathcal{A}') = ((a \wr l, b), (a' \wr l, b'))$.

Output: A list GeneratedCorners, consisting of all generated corners by $(\mathcal{A}, \mathcal{A}')$.

$(\rho, \sigma) \leftarrow \operatorname{dir}(A - A')$
if $v_{1,-1}(A') < 0$ **then**
    add $\mathcal{A}'$ to GeneratedCorners
else
    $l_1 \leftarrow \operatorname{lcm}(\rho, l)$
    gap $\leftarrow \frac{\rho}{\gcd(\rho, l)}$
    gmax $\leftarrow \min\left\{\frac{b-b'}{\text{gap}}, b-1\right\}$
    if Simple$(\mathcal{A}, \mathcal{A}')$ = TRUE **then**
        $a_1 \leftarrow \frac{al_1}{l} + (\text{gmax} - b)\frac{-\sigma l_1}{\rho}$
        $\mathcal{A}_1 \leftarrow (a_1 \wr l_1, \text{gmax})$
        if $l_1 - a_1/b_1 > 1$ **or** $\gcd(a_1, b_1) > 1$ **then**
            add $\mathcal{A}_1$ to GeneratedCorners
    else
        for $b_1 \leftarrow b' + 1$ **to** gmax **do**
            $a_1 \leftarrow \frac{al_1}{l} + (b_1 - b)\frac{-\sigma l_1}{\rho}$
            $\mathcal{A}_1 \leftarrow (a_1 \wr l_1, b_1)$
            if $v_{1,-1}(A_1) < 0$ **and** $(l_1 - a_1/b_1 > 1$ **or** $\gcd(a_1, b_1) > 1)$ **then**
                add $\mathcal{A}_1$ to GeneratedCorners
RETURN GeneratedCorners

Remark 2.2.7. Definitions 2.2.4 and 2.2.5 are motivated by the following fact: Let (P, Q) be an (m, n)-pair in $L^{(l)}$ and let $(A, (\rho, \sigma))$ be a regular corner of type II.b) of (P, Q). Let φ be the automorphism of $L^{(l_1)}$ introduced in [1, Proposition 5.18], where $l_1 := \gcd(l, \rho)$. Let $\lambda \in K^{\times}$ be as in Proposition 2.1.5(4) and set

$$A' := \frac{1}{m}\operatorname{st}_{\rho,\sigma}(P), \quad A_1 := \frac{1}{m}\operatorname{st}_{\rho,\sigma}(\varphi(P)), \quad (\rho_1, \sigma_1) := \operatorname{Pred}_{\varphi(P)}(\rho, \sigma) \quad \text{and} \quad \gamma := \frac{m_\lambda}{m}.$$

Then,

1. by Proposition 2.1.5(2) the pair $(\mathcal{A}, \mathcal{A}')$ is a valid edge,

2. since $(A, (\rho, \sigma))$ is of type II.b), we have $v_{1,-1}(A') > 0$,

3. by [1, Proposition 5.18(4)] the corner $\mathcal{A}_1$ satisfies condition (1) of Definition 2.2.4,

4. by items (3) and (4) of Proposition 2.1.5, and Remark 2.1.7, we have $b' < \gamma \leq \gamma_{\max}$.

5. by [1, Proposition 5.18(3)] we have $\mathcal{A}_{(\gamma)} = \mathcal{A}_1$,

6. by [1, Proposition 5.19] the corner $\mathcal{A}_1$ satisfies condition (2) of Definition 2.2.4.

Thus $\mathcal{A}_1 \in \mathbb{N}_{(l_1)} \times \mathbb{N}$ is a corner generated by $(\mathcal{A}, \mathcal{A}')$, $\mathcal{A}_1 \neq \mathcal{A}'$ and there exists $b' < \gamma \leq \gamma_{\max}$ such that $\mathcal{A}_1 = \mathcal{A}_{(\gamma)}$, which implies that $v_{01}(A') < v_{01}(A_1) < v_{01}(A)$.

Definition 2.2.8. Let $(\mathcal{A}, \mathcal{A}')$ and $(\mathcal{A}_1, \mathcal{A}_1')$ be valid edges and let $(\rho, \sigma) := \operatorname{dir}(A - A')$ and $(\rho_1, \sigma_1) := \operatorname{dir}(A_1 - A_1')$. We say that $(\mathcal{A}_1, \mathcal{A}_1')$ is a *child* of $(\mathcal{A}, \mathcal{A}')$ if $(\rho, \sigma) > (\rho_1, \sigma_1)$ in I and $\mathcal{A}_1$ is a corner generated by $(\mathcal{A}, \mathcal{A}')$.

The previous definition describes the main inductive construction that yields complete chains, generalizing the case when the valid edges correspond to an (m, n)-pair. This construction consists of the two steps mentioned above that are realized through Algorithms 3 and 4.

Remark 2.2.9. Let $(\mathcal{A}, \mathcal{A}')$ be a valid edge, let $(\rho, \sigma) := \operatorname{dir}(A - A')$ and let $\mathcal{A}_1 = (a_1 \wr l_1, b_1)$ be a corned generated by $(\mathcal{A}, \mathcal{A}')$. By Definition 2.2.5 we know that $v_{1,-1}(A_1) < 0$. In Algorithm 4 we obtain all the children of $(\mathcal{A}, \mathcal{A}')$ of the form $(\mathcal{A}_1, \mathcal{A}_1')$. The lower bound lo in the algorithm comes from the fact that $(\rho_1, \sigma_1) < (\rho, \sigma)$ if and only if $\mu > \frac{d_1(\rho+\sigma)}{v_{\rho,\sigma}(A_1)}$, where $d_1 := \gcd(a_1, b_1)$. The upper bound hi in lines 4 and 6 and the conditions required in line 11 come from Definition 2.1.2. By [2, Remark 3.9] we know that

$$A_1' = \left(\frac{a_1}{l_1}, b_1\right) + j\left(\operatorname{gap}(\rho_1, l_1)\frac{\sigma_1}{\rho_1}, -\operatorname{gap}(\rho_1, l_1)\right) \qquad \text{for some } 0 < j \leq \left\lfloor \frac{b_1}{\operatorname{gap}(\rho_1, l_1)} \right\rfloor.$$

Remark 2.2.10. Before running Algorithm 4 with input a corner $\mathcal{A}_1 = (a_1 \wr l_1, b_1)$ such that $l_1 - \frac{a_1}{b_1} \leq 1$, and a valid edge$(\mathcal{A}, \mathcal{A}')$, it is necessary to run Algorithm 1 with input greater than or equal to a_1.

Algorithm 4: GetCornerChildrenList

Input: A valid edge $(\mathcal{A}, \mathcal{A}')$ and a corner $\mathcal{A}_1 = (a_1 \wr l_1, b_1)$ generated by $(\mathcal{A}, \mathcal{A}')$ with $l_1 - \frac{a_1}{b_1} \leq 1$.

Output: A list CornerChildrenList, consisting of all $(\mathcal{A}_1, \mathcal{A}_1')$ that are children of $(\mathcal{A}, \mathcal{A}')$.

$(\rho, \sigma) \leftarrow \mathrm{dir}(A - A')$
$d_1 \leftarrow \gcd(a_1, b_1)$
$\mathrm{lo} \leftarrow \left\lfloor 1 + \frac{d_1(\rho+\sigma)}{v_{\rho,\sigma}(A_1)} \right\rfloor$
$\mathrm{hi} \leftarrow d_1$
if $l_1 > 1$ **then**
  $\mathrm{hi} \leftarrow \left\lfloor l_1(b_1 l_1 - a_1) + \frac{d_1}{b_1} \right\rfloor$
for $\mu \leftarrow \mathrm{lo}$ **to** hi **do**
  $\mathrm{enF} \leftarrow \frac{\mu}{d_1}\left(\frac{a_1}{l_1}, b_1\right)$
  $(\rho_1, \sigma_1) \leftarrow \mathrm{dir}(\mathrm{enF} - (1,1))$
  $\mathrm{gap} \leftarrow \frac{\rho_1}{\gcd(\rho_1, l_1)}$
  if $\mathrm{gap} \leq b_1$ **and** $d_1 \nmid \mu$ **then**
    for $j \leftarrow 1$ **to** $\left\lfloor \frac{b_1}{\mathrm{gap}} \right\rfloor$ **do**
      $A_1' \leftarrow \left(\frac{a_1}{l_1}, b_1\right) + j\left(\mathrm{gap}\,\frac{\sigma_1}{\rho_1}, -\mathrm{gap}\right)$
      if $(\ l_1 > 1$ **and** $v_{1,-1}(A_1') \neq 0\ )$ **or** $(l_1 = 1$ **and** $v_{1,-1}(A_1') < 0)$ **or**
        $(l_1 = 1,\ v_{1,-1}(A_1') > 0$ **and** $A_1' \in \mathrm{PLLC})$ **then**
          add $(\mathcal{A}_1, \mathcal{A}_1')$ to CornerChildrenList
RETURN CornerChildrenList

Definition 2.2.11. A corner $\mathcal{A} = (a \wr l, b)$ is called a *final corner* if $l - \frac{a}{b} > 1$.

In Algorithm 5 we combine Algorithms 3 and 4 in order to obtain a procedure giving the children of a valid edge $(\mathcal{A}, \mathcal{A}')$ and the final corners generated by $(\mathcal{A}, \mathcal{A}')$.

In line 1 of Algorithm 5 we use the expression "GetGeneratedCorners$(\mathcal{A}, \mathcal{A}')$" as a notation for "run GetGeneratedCorners with input $(\mathcal{A}, \mathcal{A}')$". We use similar notations in the following algorithms.

Algorithm 5: GetChildrenAndFinalList

```
Input: A valid edge (𝒜, 𝒜′).
Output: A list ChildrenList, consisting of all children of (𝒜, 𝒜′).
        A list FinalList, consisting of all final corners generated by (𝒜, 𝒜′).
GeneratedCorners ← GetGeneratedCorners(𝒜, 𝒜′)
for 𝒜₁ = (a₁ ≀ l₁, b₁) ∈ GeneratedCorners do
    if l₁ − a₁/b₁ > 1 then
        add 𝒜₁ to FinalList
    CornerChildrenList ← GetCornerChildrenList((𝒜, 𝒜′), 𝒜₁)
    for (𝒜₁, 𝒜′₁) ∈ CornerChildrenList do
        add (𝒜₁, 𝒜′₁) to ChildrenList
RETURN (ChildrenList, FinalList)
```

2.3 Main inductive step and complete chains

Now we are able to construct recursively a chain $(\mathcal{C}_0, \dots, \mathcal{C}_j)$ of valid edges $\mathcal{C}_i := (\mathcal{A}_i, \mathcal{A}'_i)$, where each $\mathcal{C}_i$ a child of the previous (except the first one). In the case of an standard (m, n)-pair (P, Q), this process terminates when the generated corner

$$\mathcal{A}_{j+1} = (a_{j+1} \wr l_{j+1}, b_{j+1})$$

is a regular corner of type I. In this case

$$l_{j+1} - \frac{a_{j+1}}{b_{j+1} > 1}.$$

Definition 2.3.1. A chain $(\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$ is called a *complete chain of length* $j+1$, if

- $\mathcal{C}_i$ is a valid edge for $i = 0, \dots, j$,

- $\mathcal{C}_{i+1}$ is a child of $\mathcal{C}_i$ for $i = 0, \dots, j-1$,

- $\mathcal{A}_{j+1}$ is generated by C_j,

- $\mathcal{A}_{j+1}$ is a final corner,

- $l_0 = 1$,

where $\mathcal{C}_i = (\mathcal{A}_i, \mathcal{A}'_i)$ and $\mathcal{A}_i = (a_i \wr l_i, b_i)$.

In Algorithm 6 we give a method for the generation of a list CompleteChains consisting of all complete chains starting with a valid edge

$$\mathcal{C}_0 = (\mathcal{A}, \mathcal{A}') = ((a, b), (a', b'))$$

and having length less than or equal to $\text{NumberOfFactors}\big(\gcd(b, (b - b')/\rho)\big) + 1$, where (ρ, σ) denotes $\text{dir}(A - A')$ and $\text{NumberOfFactors}\big(n\big)$ is an auxiliary function which returns the number of prime factors of n, counted with its multiplicity.

We use auxiliary lists OpenChains and POpenChains and an auxiliary variable Lmax. Moreover the expression $\mathscr{C} \uplus \mathcal{A}_1$ denotes the chain obtained adding $\mathcal{A}_1$ at the end of the chain $\mathscr{C}$ and similarly for $\mathscr{C} \uplus (\mathcal{A}_1, \mathcal{A}'_1)$.

Algorithm 6: GetCompleteChains

Input: A valid edge $\mathcal{C}_0 = (\mathcal{A}, \mathcal{A}') = ((a, b), (a', b'))$.

Output: A list CompleteChains, consisting of all complete chains $\mathcal{CH}$ starting in $\mathcal{C}_0$, with $\text{length}(\mathcal{CH}) \leq \text{NumberOfFactors}\Big(\gcd\Big(b, \frac{b-b'}{\rho}\Big)\Big) + 1$, where $(\rho, \sigma) := \text{dir}(A - A')$.

