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"SOLUTION OF FRACTIONAL LINEAR AND BILINEAR TIME INVARIANT SYSTEM VIA FORMAL POWER SERIES METHODS"

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RESUMEN DE LA TESIS

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Matemáticas
SOLUTION OF FRACTIONAL LINEAR AND BILINEAR TIME INVARIANT SYSTEM VIA FORMAL POWER SERIES METHODS

The area of fractional calculus is more than three centuries old but applications have only appeared in the past few decades. Differential equations of non-integer order are known to represent certain physical processes in a more precise way than using the usual differential equations with integer order. Therefore, considering fractional calculus in the context of input-output systems can be beneficial. A useful representation of an input-output map in control theory is the Chen-Fliess functional series or Fliess operator. It can be viewed as a generalization of a Taylor series, and its algebraic nature is especially well suited for several important applications. In this thesis, a general solution for a fractional linear and bilinear time invariant system via formal power series methods and Fliess operators is presented. A mathematical model (that includes a differential equation) for an input-output linear and bilinear time invariant system is very well known, both the explicit solution and the one using formal power series. However, the question of how this system behaves when a fractional differential equation (where the derivative is of a non-integer order) has not been yet studied from the power series point of view. This thesis focuses on two specific kind of derivatives, one using Riemann-Liouville fractional derivatives and the other using Caputo fractional derivatives.

PUCP

To my parents, Leonor and Jorge;

my siblings, JL, Francois and Romina;

my husband Ivan and my lovely daughter Isabella.

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CHAPTER 1

INTRODUCTION

The main goal of this dissertation is to find a general solution for a fractional linear and bilinear time invariant system via formal power series methods. A mathematical model (that includes a differential equation) for an input-output linear and bilinear time invariant system is very well known, both the explicit solution and the one using formal power series, specifically Fliess operators [5, 9, 12, 32, 33]. However, the question of how this system behaves when a fractional differential equation (where the derivative is of a non-integer order) has not been yet studied from the power series point of view. Several authors have formulated solutions to these fractional systems using ordinary differential equations of fractional order [13–15, 29]. But a solution does not appear in the literature via power series. The main contribution of this dissertation is to address this gap in the literature, mainly the analysis of these systems governed by a fractional differential equation and solved via Fliess operators. This dissertation focuses on two specific kind of derivatives, one using Riemann-Liouville fractional derivatives and the other using Caputo fractional derivatives.

This chapter is organized as follows. Section 1.1 provides the background and motivation for the research described in this dissertation. Subsequently, in Section 1.2 the problem statement is presented. Finally, Section 1.3 outlines the structure of the document.

1.1 BACKGROUND AND MOTIVATION

Systems are use to process, modify or extract sets of data or information. An input-output system is usually illustrated by a “black box” where one set of input variables $u_1(t), u_2(t), \ldots, u_m(t)$ are applied and another set of output variables $y_1(t), y_2(t), \ldots, y_\ell(t)$ are observed. A general scheme is given in Figure 1.

A linear input-output system is such that it satisfies the property of superposition. That is, given two inputs $u_1(t)$ and $u_2(t)$ that produce outputs $y_1(t)$ and $y_2(t)$, respectively, and $\alpha, \beta \in \mathbb{R}$, then the input $\alpha u_1(t) + \beta u_2(t)$ should give the output $\alpha y_1(t) + \beta y_2(t)$. If, in addition, $u(t-T)$ gives $y(t-T)$ where $T \in \mathbb{R}$, then it is called time invariant system.

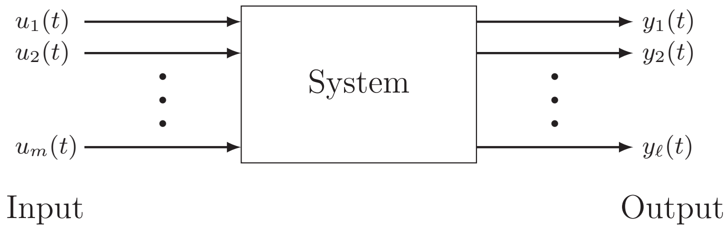

Fig. 1: Block diagram of an input-output system

A useful representation of an input-output map is the Chen-Fliess functional series or Fliess operator. It can be viewed as a generalization of a Taylor series, and its algebraic nature is especially well suited for a number of important applications [5, 12].

These operators are described by functional series which are indexed by words. First, introduce a "letter" x_i for each input $u_i(t)$ where $i = 1, 2, \cdots, m$. Also, consider a fictional input $u_0(t) := 1$ such that x_0 is the letter associated to it. A word is defined as any possible combination of letters, for example x_0x_1 is a word defined as the catenation of the letter x_0 followed by the letter x_1. In general "letters" do not commute ($x_0x_1 \neq x_1x_0$). More formally, let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet (set of all the letters) and X^* the set comprised of all words over X (including the word with no letters or empty word, $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate an $m-$input, $\ell-$output operator, F_c. Let $\mathfrak{p} \geq 1$ and $a < b$ be given. For a Lebesgue measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_\mathfrak{p}^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and define $B_\mathfrak{p}^m(R)[a, b] := \{u \in L_\mathfrak{p}^m[a, b] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. For each letter $x_i \in X$ define

$$E_{x_i}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)\, d\tau,\ i = 1, 2 \cdots, m.$$

Observe that each word $\eta \in X^*$ can be written as $\eta = x_i\bar{\eta}$ where $x_i \in X$ and $\bar{\eta} \in X^*$. Now, define inductively for each word, $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_\eta[u](t, t_0) = E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$.

The input-output operator corresponding to c is the *Fliess operator*

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0), \tag{1.1.1}$$

where $(c, \eta) \in \mathbb{R}$ denotes the value of c at $\eta \in X^*$, and is called the coefficient of η in c [5, 6].

Example 1.1.1. Observe that

$$E_{x_0}[u](t, t_0) = \int_{t_0}^{t} 1\, d\tau = t - t_0.$$

□

Example 1.1.2. When the word is $\eta = x_0^2 = x_0x_0$, the iterated integral associated to it is

$$E_{x_0^2}[u](t, t_0) = \int_{t_0}^{t} 1E_{x_0}[u](\tau, t_0)\, d\tau = \int_{t_0}^{t}\int_{t_0}^{\tau} (\tau - t_0)\, d\xi d\tau = \frac{(t - t_0)^2}{2}.$$

□

Example 1.1.3. When the word is $\eta = x_0x_1$, the iterated integral associated to it is

$$E_{x_0x_1}[u](t, t_0) = \int_{t_0}^{t} 1E_{x_1}[u](\tau, t_0)\, d\tau = \int_{t_0}^{t}\int_{t_0}^{\tau} u_1(\xi)\, d\xi d\tau.$$

□

Example 1.1.4. When the series is $c = 2x_0 + 3x_0x_1$, the input-output operator corresponding to it is the Fliess operator

$$\begin{aligned} y(t) = F_c[u](t) = F_{2x_0+3x_0x_1}[u](t) &= 2E_{x_0}[u](t,t_0) + 3E_{x_0x_1}[u](t,t_0) \\ &= 2(t-t_0) + 3\int_{t_0}^{t}\int_{t_0}^{\tau} u_1(\xi)\, d\xi d\tau. \end{aligned}$$

□

A mathematical model of an input-output system, as in Figure 1, that uses first-order differential equations, a set of inputs, outputs and state variables is called a *state-space representation*. When a state-space representation exists for an input-output system, a coordinate frame has been intrinsically assigned. On the other hand, a convenient property of any Fliess operator is that its input-output behavior is completely determined by its generating series, independent of whether or not a state-space representation is available. Therefore, the behavior of such an input-output system can be studied naturally using only combinatoric/algebraic tools [12].

A general case of a single-input single-output linear time invariant system ($m = 1$ and $\ell = 1$) is shown in Figure 2. In this case, a state-space representation is the

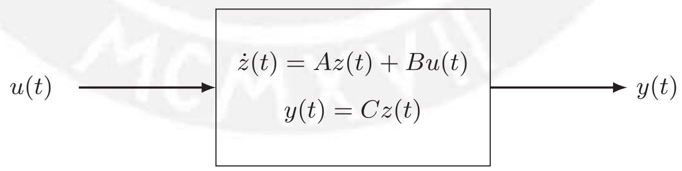

Fig. 2: Block diagram of single-input single-output linear time invariant system

first-order differential equation and output equation.

$$\begin{aligned} \dot{z} &= Az + Bu,\ z(0) = z_0 \\ y &= Cz, \end{aligned} \tag{1.1.2}$$

where $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nx1}$, $C \in \mathbb{R}^{1xn}$ and z_0 is any vector in $\mathbb{R}^n$.

The explicit solution of (1.1.2) is

$$y(t) = Cz(t) = C\left(\mathrm{e}^{At}z_0 + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau\right). \tag{1.1.3}$$

The series solution of (1.1.2) in terms of Fliess operators is obtained through iterative methods following the usual Peano-Baker formula [4, 28], namely,

$$y(t) = Cz(t) = CF_c[u](t) = C\sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t), \tag{1.1.4}$$

where $X = \{x_0, x_1\}$. The coefficients of the generating series are

$$(c,\eta) = \begin{cases} A^kB & : \eta = {x_0}^k x_1,\ k \geq 0 \\ A^k z_0 & : \eta = {x_0}^k,\ k \geq 0 \\ \ \ 0 & : \text{otherwise}. \end{cases}$$