$(\rho, \sigma) \leftarrow \text{dir}(A - A')$
Lmax $\leftarrow$ NumberOfFactors $\Big(\gcd\Big(b, \frac{b-b'}{\rho}\Big)\Big) + 1$
OpenChains $\leftarrow (\mathcal{C}_0)$
$j \leftarrow 0$
while $j <$ Lmax **do**
    POpenChains $\leftarrow \emptyset$
    for $\mathcal{CH} \in$ OpenChains **do**
        Last $\leftarrow$ Last element in $\mathcal{CH}$
        (ChildrenList, FinalList) $\leftarrow$ GetChildrenAndFinalList(Last)
        for $\mathcal{A}_1 \in$ FinalList **do**
            add $\mathcal{CH} \uplus \mathcal{A}_1$ **to** CompleteChains
        for $(\mathcal{A}_1, \mathcal{A}'_1) \in$ ChildrenList **do**
            add $\mathcal{CH} \uplus (\mathcal{A}_1, \mathcal{A}'_1)$ **to** POpenChains
    OpenChains $\leftarrow$ POpenChains
    $j \leftarrow j + 1$
RETURN CompleteChains

Theorem 2.3.2. *For each standard (m, n)-pair (P, Q), there exist*

$$\big((P_i, Q_i), (A_i, A'_i), (\rho_i, \sigma_i), l_i\big)_{0 \leq i \leq j} \quad \text{and} \quad \big((P_{j+1}, Q_{j+1}), A_{j+1}, (\rho_{j+1}, \sigma_{j+1}), l_{j+1}\big)\big),$$

where $j \in \mathbb{N}$, such that:

1. $l_0 \leq \cdots \leq l_{j+1} \in \mathbb{N}$ *with* $l_0 = 1$,

2. $(\rho_0, \sigma_0) > \ldots > (\rho_{j+1}, \sigma_{j+1})$ *in* I,

3. (P_i, Q_i) *is an* (m, n)*-pair in* $L^{(l_i)}$ *for each* $1 \leq i \leq j + 1$ *and* $(P_0, Q_0) = (P, Q)$,

4. $\ell_{\rho_h, \sigma_h}(P_i) = \ell_{\rho_h, \sigma_h}(P_{i+1})$ *for* $0 \leq h < i \leq j$,

5. $(A_h,(\rho_h,\sigma_h))$ *is a regular corner of type II.a) of* (P_i,Q_i) *for* $0\le h<i\le j+1$. *Moreover*

$$\frac{1}{m}\,\mathrm{st}_{\rho_h,\sigma_h}(P_i)=A_{h+1}.$$

6. $A_0=\frac{1}{m}\,\mathrm{en}_{10}(P)$ *and* $(A_i,(\rho_i,\sigma_i))$ *is a regular corner of type II of* (P_i,Q_i) *for* $0\le i\le j$,

7. *if* $(A_i,(\rho_i,\sigma_i))$ *is a regular corner of type II.a) of* (P_i,Q_i), *then*

$$l_{i+1}=l_i,\quad (P_{i+1},Q_{i+1})=(P_i,Q_i)\quad \textit{and}\quad A_{i+1}=A_i'=\frac{1}{m}\,\mathrm{st}_{\rho_i,\sigma_i}(P_i),$$

8. *if* $(A_i,(\rho_i,\sigma_i))$ *is a regular corner of type II.b) of* (P_i,Q_i), *then* $l_{i+1}=\mathrm{lcm}(\rho_i,l_i)$ *and there exists a root* $\lambda\in K^{\times}$ *of the polynomial* $\mathfrak{p}_i(z)$, *defined by*

$$\ell_{\rho_i,\sigma_i}(P_i)=x^{\frac{k_i}{l_i}}\,\mathfrak{p}_i(z),\quad \textit{where } z:=x^{-\sigma_i/\rho_i}y,$$

such that $m\mid m_\lambda$, *where* m_λ *is the multiplicity of* $z-\lambda$ *in* $\mathfrak{p}_i(z)$ *and*

$$\frac{1}{m}\,\mathrm{st}_{\rho_i,\sigma_i}(P_{i+1})=A_{i+1}=\Big(\frac{k_i}{ml_i},0\Big)+\frac{m_\lambda}{m}\Big(-\frac{\sigma_i}{\rho_i},1\Big)\neq A_i'=\frac{1}{m}\,\mathrm{st}_{\rho_i,\sigma_i}(P_i). \tag{2.3.5}$$

Moreover $\ell_{\rho_i,\sigma_i}(P_{i+1})=\varphi(\ell_{\rho_i,\sigma_i}(P_i))$, *where* $\varphi\in\mathrm{Aut}(L^{(l_{i+1})})$ *is defined by*

$$\varphi(x^{\frac{1}{l_{i+1}}}):=x^{\frac{1}{l_{i+1}}}\qquad \textit{and}\qquad \varphi(y):=y+\lambda x^{\frac{\sigma_i}{\rho_i}},$$

9. $(A_{j+1},(\rho_{j+1},\sigma_{j+1}))$ *is a regular corner of type I of* (P_{j+1},Q_{j+1}) *in* $L^{(l_{j+1})}$,

10. $(\mathcal{A}_{i+1},\mathcal{A}_{i+1}')$ *is a child of* $(\mathcal{A}_i,\mathcal{A}_i')$ *for* $0\le i<j$,

11. $v_{01}(A_{i+1})<v_{01}(A_i)$ *for* $0\le i\le j$,

12. *the chain*

$$\big((\mathcal{A}_0,\mathcal{A}_0'),\dots,(\mathcal{A}_j,\mathcal{A}_j'),\mathcal{A}_{j+1}\big), \tag{2.3.6}$$

is complete,

13. *if* t *is the greatest index such that* $l_t=1$, *then*

   - $\big\{(A_i,(\rho_i,\sigma_i)):0\le i\le t\big\}$ *is the set of regular corners of* (P,Q),

- *$(A_i,(\rho_i,\sigma_i))$ is a regular corner of type IIa) of (P,Q) for $0\le i<t$ and $(A_t,(\rho_t,\sigma_t))$ is a regular corner of type IIb) of (P,Q),*
- *A'_t is the last lower corner of (P,Q) (see [2, Definition 3.21]),*
- *$(P_i,Q_i)=(P,Q)$ for all $i\le t$,*

14. The set of regular corners of (P_{j+1},Q_{j+1}) is $\{(A_i,(\rho_i,\sigma_i)) : 0\le i\le j+1\}$.

Proof. Take the set

$$\{(A_0,(\rho_0,\sigma_0)),\dots,(A_t,(\rho_t,\sigma_t))\},$$

of regular corners of (P,Q), with $(\rho_i,\sigma_i)>(\rho_{i+1},\sigma_{i+1})$ for all i (note that we are using the opposed enumeration of [1, Theorem 7.6]). By [1, Remark 5.12] we know that $A_0=\frac{1}{m}\operatorname{en}_{10}(P)$. Setting $A'_i:=\frac{1}{m}\operatorname{st}_{\rho_i,\sigma_i}(P)$, we obtain a chain

$$((A_0,A'_0),\dots,(A_t,A'_t)),$$

where $A_i,A'_i\in\mathbb{N}\times\mathbb{N}_0$ by [1, Remark 5.8]. By [1, Theorem 7.6(1)],

$$\{(\rho_0,\sigma_0),\dots,(\rho_{t-1},\sigma_{t-1})\}=A(P)$$

and the 3-uple $(A_t,A'_t,(\rho_t,\sigma_t))$ is the starting triple of (P,Q). Hence, by [1, Remark 5.10] we know that $(A_i,(\rho_i,\sigma_i))$ is a regular corner of type II.a) of (P,Q) for $0\le i<t$. Therefore $v_{1,-1}(A'_i)<0$ for $0\le i<t$. Furthermore, by items (1) and (2) of Proposition 2.1.5 each one of the pairs $(\mathcal{A}_i,\mathcal{A}'_i)$, with $0\le i\le t$, is a valid edge. Moreover,

$$A_{i+1}=A'_i \quad \text{and} \quad v_{01}(A_{i+1})<v_{01}(A_i) \qquad \text{for } 0\le i<t.$$

Consequently $\mathcal{A}_{i+1}$ is a corner generated by $(\mathcal{A}_i,\mathcal{A}'_i)$ for $0\le i<t$. Therefore $(\mathcal{A}_{i+1},\mathcal{A}'_{i+1})$ is a child of $(\mathcal{A}_i,\mathcal{A}'_i)$ for $0\le i<t$. Moreover, A'_t is the last lower corner of (P,Q). For $i\le t$, set $l_i:=1$ and $(P_i,Q_i):=(P,Q)$. By [1, Remark 6.3] we know that $(A_t,(\rho_t,\sigma_t))$ is a regular corner of type II.b), and so $v_{1,-1}(\operatorname{st}_{\rho_t,\sigma_t}(P))>0$. This implies that $(\rho_t,\sigma_t)\ne(1,0)$, because (P,Q) is standard (see [1, Definition 4.3]). Since $(\rho_t,\sigma_t)\in I$ we obtain that $\rho_t>0$. Let $\lambda\in K^\times$ be as in Proposition 2.1.5(4) and let $l_{t+1}:=\rho_t$. Applying [1, Proposition 5.18 and Remark 3.9] to (P_t,Q_t) and $(A_t,(\rho_t,\sigma_t))$, we obtain an (m,n)-pair (P_{t+1},Q_{t+1}) in $L^{(l_{t+1})}$, such that

- $\operatorname{en}_{\rho_t,\sigma_t}(P_{t+1})=\operatorname{en}_{\rho_t,\sigma_t}(P_t)$ and $\ell_{\rho_h,\sigma_h}(P_{t+1})=\ell_{\rho_h,\sigma_h}(P_t)$ for $0\le h<t$,

- $(A_{t+1},(\rho_{t+1},\sigma_{t+1}))$ is a regular corner of (P_{t+1},Q_{t+1}), where

$$(\rho_{t+1},\sigma_{t+1}) \coloneqq \mathrm{Pred}_{P_{t+1}}(\rho_t,\sigma_t) \quad \text{and} \quad A_{t+1} \coloneqq \frac{1}{m}\,\mathrm{st}_{\rho_t,\sigma_t}(P_{t+1}),$$

- There exists $\lambda \in K^{\times}$ such that m divides the multiplicity m_λ of $z-\lambda$ in $\mathfrak{p}_t(z)$ and

$$A_{t+1} = \Big(\frac{k_t}{ml_t},0\Big) + \frac{m_\lambda}{m}\Big(-\frac{\sigma_t}{\rho_t},1\Big),$$

Moreover $\ell_{\rho_t,\sigma_t}(P_{t+1}) = \varphi(\ell_{\rho_t,\sigma_t}(P_t))$, where $\varphi \in \mathrm{Aut}(L^{(l_{t+1})})$ is defined by

$$\varphi(x^{\frac{1}{l_{t+1}}}) \coloneqq x^{\frac{1}{l_{t+1}}} \qquad \text{and} \qquad \varphi(y) \coloneqq y + \lambda x^{\frac{\sigma_t}{\rho_t}},$$

- $A(P_{t+1}) = A(P_t) \cup \{(\rho_t,\sigma_t)\} \cup \{(\rho,\sigma) \in A(P_{t+1}) : (\rho,\sigma) < (\rho_t,\sigma_t) \text{ in } I\}$, where $A(P_t)$ and $A(P_{t+1})$ are as in the discussion above [1, Proposition 5.2].

By Remark 2.2.7 we know that $\mathcal{A}_{t+1}$ is a corner generated by $(\mathcal{A}_t,\mathcal{A}'_t)$, that $\mathcal{A}_{t+1} \neq \mathcal{A}'_t$ and that $v_{01}(A_{t+1}) < v_{01}(A_t)$. We claim that we can assume that $(A_{t+1},(\rho_{t+1},\sigma_{t+1}))$ is of type I or II. In fact, suppose that it is a regular corner of type III and write

$$\ell_{\rho_{t+1},\sigma_{t+1}}(P_{t+1}) = x^{\frac{\kappa_{t+1}}{l_{t+1}}}\mu_0(z-\lambda_0)^{r_0} \quad \text{where } z \coloneqq x^{\frac{-\sigma_{t+1}}{\rho_{t+1}}}y,\ \mu_0,\lambda_0 \in K^{\times} \text{ and } r_0 \in \mathbb{N}.$$

Then, by [1, Theorem 7.6(1) and Remark 5.10],

$$A(P_{t+1}) = A(P_t) \cup \{(\rho_t,\sigma_t)\}$$

while, by [1, Proposition 5.17], we have $\rho_{t+1} \mid l_{t+1}$ and there exists an (m,n)-pair $(P_{t+1,1},Q_{t+1,1})$ in $L^{(l_{t+1})}$ such that,

- $\mathrm{en}_{\rho_{t+1},\sigma_{t+1}}(P_{t+1,1}) = \mathrm{en}_{\rho_{t+1},\sigma_{t+1}}(P_{t+1}) = A_{t+1} = \frac{1}{m}\,\mathrm{st}_{\rho_{t+1},\sigma_{t+1}}(P_{t+1,1})$,

- $\ell_{\rho_h,\sigma_h}(P_{t+1,1}) = \ell_{\rho_h,\sigma_h}(P_{t+1})$ for $0 \le h \le t$,

- $(A_{t+1},(\rho_{t+1,1},\sigma_{t+1,1}))$ is a regular corner of $(P_{t+1,1},Q_{t+1,1})$, where

$$(\rho_{t+1,1},\sigma_{t+1,1}) \coloneqq \mathrm{Pred}_{P_{t+1,1}}(\rho_{t+1},\sigma_{t+1}),$$

- $A(P_{t+1,1}) = A(P_{t+1}) \cup \{(\rho,\sigma) \in A(P_{t+1,1}) : (\rho,\sigma) < (\rho_{t+1},\sigma_{t+1}) \text{ in } I\}$.

Note that $(\rho_{t+1,1},\sigma_{t+1,1}) = \mathrm{Pred}_{P_{t+1,1}}(\rho_t,\sigma_t)$. As long as Case III occurs, we can find

$$(\rho_{t+1,1},\sigma_{t+1,1}) > \ldots > (\rho_{t+1,u},\sigma_{t+1,u}) > \ldots,$$

and (m,n)-pairs $(P_{t+1,u}, Q_{t+1,u})$ in $L^{(l_{t+1})}$ such that for all $u \geq 1$

- $\rho_{t+1,u} \mid l_{t+1}$,
- $\mathrm{en}_{\rho_{t+1,u},\sigma_{t+1,u}}(P_{t+1,u+1}) = \mathrm{en}_{\rho_{t+1,u},\sigma_{t+1,u}}(P_{t+1,u}) = A_{t+1} = \frac{1}{m}\,\mathrm{st}_{\rho_{t+1,u},\sigma_{t+1,u}}(P_{t+1,u+1})$,
- $(A_{t+1},(\rho_{t+1,u+1},\sigma_{t+1,u+1}))$ is a regular corner of $(P_{t+1,u+1},Q_{t+1,u+1})$, where

$$(\rho_{t+1,u+1},\sigma_{t+1,u+1}) \coloneqq \mathrm{Pred}_{P_{t+1,u+1}}(\rho_{t+1,u},\sigma_{t+1,u}) = \mathrm{Pred}_{P_{t+1,u+1}}(\rho_t,\sigma_t),$$

- $\ell_{\rho_h,\sigma_h}(P_{t+1,u+1}) = \ell_{\rho_h,\sigma_h}(P_{t+1,u})$ for $0 \leq h \leq t$,
- $A(P_{t+1,u+1}) = A(P_{t+1}) \cup \{(\rho,\sigma) \in A(P_{t+1,u+1}) : (\rho,\sigma) < (\rho_{t+1},\sigma_{t+1}) \text{ in } I\}$.

But there are only finitely many $\rho_{t+1,u}$'s with $\rho_{t+1,u} \mid l_{t+1}$. Moreover,

$$0 < -\sigma_{t+1,u} < \rho_{t+1,u},$$

since $(1,-1) < (\rho_{t+1,u},\sigma_{t+1,u}) < (1,0)$, and so there are only finitely many $(\rho_{t+1,u},\sigma_{t+1,u})$ possible. Thus, eventually cases I or II must occur, proving the claim. Note that by [1, Theorem 7.6(1) and Remarks 5.10 and 5.11]

$$(A_{t+1},(\rho_{t+1},\sigma_{t+1})) \text{ is of type II.a)} \Leftrightarrow (\rho_{t+1},\sigma_{t+1}) \in A(P_{t+1}) \Leftrightarrow A(P_t) \cup \{(\rho_t,\sigma_t)\} \subsetneq A(P_{t+1}).$$

Assume that $(A_{t+1},(\rho_{t+1},\sigma_{t+1}))$ is a regular corner of type II and set $A'_{t+1} \coloneqq \frac{1}{m}\,\mathrm{st}_{\rho_{t+1},\sigma_{t+1}}(P_{t+1})$. By Proposition 2.1.5(2) we know that $(\mathcal{A}_{t+1},\mathcal{A}'_{t+1})$ is a child of $(\mathcal{A}_t,\mathcal{A}'_t)$. If $(A_{t+1},(\rho_{t+1},\sigma_{t+1}))$ is a regular corner of type II.a), then by [1, Remark 5.11], the pair

$$\bigl(A_{t+2},(\rho_{t+2},\sigma_{t+2})\bigr) \coloneqq \bigl(A'_{t+1}, \mathrm{Pred}_{P_{t+1}}(\rho_{t+1},\sigma_{t+1})\bigr)$$

is a regular corner of $(P_{t+2},Q_{t+2}) \coloneqq (P_{t+1},Q_{t+1})$. Moreover, by definition $\mathcal{A}_{t+2}$ is generated by $(\mathcal{A}_{t+1},\mathcal{A}'_{t+1})$ and $v_{01}(A_{t+2}) < v_{01}(A_{t+1})$. On the other hand, if

$(A_{t+1},(\rho_{t+1},\sigma_{t+1}))$ is a corner of type II.b), then, arguing as above we obtain a root λ of $\mathfrak{p}_{t+1}(z)$ and an (m,n)-pair (P_{t+2},Q_{t+2}) in $L^{(l_{t+2})}$, where $l_{t+2} := \operatorname{lcm}(l_{t+1},\rho_{t+1})$, such that

- $\mathrm{en}_{\rho_{t+1},\sigma_{t+1}}(P_{t+2}) = \mathrm{en}_{\rho_{t+1},\sigma_{t+1}}(P_{t+1})$ and $\ell_{\rho_h,\sigma_h}(P_{t+2}) = \ell_{\rho_h,\sigma_h}(P_{t+1})$ for $0 \le h < t+1$,

- $(A_{t+2},(\rho_{t+2},\sigma_{t+2}))$ is a regular corner of type I or II of (P_{t+2},Q_{t+2}), where

$$(\rho_{t+2},\sigma_{t+2}) := \mathrm{Pred}_{P_{t+2}}(\rho_{t+1},\sigma_{t+1}) \quad \text{and} \quad A_{t+2} := \frac{1}{m}\,\mathrm{st}_{\rho_{t+1},\sigma_{t+1}}(P_{t+2}),$$

- $\mathcal{A}_{t+2} \neq \mathcal{A}'_{t+1}$, the pair $(\mathcal{A}_{t+1},\mathcal{A}'_{t+1})$ generates $\mathcal{A}_{t+2}$, and $v_{01}(A_{t+2}) < v_{01}(A_{t+1})$,

- there exists $\lambda \in K^{\times}$ such that m divides the multiplicity m_λ of $z-\lambda$ in $\mathfrak{p}_{t+1}(z)$ and
$$A_{t+2} = \Big(\frac{k_{t+1}}{ml_{t+1}},0\Big) + \frac{m_\lambda}{m}\Big(-\frac{\sigma_{t+1}}{\rho_{t+1}},1\Big).$$
Moreover $\ell_{\rho_{t+1},\sigma_{t+1}}(P_{t+2}) = \varphi(\ell_{\rho_{t+1},\sigma_{t+1}}(P_{t+1}))$, where $\varphi \in \mathrm{Aut}(L^{(l_{t+2})})$ is defined by
$$\varphi(x^{\frac{1}{l_{t+2}}}) := x^{\frac{1}{l_{t+2}}} \qquad \text{and} \qquad \varphi(y) := y + \lambda x^{\frac{\sigma_{t+1}}{\rho_{t+1}}},$$

- $A(P_{t+2}) = A(P_{t+1}) \cup \{(\rho_{t+1},\sigma_{t+1})\} \cup \{(\rho,\sigma) \text{ in } A(P_{t+1}) : (\rho,\sigma) < (\rho_{t+1},\sigma_{t+1}) \in I\}$.

While regular corners of type II occurs we continue with this process. Eventually a regular corner $(A_{j+1},(\rho_{j+1},\sigma_{j+1}))$ of type I must occur. Finally, by [1, Proposition 5.13], the chain (2.3.6) is complete. □

Remark 2.3.3. By Theorem 3.1.1 below, if $(A_{j+1},(\rho_{j+1},\sigma_{j+1}))$ is a regular corner of type I.a) of (P_{j+1},Q_{j+1}) in $L^{(l_{j+1})}$, then we can modify (P_{j+1},Q_{j+1}) in such a way that $(A_{j+1},(\rho_{j+1},\sigma_{j+1}))$ becomes of type I.b).

Remark 2.3.4. Let (P,Q) be a standard (m,n)-pair, let $j \in \mathbb{N}$ and let

$$\big((P_i,Q_i),(A_i,A'_i),(\rho_i,\sigma_i),l_i\big)_{0\le i\le j} \quad \text{and} \quad \big((P_{j+1},Q_{j+1}),A_{j+1},(\rho_{j+1},\sigma_{j+1}),l_{j+1}\big)$$

satisfying items (1)–(14) of Theorem 2.3.2. Let h and i be integers with $0 \le h \le i \le j$. By items (3), (5) and (6), and [1, Theorem 7.6(2)], there exists $d_h^{(i)}$ maximum such that

$$\ell_{\rho_h,\sigma_h}(P_i) = R_{hi}^{md_h^{(i)}} \qquad \text{for some } (\rho_h,\sigma_h)\text{-homogeneous } R_{hi} \in L^{(l_i)}. \tag{2.3.7}$$

By item (8) of [1, Theorem 7.6] we know that

$$\#\,\mathrm{Primefactors}(d_h^{(i)}) \geq i - h. \tag{2.3.8}$$

Write $A_h = (a_h/l_h, b_h)$, $A_{h+1} = (a_{h+1}/l_{h+1}, b_{h+1})$ and $A'_h = (a'_h/l_h, b'_h)$. We assert that

$$d_h^{(i)} \Big| D_h^{(i)} := \gcd\left(\frac{b_h - b'_h}{\mathrm{gap}(\rho_h, l_h)}, b_h, b_{h+1}, \frac{a_h l_i}{l_h}, \frac{a'_h l_i}{l_h}\right). \tag{2.3.9}$$

First note that by Theorem 2.3.2(5)

$$(a_{h+1}/l_{h+1}, b_{h+1}) = A_{h+1} = \frac{1}{m}\,\mathrm{st}_{\rho_h,\sigma_h}(P_i) = d_h^{(i)}\,\mathrm{st}_{\rho_h,\sigma_h}(R_{hi}),$$

and consequently $d_h^{(i)} | b_{h+1}$. By items (4), (7) and (8) of Theorem 2.3.2 there exists $\lambda \in K$ such that

$$\ell_{\rho_h,\sigma_h}(P_i) = \ell_{\rho_h,\sigma_h}(P_{h+1}) = \varphi(\ell_{\rho_h,\sigma_h}(P_h)),$$

where $\varphi \in \mathrm{Aut}(L^{(l_{h+1})})$ is defined by

$$\varphi(x^{\frac{1}{l_{h+1}}}) := x^{\frac{1}{l_{h+1}}} \qquad \text{and} \qquad \varphi(y) := y + \lambda x^{\frac{\sigma_h}{\rho_h}}.$$

Write $\widetilde{R}_{hi} := \varphi^{-1}(R_{hi})$. Then

$$\ell_{\rho_h,\sigma_h}(P_h) = \varphi^{-1}(\ell_{\rho_h,\sigma_h}(P_i)) = \widetilde{R}_{hi}^{md_h^{(i)}},$$

and so

$$(A_h, A'_h) = ((a_h/l_h, b_h), (a'_h/l_h, b'_h)) = \left(\mathrm{en}_{\rho_h,\sigma_h}\left(\widetilde{R}_{hi}^{d_h^{(i)}}\right), \mathrm{st}_{\rho_h,\sigma_h}\left(\widetilde{R}_{hi}^{d_h^{(i)}}\right)\right).$$

(Note that $\lambda = 0$ if and only if $(A_h, (\rho_h, \sigma_h))$ is a regular corner of type II.a) of (P_h, Q_h)). Set $z := x^{-\frac{\sigma_h}{\rho_h}} y$ and write

$$\widetilde{R}_{hi}^{d_h^{(i)}} = x^{\frac{a'_h}{l_h}} y^{b'_h} f_{hi}(z) \quad \text{and} \quad \widetilde{R}_{hi} = x^{\frac{u'_h}{l_i}} y^{v'_h} g_{hi}(z),$$

where f_{hi} and g_{hi} are polynomials such that $f_{hi}(0) \neq 0$ and $g_{hi}(0) \neq 0$. Clearly

$$d_h^{(i)} \Big| b'_h, \quad d_h^{(i)} \Big| b_h, \quad d_h^{(i)} \Big| \frac{a'_h l_i}{l_h}, \quad d_h^{(i)} \Big| \frac{a_h l_i}{l_h} \quad \text{and} \quad f_{hi} = g_{hi}^{d_h^{(i)}}. \tag{2.3.10}$$

Thus $d_h^{(i)}$ divides $b_h - b'_h$. We next prove that

$$d_h^{(i)} \Big| \frac{b_h - b'_h}{\operatorname{gap}(\rho_h, l_h)}. \tag{2.3.11}$$

Assume for a moment that $\operatorname{gap}(\rho_h, l_h) \mid t_{hi}$ where $t_{hi} := \deg g_{hi}$ and write $t_{hi} = \operatorname{gap}(\rho_h, l_h) t'_{hi}$. From

$$x^{\frac{a_h - a'_h}{l_h}} y^{b_h - b'_h} = z^{t_{hi} d_h^{(i)}} = x^{-\frac{t_{hi} d_h^{(i)} \sigma_h}{\rho_h}} y^{\operatorname{gap}(\rho_h, l_h) t'_{hi} d_h^{(i)}},$$

we obtain that

$$\operatorname{gap}(\rho_h, l_h) d_h^{(i)} \mid b_h - b'_h,$$

from which (2.3.11) follows. Consequently, we are reduced to prove that $\operatorname{gap}(\rho_h, l_h) \mid t_{hi}$. Suppose this is false and write

$$g_{hi} = \sum_{u=0}^{t_{hi}} a_u z^u$$

Let v be the minimum u such that $a_u \neq 0$ and $\operatorname{gap}(\rho_h, l_h) \nmid u$. A direct computation using that $\operatorname{gap}(\rho_h, l_h) \nmid v$ and that $\operatorname{gap}(\rho_h, l_h) \mid u$ for all $u < v$ such that $a_u \neq 0$, shows that the coefficient of z^v in $g_{hi}^{m d_h^{(i)}}(z)$ is $m d_h^{(i)} a_0^{m d_h^{(i)} - 1} a_v \neq 0$. But this is impossible, since

$$x^{\frac{m a'_h}{l_h}} y^{m b'_h} g_{hi}^{m d_h^{(i)}}(z) = \widetilde{R}_{hi}^{m d_h^{(i)}} = \ell_{\rho_h, \sigma_h}(P_h) \in L^{(l_h)} \qquad \text{and} \qquad z^v = x^{-\frac{\sigma_h v}{\rho_h}} y^v \notin L^{(l_h)}.$$

This proves (2.3.11) and thus finishes the proof of (2.3.9).

Remark 2.3.5. From inequality (2.3.8) and condition (2.3.9) (both with $h = 0$ and $i = j$), we obtain that $j \leq \#\operatorname{Primefactors}(D)$, where $D := \gcd(b_0, (b_0 - b'_0)/\rho_0)$.

2.4 Divisibility conditions and admissible complete chains

In this subsection we first prove that if a complete chain $\mathscr{C} = (\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$ is constructed from a standard (m,n)-pair (P,Q) as in Theorem 2.3.2, then $\mathscr{C}$ satisfies certain arithmetic conditions. In Definition 2.4.2 we name arbitrary complete chains that satisfy these properties "admissible complete chains". Then we obtain a procedure in order to determine if a given complete chain is admissible.

Let (P,Q) be an standard (m,n)-pair, let $j \in \mathbb{N}$ and let

$$\big((P_i,Q_i),(A_i,A_i'),(\rho_i,\sigma_i),l_i\big)_{0\le i\le j} \quad \text{and} \quad \big((P_{j+1},Q_{j+1}),A_{j+1},(\rho_{j+1},\sigma_{j+1}),l_{j+1}\big),$$

be as in Remark 2.3.4. By items (3), (5) and (6) of Theorem 2.3.2 and [1, Theorem 7.6(3)] (which applies since $v_{\rho_h,\sigma_h}(P_h) > 0$ by [1, Corollary 5.7(1)]) for $h \le j$ there exist $p_h, q_h \in \mathbb{N}$ coprime and a (ρ_h,σ_h)-homogeneous element $F_h \in L^{(l_h)}$ such that,

$$v_{\rho_h,\sigma_h}(F_h) = \rho_h + \sigma_h, \quad [F_h, \ell_{\rho_h,\sigma_h}(P_h)] = \ell_{\rho_h,\sigma_h}(P_h) \quad \text{and} \quad \mathrm{en}_{\rho,\sigma}(F_h) = \frac{p_h}{q_h}\frac{1}{m}\,\mathrm{en}_{\rho_h,\sigma_h}(P_h).$$

Let $\varphi \in \mathrm{Aut}(L^{(l_{h+1})})$ be as in Remark 2.3.4. Since φ is (ρ_h,σ_h)-homogeneous,

$$v_{\rho_h,\sigma_h}(\varphi(F_h)) = \rho_h + \sigma_h.$$

Moreover, by [1, Remark 3.10] and items (7) and (8) of Theorem 2.3.2,

$$[\varphi(F_h), \ell_{\rho_h,\sigma_h}(P_{h+1})] = [\varphi(F_h), \varphi(\ell_{\rho_h,\sigma_h}(P_h))] = \varphi(\ell_{\rho_h,\sigma_h}(P_h)) = \ell_{\rho_h,\sigma_h}(P_{h+1}).$$

Thus, by item (4) of Theorem 2.3.2

$$[\varphi(F_h), \ell_{\rho_h,\sigma_h}(P_i)] = \ell_{\rho_h,\sigma_h}(P_i) \qquad \text{for } h < i \le j. \tag{2.4.12}$$

Since $\rho_h > 0$, the end point of each (ρ_h,σ_h)-homogeneous element F of $L^{(l_i)}$ is the support of the monomial of greatest degree in y of F. Consequently

$$\mathrm{en}_{\rho_h,\sigma_h}(F_h) = \mathrm{en}_{\rho_h,\sigma_h}(\varphi(F_h)),$$

because the monomials of greatest degree in y of F_h and $\varphi(F_h)$ coincide. Note that since $(A_h,(\rho_h,\sigma_h))$ is a regular corner of type II) of P_i the hypothesis of [1, Proposition 2.11(5)] are fulfilled, and so $\varphi(F_h)$ is the unique (ρ_h,σ_h)-homogeneous element of $L^{(l_i)}$ that satisfies equality (2.4.12).

Remark 2.4.1. By items (4), (5), (6) and (8) of [1, Theorem 7.6] the following conditions hold:

- $q_h \nmid d_h^{(i)}$ for all $0 \le h \le i \le j$.
- $q_k \mid d_h^{(i)}$ for all $0 \le h < k \le i \le j$.
- $q_h \nmid q_k$ for all $0 \le h < k \le j$.

Note that since

$$\gcd(p_h,q_h)=1 \quad \text{and} \quad \frac{p_h}{q_h}=\frac{\rho_h+\sigma_h}{v_{\rho_h,\sigma_h}(A_h)},$$

we have

$$p_h=\frac{\rho_h+\sigma_h}{\gcd(\rho_h+\sigma_h,v_{\rho_h,\sigma_h}(A_h))} \quad \text{and} \quad q_h=\frac{v_{\rho_h,\sigma_h}(A_h)}{\gcd(\rho_h+\sigma_h,v_{\rho_h,\sigma_h}(A_h))}. \tag{2.4.13}$$

Let $(\mathcal{C}_0,\dots,\mathcal{C}_j,\mathcal{A}_{j+1})$ be a complete chain (see Definition 2.3.1). For $0 \le i \le j$, write

$$\mathcal{C}_i=(\mathcal{A}_i,\mathcal{A}'_i),\quad \mathcal{A}_i=(a_i \wr l_i,b_i),\quad \mathcal{A}'_i=(a'_i \wr l_i,b'_i) \quad \text{and} \quad (\rho_i,\sigma_i):=\operatorname{dir}(A_i-A'_i),$$

and write

$$\mathcal{A}_{j+1}=(a_{j+1}\wr l_{j+1},b_{j+1}).$$

Now for $0 \le h \le j$, we can define p_h and q_h by equalities (2.4.13), and we do it. Moreover, as in Remark 2.3.4, we set

$$D_h^{(i)}:=\gcd\left(\frac{b_h-b'_h}{\operatorname{gap}(\rho_h,l_h)},b_h,b_{h+1},\frac{a_hl_i}{l_h},\frac{a'_hl_i}{l_h}\right).$$

Definition 2.4.2. A complete chain is called an *admissible complete chain* if for all $0 \le h < i \le j$ it satisfies

$$q_i \mid D_h^{(i)},\quad q_h \nmid q_i \quad \text{and} \quad \#\operatorname{Primefactors}(D_h^{(i)}) \ge i-h.$$

By Remark 2.4.1, inequality (2.3.8) and condition (2.3.9) every complete chains arising from a standard (m,n)-pair (P,Q) is admissible. In Algorithm 7 we give a procedure to verify if an arbitrary complete chain is admissible.

Algorithm 7: GetIsAdmissible

Input: A complete chain $\mathcal{C} = (\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$ with
$\mathcal{C}_i = (\mathcal{A}_i, \mathcal{A}'_i) = \big((a_i \wr l_i, b_i), (a'_i \wr l_i, b'_i)\big)$.

Output: A boolean variable IsAdmissible.

$h \leftarrow 0$
$i \leftarrow 1$
IsAdmissible $\leftarrow$ TRUE
while $h < j$ **and** IsAdmissible $=$ TRUE **do**
    $(\rho, \sigma) \leftarrow \mathrm{dir}(A_h - A'_h)$
    gap $\leftarrow \frac{\rho}{\gcd(\rho, l_h)}$
    $q \leftarrow \frac{v_{\rho,\sigma}(A_h)}{\gcd(\rho+\sigma, v_{\rho,\sigma}(A_h))}$
    while $i \leq j$ **and** IsAdmissible $=$ TRUE **do**
        $(\rho', \sigma') \leftarrow \mathrm{dir}(A_i - A'_i)$
        $q' \leftarrow \frac{v_{\rho',\sigma'}(A_i)}{\gcd(\rho'+\sigma', v_{\rho',\sigma'}(A_i))}$
        $D \leftarrow \gcd\left(\frac{b_h - b'_h}{\text{gap}}, b_h, b_{h+1}, \frac{a_h l_i}{l_h}, \frac{a'_h l_i}{l_h}\right)$
        if $\#\,\mathrm{Primefactors}(D) \geq i - h$ **and** $q' \mid D$ **and** $q \nmid q'$ **then**
            $i \leftarrow i + 1$
        else
            IsAdmissible $\leftarrow$ FALSE
    $h \leftarrow h + 1$
    $i \leftarrow h + 1$
RETURN IsAdmissible

In Algorithm 8 we obtain all admissible complete chains starting from a valid edge $(\mathcal{A}, \mathcal{A}')$ with $v_{11}(A) \leq M$ for a given upper bound M. Due to all the previous algorithms, this main procedure is short.

Algorithm 8: Main algorithm

Input: A positive integer M.

Output: A list AdmissibleCompleteChains of all admissible complete chains $(\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$, with $v_{11}(A_0) \leq M$, where $\mathcal{A}_0$ is the first coordinate of $\mathcal{C}_0$.

PLLC $\leftarrow$ GetPossibleLastLowerCorners$\left(\left\lfloor \frac{M}{2} \right\rfloor\right)$
for $a = 2$ **to** $\left\lfloor \frac{M}{2} \right\rfloor$ **do**
    for $b = a + 1$ **to** $M - a$ **do**
        StartingEdges $\leftarrow$ GetStartingEdges$((a, b), \text{PLLC})$
        for $(\mathcal{A}, \mathcal{A}') \in$ StartingEdges **do**
            CompleteChains $\leftarrow$ GetCompleteChains$(\mathcal{A}, \mathcal{A}')$
            for $\mathcal{CH} \in$ CompleteChains **do**
                IsAdmissible $\leftarrow$ GetIsAdmissible$(\mathcal{CH})$
                if IsAdmissible = TRUE **then**
                    add $\mathcal{CH}$ to AdmissibleCompleteChains
RETURN AdmissibleCompleteChains

We want to apply Algorithm 8 in order to obtain limitations on the possible counterexamples to the Jacobian Conjecture. Assume then that this conjecture is false. By [1, Corollary 5.21] we know there exists a counterexample (P, Q) and $m, n \in \mathbb{N}$ coprime such that (P, Q) is a standard (m, n)-pair and a minimal pair, which means that $\gcd(v_{1,1}(P), v_{1,1}(Q)) = B$, where B is as in (2.0.1).

Let A_0 be as in Remark 2.3.4. By [1, Proposition 5.2 and Corollary 5.21(3)]

$$A_0 = \frac{1}{m} \operatorname{en}_{10}(P) \quad \text{and} \quad \gcd(v_{11}(P), v_{11}(Q)) = v_{11}(A_0).$$

By Theorem 2.3.2 and Remark 2.4.1 we know that $\mathcal{A}_0$ is the first coordinate of $\mathcal{C}_0$ for one of the admissible complete chains obtained running Algorithm 8 with $M \geq B$.

Chapter 3

Generation of (m, n)-families parameterized by $\mathbb{N}_0$

3.1 (m, n)-families

In this section, for a complete chain $\mathscr{C} := (\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$, we obtain restrictions on all the possible m and n such that there could exist an (m, n)-pair (P, Q) that generates $\mathscr{C}$ as in Theorem 2.3.2.

Proposition 3.1.1. *If an (m, n)-pair (P, Q) in $L^{(l)}$ has a regular corner $(A, (\rho, \sigma))$ of type I.a), then $\rho \mid l$ and there exists $\varphi \in \mathrm{Aut}(L^{(l)})$, such that $(\varphi(P), \varphi(Q))$ is an (m, n)-pair and $(A, (\rho, \sigma))$ is a regular corner of type I.b) of $(\varphi(P), \varphi(Q))$. Moreover, the regular corners of (P, Q) and the regular corners of $(\varphi(P), \varphi(Q))$, coincide.*

Proof. Let $A' := \frac{1}{m}\operatorname{st}_{\rho,\sigma}(P)$ and write $A = (a/l, b)$ and $A' = (a'/l, b')$. By [1, Proposition 5.13a)] we know that $b' = 0$. Write

$$\ell_{\rho,\sigma}(P) = x^{\frac{ma'}{l}} p(z) \quad \text{with } z := x^{-\frac{\sigma}{\rho}} y,\ p(z) = \sum a_i z^i \in K[z] \text{ and } a_0 \neq 0,$$

and

$$\ell_{\rho,\sigma}(Q) = x^{\frac{na'}{l}} q(z) \quad \text{with } z := x^{-\frac{\sigma}{\rho}} y,\ q(z) = \sum b_i z^i \in K[z] \text{ and } b_0 \neq 0.$$

A direct computation shows that there exists $S \in L^{(l)}$, such that

$$[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = \frac{a'}{l}(ma_0 b_1 - na_1 b_0) x^{\frac{ma'+na'}{l} - \frac{\sigma}{\rho} - 1} + yS.$$

Since $(A,(\rho,\sigma))$ of type I, we have $[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]\neq 0$. So, by [1, Proposition 1.13]

$$[\ell_{\rho,\sigma}(P),\ell_{\rho,\sigma}(Q)]=\ell_{\rho,\sigma}([P,Q])\in K^{\times}. \qquad (3.1.1)$$

Thus, necessarily $\frac{(m+n)a'}{l}-\frac{\sigma}{\rho}=1$ and $a_1\neq 0$ or $b_1\neq 0$. If $a_1\neq 0$, then

$$\Bigl(\frac{ma'}{l}-\frac{\sigma}{\rho},1\Bigr)\in \operatorname{Supp}(\ell_{\rho,\sigma}(P))\subseteq \frac{1}{l}\mathbb{Z}\times \mathbb{N}_0.$$

Since $\bigl(\frac{ma'}{l},0\bigr)$ also is in $\operatorname{Supp}(\ell_{\rho,\sigma}(P))\subseteq \frac{1}{l}\mathbb{Z}\times\mathbb{N}_0$, we conclude that $\frac{\sigma}{\rho}\in\frac{1}{l}\mathbb{Z}$, which implies $\rho\mid l$. Similarly, if $b_1\neq 0$, then we also obtain $\rho\mid l$, as desired. Now let $z-\lambda$ be a linear factor of $p(z)$. Define $\varphi\in\operatorname{Aut}(L^{(l)})$ by

$$\varphi(x^{1/l}):=x^{1/l}\quad\text{and}\quad \varphi(y):=y+\lambda x^{\sigma/\rho}.$$

Then

$$\varphi(\ell_{\rho,\sigma}(P))=x^{\frac{ma'}{l}}p(z+\lambda)=x^{\frac{ma'}{l}}\overline{p}(z)\quad\text{and}\quad \varphi(\ell_{\rho,\sigma}(Q))=x^{\frac{na'}{l}}q(z+\lambda)=x^{\frac{na'}{l}}\overline{q}(z),$$

where $\overline{p}(z)=p(z+\lambda)$ and $\overline{q}(z)=q(z+\lambda)$. By [1, Proposition 3.9] we know that, for all $H\in L^{(l)}$,

$$\ell_{\rho,\sigma}(\varphi(H))=\varphi(\ell_{\rho,\sigma}(H)),\qquad \operatorname{en}_{\rho,\sigma}(\varphi(H))=\operatorname{en}_{\rho,\sigma}(H)$$

and

$$\ell_{\rho_1,\sigma_1}(\varphi(H))=\ell_{\rho_1,\sigma_1}(H)\quad\text{for all }(\rho,\sigma)<(\rho_1,\sigma_1)\leq(1,1). \qquad (3.1.2)$$

Using this with $H=P$ and $H=Q$, we obtain that

$$\frac{v_{11}(\varphi(P))}{v_{11}(\varphi(Q))}=\frac{v_{10}(\varphi(P))}{v_{10}(\varphi(Q))}=\frac{m}{n}\quad\text{and}\quad v_{1,-1(\operatorname{en}_{10}(\varphi(P)))}<0.$$

Hence $(\varphi(P),\varphi(Q))$ is an (m,n)-pair, since $[\varphi(P),\varphi(Q)]=[P,Q]\in K^{\times}$, by [1, Proposition 3.10]. We claim that $(\rho,\sigma)\in\operatorname{Dir}(\varphi(P))$. In fact since

$$\ell_{\rho,\sigma}(\varphi(P))=\varphi(\ell_{\rho,\sigma}(P))=x^{\frac{ma'}{l}}\overline{p}(z),$$

in order to see this it suffices to show that $\overline{p}$ is not a monomial, which follows easily from the fact that $\deg(p)=m(b-b')>1$ and λ is a simple root of p by Remark 2.1.6.

Write $\overline{p}(z) = \sum_i \overline{a}_i z^i$ and $\overline{q}(z) = \sum_i \overline{b}_i z^i$. By [1, Proposition 3.10] and (3.1.1), we have

$$[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = [\varphi(\ell_{\rho,\sigma}(P)), \varphi(\ell_{\rho,\sigma}(Q))] = \varphi([\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)]) \in K^{\times}.$$

Using this and the fact that $\overline{a}_0 = p(\lambda) = 0$ we obtain

$$-\frac{na'}{l}\overline{a}_1\overline{b}_0 = [\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] \in K^{\times}.$$

Hence

$$\mathrm{st}_{\rho,\sigma}(\varphi(P)) = \Big(\frac{ma'}{l} - \frac{\sigma}{\rho}, 1\Big) \quad \text{and} \quad \mathrm{st}_{\rho,\sigma}(\varphi(Q)) = \Big(\frac{na'}{l}, 0\Big),$$

and so $(A, (\rho, \sigma))$ is a regular corner of type I.b) of $(\varphi(P), \varphi(Q))$. Using this, that $(A, (\rho, \sigma))$ is a regular corner of type I) of (P, Q), equalities (3.1.2) with $H = P$, and [1, Remark 5.10 and Theorem 7.6(1)], we obtain that (P, Q) and $(\varphi(P), \varphi(Q))$ have the same regular corners. □

Let $((a/l, b), (\rho, \sigma))$ be a regular corner of type I.b) of an (m, n)-pair (P, Q) in $L^{(l)}$. According to [1, Proposition 5.13b)] there exists $k \in \mathbb{N}$, with $k < l - \frac{a}{b}$ such that

$$\{\mathrm{st}_{\rho,\sigma}(P), \mathrm{st}_{\rho,\sigma}(Q)\} = \left\{\left(\frac{k}{l}, 0\right), \left(1 - \frac{k}{l}, 1\right)\right\}, \tag{3.1.3}$$

Proposition 3.1.2. *Let* $e_k \coloneqq \gcd(k, bl - a)$. *If* $\mathrm{st}_{\rho,\sigma}(Q) = (k/l, 0)$, *then* $\frac{k}{e_k} \mid n$ *and*

$$(m + n)b - \frac{ne_k}{k}\frac{bl - a}{e_k} = 1, \tag{3.1.4}$$

while if $\mathrm{st}_{\rho,\sigma}(P) = (k/l, 0)$, *then* $\frac{k}{e_k} \mid m$ *and*

$$(m + n)b - \frac{me_k}{k}\frac{bl - a}{e_k} = 1. \tag{3.1.5}$$

Proof. Assume first that $\mathrm{st}_{\rho,\sigma}(Q) = (k/l, 0)$. Since, by [1, Corollary 5.7(2)],

$$\mathrm{en}_{\rho,\sigma}(P) = m\left(\frac{a}{l}, b\right) \qquad \text{and} \qquad \mathrm{en}_{\rho,\sigma}(Q) = n\left(\frac{a}{l}, b\right),$$

we have

$$\rho - \frac{\rho k}{l} + \sigma = v_{\rho,\sigma}(\mathrm{st}_{\rho,\sigma}(P)) = v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(P)) = m\left(\frac{a\rho}{l} + b\sigma\right)$$

and

$$\frac{\rho k}{l} = v_{\rho,\sigma}(\mathrm{st}_{\rho,\sigma}(Q)) = v_{\rho,\sigma}(\mathrm{en}_{\rho,\sigma}(Q)) = n\left(\frac{a\rho}{l} + b\sigma\right),$$

which leads to

$$1 - \frac{k}{l} + \frac{\sigma}{\rho} = \frac{ma}{l} + mb\frac{\sigma}{\rho} \qquad \text{and} \qquad \frac{\sigma}{\rho} = \frac{k-na}{nlb}. \tag{3.1.6}$$

Hence,

$$\frac{ma}{l} + mb\frac{k-na}{nlb} = 1 - \frac{k}{l} + \frac{k-na}{nlb},$$

which gives

$$(m+n)bk - n(bl-a) = k.$$

Therefore $k \mid n(bl-a)$. Since $\frac{k}{e_k}$ and $\frac{bl-a}{e_k}$ are coprime, necessarily $\frac{k}{e_k} \mid n$. So, equality (3.1.4) is true. The case $\mathrm{st}_{\rho,\sigma}(P) = (k/l, 0)$ is similar. $\square$

Let $\mathcal{A} := (a\wr l, b) \in \mathbb{N}_{(l)} \times \mathbb{N}_0$ be a final corner and let $k \in \mathbb{N}$ be such that $k < l - \frac{a}{b}$. We want to find all the $(m,n) \in \mathbb{N}^2$ such that one of the equalities (3.1.4) or (3.1.5) is satisfied. By symmetry it suffices to find the set of all those $(m,n) \in \mathbb{N}^2$ such that equality (3.1.4) is satisfied and then to add to this set the pairs obtained by swapping m with n. For the first task we proceed as follows: we first check that

$$\gcd\left(b, \frac{bl-a}{e_k}\right) = 1, \quad \text{where } e_k := \gcd(k, bl-a).$$

If this is the case we determine the Bezout coefficients M, N with $N \geq 1$ in

$$Mb - N\frac{bl-a}{e_k} = 1.$$

For each solution (M,N) we set $n := \frac{Nk}{e_k}$ and $m := M - n$. Since $b < \frac{bl-a}{k}$, we have

$$mb = Mb - \frac{Nk}{e_k}b > Mb - \frac{Nk}{e_k}\frac{bl-a}{k} = 1,$$

which implies that $m \geq 1$ as desired. Then we keep all the pairs (m,n) that also satisfy $m > 1$, $n > 1$ and $\gcd(m,n) = 1$.

Definition 3.1.3. Let $\mathcal{A} := (a\wr l, b) \in \mathbb{N}_{(l)} \times \mathbb{N}_0$ be a final corner and let

$$I(\mathcal{A}) := \left\{k \in \mathbb{N} : 1 \leq k < l - \frac{a}{b} \text{ and } \gcd\left(b, \frac{bl-a}{\gcd(k, bl-a)}\right) = 1\right\}.$$

For each $k \in I(\mathcal{A})$ we set

$$\mathrm{MN}_k(\mathcal{A}) := \left\{(m,n) \in \mathbb{N}^2 : m,n > 1,\ \gcd(m,n) = 1 \text{ and } (m+n)bk - n(bl-a) = k\right\},$$

and we define the set $\mathrm{MN}(\mathcal{A})$, of *possible* (m,n) *for* $\mathcal{A}$, by

$$\mathrm{MN}(\mathcal{A}) := \bigcup_{k \in I(\mathcal{A})} \mathrm{MN}_k(\mathcal{A}).$$

Next we describe these values as unions of infinite families of (m,n)'s, parameterized by $\mathbb{N}_0$.

Let $k \in \mathbb{N}$ be such that $1 \le k < l - \frac{a}{b}$ and set $e_k := \gcd(k, bl-a)$. Assume $\gcd\left(b, \frac{bl-a}{e_k}\right) = 1$ and let M_k and N_k with $N_k \in \mathbb{N}$ minimum satisfying

$$M_k b - N_k \frac{bl-a}{e_k} = 1.$$

Then

$$\left\{(M,N) \in \mathbb{Z} \times \mathbb{N} : Mb - N\frac{bl-a}{e_k} = 1\right\} = \left\{\left(M_k + j\frac{bl-a}{e_k}, N_k + jb\right) : j \in \mathbb{N}_k\right\}.$$

Set

$$m'_{kj} := M_k + j\frac{bl-a}{e_k} - \frac{(N_k + jb)k}{e_k} \qquad \text{and} \qquad n'_{kj} := \frac{(N_k + jb)k}{e_k}.$$

Thus

$$m'_{kj} = m'_{k0} + j\Delta_k^{(1)} \quad \text{and} \quad n'_{kj} = n'_{k0} + j\Delta_k^{(2)}, \quad \text{where } \Delta_k^{(1)} := \frac{bl - bk - a}{e_k} \text{ and } \Delta_k^{(2)} := \frac{bk}{e_k}.$$

So,

$$m'_{k,j+1} > m'_{kj} \quad \text{and} \quad n'_{k,j+1} > n'_{kj} \qquad \text{for all } j \in \mathbb{N}_0.$$

Hence, by the comments above Definition 3.1.3, we have $1 \le m'_{k0}, n'_{k0}$. Since we only want consider the m'_{kj}'s and n'_{kj}'s greater than 1, we set

$$m_{kj} := \begin{cases} m'_{kj} & \text{if } n'_{k0} > 1 \text{ and } m'_{k0} > 1, \\ m'_{k,j+1} & \text{otherwise,} \end{cases} \quad \text{and} \quad n_{kj} := \begin{cases} n'_{kj} & \text{if } n'_{k0} > 1 \text{ and } m'_{k0} > 1, \\ n'_{k,j+1} & \text{otherwise.} \end{cases}$$

Clearly

$$m_{kj} = m_{k0} + j\Delta_k^{(1)} \qquad \text{and} \qquad n_{kj} = n_{k0} + j\Delta_k^{(2)}. \tag{3.1.7}$$

With these notations,

$$\mathrm{S}(\mathcal{A}, k) \coloneqq \Big\{(m,n) \in \mathbb{N}^2 : m, n > 1 \text{ and } (m+n)bk - n(bl-a) = k\Big\} = \{(m_{kj}, n_{kj}) : j \in \mathbb{N}_0\}.$$

Since

$$\mathrm{MN}_k(\mathcal{A}) = \{(m,n) \in \mathrm{S}(\mathcal{A}, k) : \gcd(m,n) = 1\},$$

we must choose the (m,n)'s in $\mathrm{S}(\mathcal{A},k)$ such that $\gcd(m,n) = 1$. Note that

$$mb\frac{k}{e_k} + n\Big(b\frac{k}{e_k} - \frac{bl-a}{e_k}\Big) = \frac{k}{e_k},$$

and so $\gcd(m,n) \mid \frac{k}{e_k}$. For $i \in \big\{0, \dots, \frac{k}{e_k} - 1\big\}$ we define

$$\mathrm{MN}_{ki}(\mathcal{A}) \coloneqq \Big\{\Big(m_{k,i+j\frac{k}{e_k}}, n_{k,i+j\frac{k}{e_k}}\Big) : j \in \mathbb{N}_0\Big\} = \Big\{\Big(m_{ki} + j\frac{k}{e_k}\Delta_k^{(1)}, n_{ki} + j\frac{k}{e_k}\Delta_k^{(2)}\Big) : j \in \mathbb{N}_0\Big\}.$$

Lemma 3.1.4. *For all* $i \in \big\{0, \dots, \frac{k}{e_k} - 1\big\}$ *and all* $(m,n) \in \mathrm{MN}_{ki}(\mathcal{A})$*, we have*

$$\gcd(m,n) = \gcd(m_{ki}, n_{ki}).$$

Moreover, there exists i *such that* $\gcd(m_{ki}, n_{ki}) = 1$.

Proof. Clearly $\mathrm{MN}_{ki}(\mathcal{A}) \subseteq \mathrm{S}(\mathcal{A},k)$ and so, if $(m,n) \in \mathrm{MN}_{ki}(\mathcal{A})$, then $\gcd(m,n) \mid \frac{k}{e_k}$. Consequently, for $d_{ki} \coloneqq \gcd(m_{ki}, n_{ki})$ we have

$$d_{ki} \mid m_{ki} + j\frac{k}{e_k}\Delta_k^{(1)} \quad \text{and} \quad d_{ki} \mid n_{ki} + j\frac{k}{e_k}\Delta_k^{(2)} \qquad \text{for all } j,$$

and hence $d_{ki} \mid \gcd(m,n)$ for all $(m,n) \in \mathrm{MN}_{ki}(\mathcal{A})$. Similarly one shows $\gcd(m,n) \mid d_{ki}$, which proves the first assertion. On the other hand, since $\gcd\Big(\Delta_k^{(1)}, \frac{k}{e_k}\Big) = 1$, the class $\big[\Delta_k^{(1)}\big]$ of $\Delta_k^{(1)}$ in $\mathbb{Z}/\frac{k}{e_k}\mathbb{Z}$ is invertible, and so

$$\Big\{[m_{ki}] : i = 0, \dots, \frac{k}{e_k} - 1\Big\} = \frac{\mathbb{Z}}{\frac{k}{e_k}\mathbb{Z}},$$

where $[m_{ki}]$ denotes the class of $m_{ki} = m_{k0} + i\Delta_k^{(1)}$ in $\mathbb{Z}/\frac{k}{e_k}\mathbb{Z}$. It follows that there exists an i such that $m_{ki} \equiv 1 \pmod{\frac{k}{e_k}}$. Since $d_{ki} \mid m_{ki}$ and $d_{ki} \mid \frac{k}{e_k}$, we obtain

$d_{ki} = 1$, as desired. □

For each $k \in I(\mathcal{A})$ we let $J_k(\mathcal{A})$ denote $\left\{0 \leq i < \frac{k}{e_k} : \gcd(m_{ki}, n_{ki}) = 1\right\}$, where m_{ki} and n_{ki} are as in (3.1.7). Using the previous results we obtain the following description of the set $\mathrm{MN}(\mathcal{A})$,

$$\mathrm{MN}(\mathcal{A}) = \bigcup_{k \in I(\mathcal{A})} \mathrm{MN}_k(\mathcal{A}) \quad \text{and} \quad \mathrm{MN}_k(\mathcal{A}) = \bigcup_{i \in J_k(\mathcal{A})} \mathrm{MN}_{ki}(\mathcal{A}).$$

Remark 3.1.5. Note that for a final corner $\mathcal{A}$ the set $I(\mathcal{A})$ can be empty (for example take $\mathcal{A} = (16 \wr 3, 10))$. However, if $k \in I(\mathcal{A})$, then by Lemma 3.1.4 there exists at least one (m, n)-family associated to $\mathcal{A}$. It follows that a final corner $\mathcal{A} = (a \wr l, b)$ has at least one (m, n)-family attached to it, if and only if there exists $k \in \mathbb{N}$ with $l - a/b > k \geq 1$, such that

$$\gcd\left(b, \frac{bl - a}{\gcd(k, bl - a)}\right) = 1.$$

In Algorithm 9 we obtain the set $\mathrm{MN}(\mathcal{A})$. To achieve this we use the auxiliary function BezoutCoefficients(x, y) which, for coprime positive integers x and y, returns the ordered pair (M, N) of positive integers such that $Mx - Ny = 1$ and N is minimal.

Algorithm 9: GetmnFamilies

Input: A final corner $\mathcal{A} = (a \wr l, b)$.

Output: A list mnFamilies of triples $\big((k,i),(m_{ki},n_{ki}),(\Delta^{(1)},\Delta^{(2)})\big)$ such that $k \in I(\mathcal{A})$, $i \in J_k(\mathcal{A})$ and $\mathrm{MN}(\mathcal{A}) = \bigcup_{k,i}\big\{(m_{ki}+j\Delta^{(1)}, n_{ki}+j\Delta^{(2)}) : j \in \mathbb{N}_0\big\}$.

for $k = 1$ **to** $\lceil l - \frac{a}{b} \rceil - 1$ **do**
    $e \leftarrow \gcd(k, bl - a)$
    if $\gcd(b, \frac{bl-a}{e}) = 1$ **then**
        $(M, N) \leftarrow \mathrm{BezoutCoefficients}\big(b, \frac{bl-a}{e}\big)$
        $n \leftarrow \frac{Nk}{e}$
        $m \leftarrow M - n$
        $\Delta^{(1)} \leftarrow \frac{bl-a-bk}{e}$
        $\Delta^{(2)} \leftarrow \frac{bk}{e}$
        if $m = 1$ **or** $n = 1$ **then**
            $(m, n) \leftarrow (m, n) + (\Delta^{(1)}, \Delta^{(2)})$
        $\overline{k} \leftarrow \frac{k}{e}$
        if $\overline{k} = 1$ **then**
            add $\big((k,0),(m,n),(\Delta^{(1)},\Delta^{(2)})\big)$ **to** mnFamilies
        else
            for $i = 0$ **to** $\overline{k} - 1$ **do**
                $m_i \leftarrow m + i\Delta^{(1)}$
                $n_i \leftarrow n + i\Delta^{(2)}$
                if $\gcd(m_i, n_i) = 1$ **then**
                    add $\big((k,i),(m_i,n_i),(\overline{k}\Delta^{(1)},\overline{k}\Delta^{(1)})\big)$ **to** mnFamilies

RETURN mnFamilies

3.2 Program and graphic display

A website based on these algorithms is available at https://ituvox.github.io/jacobianshape/, making it possible to visualize the construction of chains starting from points below a given upper bound.

The infrastructure for it consists of three parts:

1. A C++ implementation of the described pseudocode, along with additional routines to export the information (corners, edges, open and complete chains) to text files formatted for input into an SQL database.

2. An SQL database instance, implemented in PostgreSQL, which organizes the data generated by the C++ program in order to enable easy access by SQL queries.

3. A website mainly developed in the JavaScript language, using the D3.js library for the graphical interface, along with PHP scripts to query the database.

This structure allows a clear separation of responsibilities: the JavaScript code is only concerned with showing the information, assuming it is already suitably formatted, while the C++ program is only concerned with generating the information. It also allows for fast updates to any part of the infrastructure, since each part only depends on the output generated by the others and not on their implementation. The website consists of a single widget, which contains the following controls:

1. An options bar, near the top and below the title. This includes a button to load all points (x, y) with $v_{11}(x, y) <$ deg, for some specified value of deg, and checkboxes for options.

2. A numbered two-dimensional grid, with the ability to zoom and pan, which displays the current items (a collection of corners and edges). A corner A can be clicked to display an edge $(\mathcal{A}, \mathcal{A}')$, and the bottom point A' of an edge can be clicked to display the corners generated by it.

3. A collection of *filters* in a right hand panel. These are checkboxes that can be used to only show specific corners. For example, only corners of Type I and Type II, or only corners leading to admissible complete chains.

3.3 Admissible complete chains with $v_{11}(A_0) \leq 35$

Applying Algorithm 8 with $M = 35$ we obtain the admissible complete chains $(\mathcal{C}_0, \dots, \mathcal{C}_j, \mathcal{A}_{j+1})$ with $v_{11}(A_0) \leq 35$, where $\mathcal{A}_0$ is the first coordinate of $\mathcal{C}_0$. This procedure yields 14 admissible complete chains of length 1 and 2 admissible complete chains of length 2. Applying now Algorithm 9 with input the final corner $\mathcal{A}_{j+1}$ of any of these chains we obtain the corresponding (m,n)-families $\mathrm{MN}_k(\mathcal{A}_{j+1})$ (see Definition 3.1.3). We obtain a two tables. The first consists of 17 families of length 1, and the second one, of 7 families of length 2. We only list the cases satisfying equality (3.1.4). The other cases (satisfying (3.1.5)) can be obtained by swapping m with n.

Family	$\mathcal{A}_0$	$\mathcal{A}'_0$	$\mathcal{A}_1$	k	m	n
F_1	$(4,12)$	$(1,0)$	$(7\wr 4,3)$	1	$2j+3$	$3j+4$
F_2	$(5,20)$	$(1,0)$	$(7\wr 5,2)$	1	$j+2$	$2j+3$
F_3	$(5,20)$	$(1,0)$	$(8\wr 5,3)$	1	$4j+3$	$3j+2$
F_4	$(5,20)$	$(1,0)$	$(8\wr 5,3)$	2	$2j+3$	$12j+16$
F_5	$(5,20)$	$(1,0)$	$(9\wr 5,4)$	1	$7j+9$	$4j+5$
F_6	$(5,20)$	$(1,0)$	$(9\wr 5,4)$	2	$3j+4$	$8j+10$
F_7	$(6,15)$	$(1,0)$	$(7\wr 3,4)$	1	$j+2$	$4j+7$
F_8	$(6,15)$	$(1,0)$	$(8\wr 3,5)$	1	$2j+3$	$5j+7$
F_9	$(7,21)$	$(1,0)$	$(11\wr 7,2)$	1	$j+2$	$2j+3$
F_{10}	$(7,21)$	$(1,0)$	$(13\wr 7,3)$	1	$5j+7$	$3j+4$
F_{11}	$(7,21)$	$(1,0)$	$(13\wr 7,3)$	2	$j+2$	$3j+5$
F_{12}	$(8,24)$	$(2,0)$	$(13\wr 4,5)$	1	$2j+3$	$5j+7$
F_{13}	$(9,21)$	$(2,0)$	$(13\wr 3,7)$	1	$j+2$	$7j+13$
F_{14}	$(9,24)$	$(1,0)$	$(7\wr 3,4)$	1	$j+2$	$4j+7$
F_{15}	$(9,24)$	$(1,0)$	$(8\wr 3,5)$	1	$2j+3$	$5j+7$
F_{16}	$(9,24)$	$(1,0)$	$(10\wr 3,7)$	1	$4j+3$	$7j+5$
F_{17}	$(9,24)$	$(1,0)$	$(11\wr 3,8)$	1	$5j+2$	$8j+3$

and

Family	$\mathcal{A}_0$	$\mathcal{A}_0'$	$\mathcal{A}_1$	$\mathcal{A}_1'$	$\mathcal{A}_2$	k	m	n
F_{18}	$(6,18)$	$(6,15)$	$(6,15)$	$(1,0)$	$(7\wr 3,4)$	1	$j+2$	$4j+7$
F_{19}	$(6,18)$	$(6,15)$	$(6,15)$	$(1,0)$	$(8\wr 3,5)$	1	$2j+3$	$5j+7$
F_{20}	$(6,24)$	$(6,15)$	$(6,15)$	$(1,0)$	$(7\wr 3,4)$	1	$j+2$	$4j+7$
F_{21}	$(6,24)$	$(6,15)$	$(6,15)$	$(1,0)$	$(8\wr 3,5)$	1	$2j+3$	$5j+7$
F_{22}	$(8,24)$	$(2,0)$	$(14\wr 4,6)$	$(5\wr 4,2)$	$(5\wr 4,2)$	1	$j+2$	$2j+3$
F_{23}	$(8,24)$	$(2,0)$	$(14\wr 4,6)$	$(11\wr 4,4)$	$(11\wr 4,4)$	1	$j+2$	$4j+7$
F_{24}	$(8,24)$	$(2,0)$	$(14\wr 4,6)$	$(5\wr 4,0)$	$(19\wr 8,3)$	1	$2j+3$	$3j+4$

For each one of these chains let $(a\wr l, b)$ be its final corner and let $e_k = \gcd(k, bl-a)$. In all the cases except F_4, we have $k/e_k = 1$. In case F_4 we have $k/e_k = 2$ and $J_k(8\wr 5,3) = \{1\}$.

We claim that the families F_{18}, F_{19}, F_{20} and F_{21} can not be obtained from a standard (m,n)-pair (P,Q) as in Theorem 2.3.2. Note that with the notations used in that theorem for the four families we have

$$(\rho_0,\sigma_0) = \operatorname{dir}(A_0 - A_0') = (1,0) \qquad \text{and} \qquad (\rho_1,\sigma_1) = \operatorname{dir}(A_1 - A_1') = (3,-1).$$

Hence, by the second equality in (2.4.13) we have $q_1 = 3$. If there were an (m,n)-pair (P,Q) for one the families, then by equality (2.3.7) and Remark 2.4.1 with $h = 0$ and $i = k = 1$ there exists $R \in L$ such that $\ell_{10}(P) = R^{3m}$. Let $(a,b) = A_0$ and $(a',b') = A_0'$. Since

$$\ell_{10}(P) = x^{a'm} y^{b'm} p(y) \qquad \text{where } p(0) \neq 0 \text{ and } \deg(p) = mb - mb',$$

in the first two cases there exist $\lambda_P, \lambda \in K^\times$ such that

$$\ell_{10}(P) = \lambda_p (x^2 y^5 (y-\lambda))^{3m},$$

while in the last two cases there exist $\lambda_P, \lambda, \lambda', \lambda'' \in K^\times$ such that

$$\ell_{10}(P) = \lambda_p (x^2 y^5 (y-\lambda)(y-\lambda')(y-\lambda''))^{3m} \quad \text{and} \quad \lambda \notin \{\lambda', \lambda''\} \text{ or } \lambda = \lambda' = \lambda''.$$

Define $\varphi \in \operatorname{Aut}(L)$ by

$$\varphi(x) := x \quad \text{and} \quad \varphi(y) := y + \lambda.$$

By [1, Proposition 3.9] we know that, for all $H \in L$,

$$\ell_{10}(\varphi(H)) = \varphi(\ell_{10}(H)), \qquad \operatorname{en}_{10}(\varphi(H)) = \operatorname{en}_{10}(H)$$

and

$$\ell_{\rho_1,\sigma_1}(\varphi(H)) = \ell_{\rho_1,\sigma_1}(H) \quad \text{for all } (1,0) < (\rho_1,\sigma_1) < (-1,0).$$

Using this with $H = P$ and $H = Q$, we obtain that

$$\frac{v_{11}(\varphi(P))}{v_{11}(\varphi(Q))} = \frac{v_{10}(\varphi(P))}{v_{10}(\varphi(Q))} = \frac{m}{n} \quad \text{and} \quad v_{1,-1}(\operatorname{en}_{10}(\varphi(P))) < 0.$$

Hence $(\varphi(P), \varphi(Q))$ is an (m,n)-pair, since, by [1, Proposition 3.10],

$$[\varphi(P), \varphi(Q)] = [P, Q] \in K^{\times}.$$

Moreover

$$\ell_{10}(\varphi(P)) = \varphi(\ell_{10}(P)) = \lambda_p (x^2 (y+\lambda)^5 y)^{3m} = \lambda_p x^{6m} y^{3m} (y+\lambda)^{15m}$$

in the first two cases, and

$$\ell_{10}(\varphi(P)) = \varphi(\ell_{10}(P)) = \lambda_p x^{6m} y^{3m} (y+\lambda-\lambda')^{3m} (y+\lambda-\lambda'')^{3m} (y+\lambda)^{15m}$$

in the last two cases. So, in the first two cases

$$\frac{1}{m} \operatorname{st}_{10}(\varphi(P)) = (6,3),$$

and the same occurs in the last two cases if $\lambda \notin \{\lambda', \lambda''\}$. Hence, by [2, Remark 3.2] the point $(6,3)$ is a last lower corner. But this is impossible by [2, Remark 3.29]. On the other hand if in the last two cases $\lambda = \lambda' = \lambda''$, then

$$\frac{1}{m} \operatorname{st}_{10}(\varphi(P)) = (6,9),$$

and so $(\varphi(P), \varphi(Q))$ is a standard (m,n)-pair. Let $(A, A', (\rho, \sigma))$ be the starting triple of $(\varphi(P), \varphi(Q))$. Since

$$(1,-1) < (\rho,\sigma) \leq \mathrm{Pred}_{\varphi(P)}(1,0),$$

arguing as in the proof of [1, Proposition 6.1(9)] we obtain that

$$v_{11}(A) \leq v_{11}(6,9) = 15.$$

But this is impossible by [1, Proposition 6.5].

Remark 3.3.1. The possible counterexample in F_{13} with $j = 1$ was analyzed extensively by Orevkov in [11] (see [11, Lemma 4.1(a)]).

3.4 Possible counterexamples with $\max(\deg(P), \deg(Q)) \leq 150$

In [10] there are listed four cases (which correspond to six cases in our terminology) of possible counterexamples with $\max(\deg(P), \deg(Q)) \leq 100$. They are discarded by hand. Here we describe the shape of the 34 possible counterexamples with $\max(\deg(P), \deg(Q)) \leq 150$. We only list the cases satisfying equality (3.1.4). The other cases (satisfying (3.1.5)) can be obtained by swapping m with n. Thirteen of them correspond to a choice of (m, n) in some of the families listed in the previous section, as can be seen in the following table, where the red pairs correspond to possible counterexamples with $\max(\deg(P), \deg(Q)) \leq 100$.

Family	(m, n)	$\max\{\deg(P), \deg(Q)\}$
F_1	(3,4)	64
F_1	(5,7)	112
F_2	(2,3)	75
F_2	(3,5)	125
F_3	(3,2)	75
F_7	(2,7)	147
F_8	(3,7)	147
F_9	(2,3)	84
F_9	(3,5)	140
F_{11}	(2,5)	140
F_{17}	(2,3)	99
F_{22}	(2,3)*	96
F_{24}	(3,4)	128

Five of them correspond to the six cases found by Moh, one of the cases of Moh was discarded by the algorithm because it featured $(A_0, A_0') = ((7, 21), (2, 1))$, and $(2, 1) \notin$ PLLC. The sixth red case, marked with a star, corresponds to F_{22}. This case was probably discarded as a possible counterexample by Heitmann (with no mention to it) by symmetry reasons. This case corresponds to the first case listed in [9, pag. 426] with $\delta_3 = 1/4$, $\delta_2 = 9/16$ and $\delta_1 = 7/12$. In Proposition 3.4.1 we show that we can discard it.

There are 9 other possible pairs with a complete chain of length 1, which we list in the following table:

A_0	A_1	(m,n)	$\max\{\deg(P),\deg(Q)\}$
(7,35)	(19/7,5)	(2,3)	126
(7,42)	(13/7,6)	(3,2)	147
(7,42)	(13/7,6)	(2,3)	147
(8,28)	(7/4,3)	(3,4)	144
(8,28)	(11/4,7)	(3,2)	108
(9,36)	(17/9,4)	(3,2)	135
(9,36)	(17/9,4)	(2,3)	135
(11,33)	(19/4,8)	(2,3)	132
(12,33)	(11/3,8)	(2,3)	135

There are also 11 other possible pairs with a complete chain of length 2, which we list in the following table:

A_0	A_1	A_2	(m,n)	$\max\{\deg(P),\deg(Q)\}$
(8,32)	(8,28)	(11/4,7)	(3,2)	120
(8,40)	(8,28)	(11/4,7)	(3,2)	144
(9,27)	(9,24)	(11/3,8)	(2,3)	108
(9,36)	(9,24)	(11/3,8)	(2,3)	135
(10,40)	(16/5,6)	(23/10,3)	(3,2)	150
(10,40)	(18/5,8)	(8/5,3)	(3,2)	150
(12,30)	(16/3,10)	(11/6,3)	(3,2)	126
(12,36)	(12,33)	(11/3,8)	(2,3)	144
(12,36)	(9,24)	(11/3,8)	(2,3)	144
(12,36)	(21/4,9)	(19/4,8)	(2,3)	144
(12,36)	(21/4,9)	(12/4,5)	(2,3)	144

Finally there is another possible pair with a complete chain of length 3:

A_0	A_1	A_2	A_3	(m,n)	$\max\{\deg(P),\deg(Q)\}$
(12,36)	(12,30)	(16/3,10)	(11/6,3)	(3,2)	144

Proposition 3.4.1. *The example corresponding to F_{22} with $(m,n) = (2,3)$ can not be obtained from a standard (m,n)-pair (P,Q) as in Theorem 2.3.2.*

Proof. With the notations used in Theorem 2.3.2, we have

$$\mathcal{A}_1 = (14\wr 4, 6), \qquad \mathcal{A}_1' = \mathcal{A}_2 = (5\wr 4, 2) \qquad \text{and} \qquad (\rho_1,\sigma_1) = \operatorname{dir}(A_1 - A_1') = (16,-9).$$

Consequently,

$$\ell_{16,-9}(P_1) = x^{\frac{5m}{4}} y^{2m} p(z) \quad \text{with } z := x^{\frac{9}{16}} y,\ p \in K[z] \text{ and } p(0) \neq 0.$$

Combining this with equality (2.3.7) and the fact that $\operatorname{gap}(16,4) = 4$ we obtain that

$$\ell_{16,-9}(P_1) = \lambda_p x^{\frac{5m}{4}} y^{2m} (z^4 - \lambda')^m \qquad \text{where } \lambda', \lambda_p \in K^{\times}.$$

Hence

$$\ell_{16,-9}(P_1) = \lambda_p x^{\frac{5m}{4}} y^{2m} (z^4 - \lambda^4)^m = \lambda_p x^{\frac{5m}{4}} y^{2m} (z-\lambda)^m (z^3 + z^2\lambda + z\lambda^2 + \lambda^3)^m$$

where $\lambda \in K^{\times}$ is such that $\lambda^4 = \lambda'$. Thus the multiplicity m_λ of λ as a root of $p(z)$ equals m. Define $\varphi \in \operatorname{Aut}(L^{(16)})$ by $\varphi(x) := x$ and $\varphi(y) := y + \lambda x^{-9/16}$. By [1, Proposition 3.9] we know that,

$$\ell_{16,-9}(\varphi(H)) = \varphi(\ell_{16,-9}(H)), \qquad \operatorname{en}_{16,-9}(\varphi(H)) = \operatorname{en}_{16,-9}(H)$$

and

$$\ell_{\rho_1,\sigma_1}(\varphi(H)) = \ell_{\rho_1,\sigma_1}(H) \quad \text{for all } (16,-9) < (\rho_1,\sigma_1) < (-16,9),$$

for all $H \in L^{(16)}$. Using this with $H = P_1$ and $H = Q_1$, we obtain that

$$\frac{v_{11}(\varphi(P_1))}{v_{11}(\varphi(Q_1))} = \frac{v_{10}(\varphi(P_1))}{v_{10}(\varphi(Q_1))} = \frac{m}{n} \quad \text{and} \quad v_{1,-1}(\operatorname{en}_{16,-9}(\varphi(P_1))) < 0.$$

Hence $(\varphi(P_1), \varphi(Q_1))$ is an (m,n)-pair, since $[\varphi(P_1), \varphi(Q_1)] = [P_1, Q_1] \in K^{\times}$, by [1, Proposition 3.10]. Moreover

$$\begin{aligned}\ell_{16,-9}(\varphi(P_1)) &= \varphi(\ell_{16,-9}(P_1))\\ &= \lambda_p x^{\frac{5m}{4}}(y+\lambda x^{\frac{-9}{16}})^{2m}((z+\lambda)^4-\lambda^4))^m\\ &= \lambda_p x^{\frac{11m}{16}} y^m (z+\lambda)^{2m}(z^3+4z^2\lambda+6z\lambda^2+4\lambda^3)^m,\end{aligned}$$

and so $\left(\frac{11}{16},1\right) = \frac{1}{m}\operatorname{st}_{16,-9}(\varphi(P_1))$. Now note that the inequality (5.9) in [1, Proposition 5.18] is satisfied for $a=20$, $b=6$, $l=16$, $\rho=16$ and $\sigma=-9$. Consequently, by that proposition, the (m,n)-pair $(\varphi(P_1),\varphi(Q_1))$ has a regular corner at $(11/16,1)$. Since $\gcd(11,1)=1$, by [1, Proposition 5.19] there exists a (possibly different) (m,n)-pair (P',Q') in $L^{(16)}$ such that $(11/16,1)$ is the first entry of a regular corner of type I of (P',Q'). By Proposition 3.1 we can assume that $(11/16,1)$ is the first entry of a regular corner of type I.b) of (P',Q'). Then $a=11$, $b=1$, $l=16$, $k\in\{1,2,3,4\}$, $e_k=1$ and $\{m,n\}=\{2,3\}$ in the setting of Proposition 3.1.2. Hence

$$1=(m+n)b-\frac{me_k}{k}\frac{bl-a}{e_k}=5-\frac{m}{k}5=5\frac{k-m}{k}$$

or

$$1=(m+n)b-\frac{ne_k}{k}\frac{bl-a}{e_k}=5-\frac{n}{k}5=5\frac{k-n}{k}.$$

But both equalities are evidently false for $n,m\in\{2,3\}$ and $k\in\{1,2,3,4\}$, since $5\nmid k$. □

3.5 Increasing the lower bound

Based on the tables obtained in the last sections, we begin with the study of the cases with $\max\{\deg(P),\deg(Q)\}<125$. The aim is to prove the following result:

Conjecture 3.5.1. *If (P,Q) is a counterexample to the Jacobian Conjecture, then $\max\{\deg(P),\deg(Q)\}\geq 125$.*

The following table presents all the cases under consideration. All but 3 of them have been shown to be impossible in various cited works:

A_0	(m,n)	$\max\{\deg(P),\deg(Q)\}$	Discarded?
$(4,12)$	(3,4)	64	[4]
$(4,12)$	(5,7)	112	[4]
$(5,20)$	(2,3)	75	[3, section 5]
$(5,20)$	(3,2)	75	[3, section 5]
$(7,21)$	(2,3)	84	[6]
$(8,24)$	(2,3)	96	Section 3.4.
$(8,28)$	(3,2)	108	-
$(8,32)$	(3,2)	120	[1]
$(9,24)$	(2,3)	99	-
$(9,27)$	(2,3)	108	-

Let us analyze the three remaining cases. We can apply some automorphisms reminiscent of the procedure in [1, Section 8] to their Newton Polygons in order to greatly reduce their sizes.

Proposition 3.5.2 (Case (9,27)). *If there is a counterexample to the Jacobian Conjecture in the case* $(9,27)$*, then there exist* $P,Q \in L^{(1)}$ *with* $[P,Q]=x$ *and*

$$N(P) = \{(0,0),(1,1),(6,16),(6,18),(0,18)\}$$
$$N(Q) = \{(0,0),(1,0),(9,24),(9,27),(0,27)\}$$

Proof. The corners of the polygons of P and Q are $\{(0,0),(1,0),(9,24),(9,27),(0,9)\}$ multiplied by $(m,n)=(2,3)$ respectively. The edge $\{(9,27),(0,9)\}$ is given by $y^9(xy^2-\alpha_1)^9$, because enF $=\frac{1}{9}(9,27)$ when looking at its corresponding direction. After applying the automorphism ϕ_1 with $\phi_1(x)=y$ and $\phi_1(y)=x$ and then the automorphism ϕ_2 with $\phi_2(x)=x$ and $\phi_2(y)=y+\alpha_1 x^{-2}$ we transform the corners of the polygons to $\{(0,0),(27,9),(24,9),(0,1),(-2,0)\}$, again multiplied by $(2,3)$ respectively.

The edge $\{(24,9),(0,1)\}$ is given by $y(yx^3-\alpha_2)^8$ by some α_2 (this corresponds to the edge $\{(1,0),(9,24)\}$ in the original polygon which is of this form). Apply the automorphism ϕ_3 given by $\phi_3(x) = x$ and $\phi_3(y) = y + \alpha_2 x^{-3}$ to reduce this edge to $\{(24, 9), (21, 8)\}$. Let us analyze the possibilities for the opposite vertex in the other edge containing $(21, 8)$.

To do this, set $(\rho_2, \sigma_2) = \min\{\mathrm{Succ}_P(-1, 3), \mathrm{Succ}_Q(-1, 3)\}$ Then if $\mathrm{en}_{\rho_2,\sigma_2}(P) \sim \mathrm{en}_{\rho_2,\sigma_2}(Q)$, the point $(a', b') = \frac{1}{2}\,\mathrm{en}_{\rho_2,\sigma_2}(P) = \frac{1}{3}\,\mathrm{en}_{\rho_2,\sigma_2}(Q)$ could be at any of $\{(-2, 0), (-1, 0), (1, 1), (2, 1), (4, 2), (5, 2), (7, 3), (10, 4), (13, 5)\}$. To discard all but $(5, 2)$ and $(13, 5)$, it is enough to check that there cannot exist an element F in the other cases. The corner of F which is not $(1, 1)$ must be of the form $(1, 1) + c(21 - a', 8 - b')/\gcd(21 - a', 8 - b')$ for some positive integer c, so that

$$(1, 1) + c\frac{(21 - a', 8 - b')}{\gcd(21 - a', 8 - b')} = \frac{p}{q}(21, 8).$$

Taking $v_{-8,21}$ of this equality gives $13 \gcd(21 - a', 8 - b') + c(8a' - 21b') = 0$ which implies that $21b' - 8a' | 13 \gcd(21 - a', 8 - b')$. For each of the possibilities for (a', b') above, we can discard $(1, 1)$ as (a', b') cannot lie in the diagonal, and for the rest, only $(5, 2)$ and $(13, 5$ satisfy this divisibility condition, as can be seen in the table below.

(a', b')	$21b' - 8a'$	$13 \gcd(21 - a', 8 - b')$
$(-2, 0)$	16	13
$(-1, 0)$	8	26
$(2, 1)$	5	13
$(4, 2)$	10	13
$(5, 2)$	2	26
$(7, 3)$	7	13
$(10, 4)$	4	13
$(13, 5)$	1	13

Whether we continue assuming $(a', b') \in \{(5, 2), (13, 5)\}$ or we consider instead the case $\mathrm{en}_{\rho_2,\sigma_2}(P) \not\sim \mathrm{en}_{\rho_2,\sigma_2}(Q)$, we can apply [1, Proposition 8.2] and get the existence of $k \in \mathbb{N}$ with

$$(k+1)b < a \qquad \text{and} \qquad \{\mathrm{en}_{\rho,\sigma}(P), \mathrm{en}_{\rho,\sigma}(Q)\} = \{(-k,0),(k+1,1)\}.$$

where (a,b) is one of $\{(21,8),(13,5),(5,2)\}$ and (ρ,σ) is the direction corresponding to the edge in question. In all cases this gives $k=1$ and so $\{\mathrm{en}_{\rho,\sigma}(P), \mathrm{en}_{\rho,\sigma}(Q)\}$ $= \{(-1,0),(2,1)\}$, with the same direction $(\rho,\sigma) = (-3,8)$. (Note that if we started assuming $(a',b') \in \{(5,2),(13,5)\}$ then we have reached a contradiction, as we have a different end for direction $(-3,8)$.) Since $\mathrm{st}_{-3,8}(P) = (42,16)$ and $\mathrm{st}_{-3,8}(Q) = (63,24)$, we get that $\mathrm{en}_{-3,8}(P) = (2,1)$ and $\mathrm{en}_{-3,8}(Q) = (-1,0)$. In fact, $(-3,8) \times ((42,16)-(-1,0)) \neq 0$.

For convenience, we now apply the morphism φ such that $\varphi(x) = x^{-1}$ and $\varphi(y) = x^3y$. Note that this is not an automorphism, and by the chain rule we have $[\varphi(P), \varphi(Q)] = \varphi[P,Q][\varphi(x),\varphi(y)] = -[P,Q]x$. This transforms the polygons of P and Q into

$$N(P) = \{(0,0),(1,1),(6,16),(6,18),(0,18)\}$$
$$N(Q) = \{(0,0),(1,0),(9,24),(9,27),(0,27)\}$$

as desired. □

Proposition 3.5.3 (Case (9,24)). *If there is a counterexample to the Jacobian Conjecture in the case* $(9,24)$*, then there exist* $P, Q \in L^{(1)}$ *with* $[P,Q] = x$ *and*

$$N(P) = \{(0,0),(1,1),(6,16),(6,18),(0,12)\}$$
$$N(Q) = \{(0,0),(1,0),(9,24),(9,27),(0,18)\}$$

Proof. The corners of the polygons of P and Q are $\{(0,0),(1,0),(9,24),(9,27),(0,h)\}$ multiplied by $(m,n) = (2,3)$ respectively, where $h \in \{3,6,9,12\}$ (in fact, $\mathrm{enF} = \frac{2}{3}(9,24)$ for the relevant direction). To discard 9 and 12 as possibilities, let us apply the automorphism ϕ_1 with $\phi_1(x) = y$, $\phi_1(y) = x$ and then apply Proposition 1.0.6, with $(a,b) = (24,9), (\rho,\sigma) \sim (9,h-24)$ and $(r,s) = (h,0)$. The values for $(\vartheta, \gcd(a-r,b-s), \gcd(r,s))$ for $h=12$, $h=9$ and $h=6$ are $(36,3,12)$, $(27,3,9)$ and $(6,9,6)$ respectively, showing that $h \leq 6$ and we may assume $h=6$.

The edge $\{(6,0),(24,9)\}$ is of the form $x^6(x^2y-\alpha_1)^3(x^2y-\alpha_2)^3(x^2y-\alpha_3)^3$. Apply an automorphism ϕ_2 with $\phi_2(x)=x$ and $\phi_2(y)=y+\alpha_1x^{-2}$ and we transform the corners of the polygons to $\{(0,0),(18,6),(24,9),(0,1),(-2,0)\}$, again multiplied by $(2,3)$ respectively. In fact, the edge ending at $(18,6)$ cannot begin at $(3,0)$ by Proposition 1.0.6, and we can also assume that $\alpha_2=\alpha_3$. To see this, if the three roots were different, then set $(c,d)=(12,3)$, $(a,b)=(24,9)$ and $s=\vartheta=N_1=6$ and $N_2=3$ in [2, Proposition 3.12]. By [2, Proposition 3.12 (2)], there is a linear factor in the edge with multiplicity $s=\vartheta=6$, showing that two roots must have been equal.

After doing the transformations on the edge $\{(24,9),(0,1)\}$ exactly as in the case $(9,27)$, one obtains the desired form:

$$N(P)=\{(0,0),(1,1),(6,16),(6,18),(0,12)\}$$
$$N(Q)=\{(0,0),(1,0),(9,24),(9,27),(0,18)\}$$

□

Proposition 3.5.4 (Case (8,28)). *If there is a counterexample to the Jacobian Conjecture in the case* $(8,28)$, *then there exist* $P,Q\in L^{(1)}$ *with* $[P,Q]=x^2$ *and*

$$N(P)=\{(0,0),(1,0),(8,14),(8,16)\}$$
$$N(Q)=\{(0,0),(2,1),(12,21),(12,24)\}$$

Proof. The corners of the polygons of P and Q in this case are $\{(0,0),(1,0),(8,28),(0,h)\}$ multiplied by $(m,n)=(3,2)$ respectively, where $h\in\{4,8,12,16\}$ (this time, enF $=\frac{3}{4}(8,28)$ for the relevant direction). Applying Proposition 1.0.6 as in 3.5.3, we obtain that $h=4$. After the automorphism ϕ_1 with $\phi_1(x)=y$ and $\phi_1(y)=x$, we have the polygon $\{(0,0),(4,0),(28,8),(0,1)\}$.

The edge $(4,0)-(28,8)$ is of the form $x^4(x^3y-\alpha_1)^4(x^3y-\alpha_2)^4$. Let us apply the automorphism ϕ_2 with $\phi_2(x)=x$ and $\phi_2(y)=y+\alpha_1x^{-3}$. If $\alpha_1\neq\alpha_2$, then this would give edges $(0,0)-(16,4)-(28,8)$ which is not possible. We then must have $\alpha_1=\alpha_2$ so that the polygon is reduced to $\{(0,0),(28,8),(0,1),(-3,0)\}$. The edge $\{(28,8),(1,0)\}$ must be of the form $y(x^4y-\alpha)^7$, corresponding to its form before the transformations. Apply then the automorphism ϕ_3 given by $\phi_3(x)=x$, $\phi_3(y)=y+\alpha_2x^{-3}$, reducing the edge $\{(8,28),(0,1)\}$ to $\{(7,24),(8,28)\}$. As in Proposition 3.5.2, one can analyze the possibilities for the opposite vertex (a,b) in the other edge containing $(7,24)$ and obtain that $\mathrm{Succ}_P(-1,4) = \mathrm{Succ}_Q(-1,4) = (-2,7)$. We can apply [1, Proposition 8.2] and get the existence of $k \in \mathbb{N}$ with

$$(k+1)b < a \qquad \text{and} \qquad \{\mathrm{en}_{\rho,\sigma}(P), \mathrm{en}_{\rho,\sigma}(Q)\} = \{(-k,0),(k+1,1)\}.$$

where (a,b) is one of $\{(24,7),(17,5),(10,3),(3,1)\}$ and (ρ,σ) is the direction corresponding to the edge in question, obtaining $k \in \{1,2\}$. The case $k=2$ is impossible, as the edges of P and Q would have no way of being parallel. In the case $k=1$, we can set $(\mathrm{en}_{\rho,\sigma}(Q), \mathrm{en}_{\rho,\sigma}(P)) = ((2,1),(-1,0))$. Apply the the morphism φ with $\varphi(x) = x^{-1}$ and $\varphi(y) = x^4 y$. As in Proposition 3.5.2, this is not an automorphism and the chain rule gives $[\varphi(P),\varphi(Q)] = -[P,Q]x^2$. The Newton Polygons of P and Q become, respectively

$$N(P) = \{(0,0),(1,0),(8,14),(8,16)\}$$
$$N(Q) = \{(0,0),(2,1),(12,21),(12,24)\}$$

as desired. □

This simple form for the three cases allows them to be attacked with the techniques developed in [3]. For example, consider the cases $(9,24)$ and $(9,27)$. Using the notation and arguments of [3, Section 1], we can write $P = C^2$ and $Q = C^3 + \alpha_2 C^2 + \alpha_1 C + \alpha_0 + \lambda C^{-1} + F$, where $C, F \in K((y))((x^{-1}))$ with $\deg_x(F) = -4$. We may set $P := P - \beta$ and $Q := Q - \alpha_2 P - \alpha_0$ as desired without altering the support of Q or the value of $[P,Q]$, and by using such manipulations we may assume $\alpha_2 = \alpha_1 = \alpha_0 = 0$ so that $Q = C^3 + \lambda C^{-1} + F$.

One could analyze these conditions more closely, or assume the field to be $\mathbb{C}$ and use a computer algebra system, in order to verify that such a system cannot have a solution, as it is well known that no loss of generality is incurred by assuming the field to be $\mathbb{C}$.

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