In order to see that these two solutions (1.1.3) and (1.1.4) are equivalent, consider only the part of the series (1.1.4) related to the words of the form $\eta = {x_0}^k, k \geq 0$. Calling it y_1 and writing every term explicitly gives

$$\begin{aligned} y_1(t) &= Cz_0E_\emptyset[u](t) + CAz_0E_{x_0}[u](t) + CA^2z_0E_{{x_0}^2}[u](t) + CA^3z_0E_{{x_0}^3}[u](t) + \ldots \\ &= C\left(1 + AE_{x_0}[u](t) + A^2E_{{x_0}^2}[u](t) + A^3E_{{x_0}^3}[u](t) + \ldots\right)z_0. \end{aligned} \tag{1.1.5}$$

Observe that for any $k \geq 0$

$$E_{{x_0}^k}[u](t) = \frac{(t-t_0)^k}{k!}.$$

It is sufficient to repeat the procedure in Examples 1.1.1 and 1.1.2 inductively. Substituting the expression above into (1.1.5) yields

$$y_1(t) = C\left(1 + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots\right)z_0 = C\mathrm{e}^{At}z_0, \tag{1.1.6}$$

since by definition $\mathrm{e}^{At} = \sum_{k=0}^{\infty}(At)^k$. Now, it is easy to see that the term y_1 given on (1.1.6) appears explicitly in (1.1.3). Analogously the same identification can be made for the part of the series (1.1.4) related to the words of the form $\eta = {x_0}^k x_1, k \geq 0$ and the second term of the explicit solution (1.1.3).

The behaviour of this state-space representation is very well known [9, 10]. However, the question of how this system behaves when the differential equation (1.1.2) is modeled by a fractional differential equation where the derivative is of a non-integer order has not yet been studied from the power series point of view in terms of Fliess operators. Additional mathematical machinery given by fractional calculus is needed to properly describe a fractional differential equation since fractional integrals and several forms of fractional derivatives are available in the literature.

The area of fractional calculus is more than three centuries old, but applications have only appeared in the past few last decades. Fractional differential equations of non-integer order represent in a better way to model certain processes involving long-range dependencies, power laws, long-term memory effects, diffusion processes in semi-infinite media and thermal systems [23, 25, 29]. Basically, considering fractional derivatives in an input-output system can be beneficial for many control loops that take advantage of important properties for robustness and dynamic performance [18].

In this thesis, two approaches regarding the fractional differential equation generalized from (1.1.2) are studied. The first one considers the Riemann-Liouville fractional derivative as shown in (1.1.7), and the second one is involves the Caputo fractional derivative as shown in (1.1.8).

$$\begin{aligned} D^{\alpha} z &= Az + Bu, \ z(0) = z_0 \\ y &= Cz. \end{aligned} \tag{1.1.7}$$

$$\begin{aligned} {}^{C}D^{\alpha} z &= Az + Bu, \ z(0) = z_0 \\ y &= Cz. \end{aligned} \tag{1.1.8}$$

where $\alpha \in \mathbb{R}$ and $0 < \alpha \leq 1$. As expected when $\alpha = 1$ both systems are equivalent to the original (1.1.2).

Several authors have formulated solutions to (1.1.7) and (1.1.8) using ordinary differential equations of fractional order. In particular, Samko and Kilbas gave a detailed description of the situation for the Caputo fractional derivative [13–15, 29]. However, a solution does not appear in the literature from the power series point of view. Hence, the primary goal of this thesis is to address these specific gaps in the literature.

1.2 PROBLEM STATEMENT

The main goals of this thesis are to:

$i.$ Compute a general solution for a fractional linear and bilinear time invariant system via formal power series in terms of Fliess operators.

$ii.$ Define a fractional extension of Fliess operators in two specific cases: using Riemann-Liouville fractional derivatives and Caputo fractional derivatives.

$iii.$ Characterize a fractional extension of iterated integrals using Riemann-Liouville fractional integrals.

$iv.$ Provide for each approach independents checks of the solutions obtained and verify that they agree with the known literature.

1.3 DISSERTATION OUTLINE

This thesis is organized as follows. In Chapter 2, the fractional calculus framework is presented. First, the basic special functions used in the main definitions are reviewed. Then, the Riemann-Liouville fractional integral, the Riemann-Liouville and Caputo fractional derivatives are studied. The most important properties needed in the following chapters are presented and proved. In Chapter 3, a basic introduction to formal power series is presented. In Chapter 4, Fliess operators with fractional behaviour are considered. A characterization of a fractional extension of the iterated integrals using Riemann-Liouville fractional integrals is provided and a fractional extension of Fliess operators in two specific cases is given using Riemann-Liouville fractional derivatives and Caputo fractional derivatives. Subsequently, a general solution for a linear and bilinear time invariant system is determined using this fractional extension of Fliess operators. Separate analyses are done for each system, and independent checks of the solutions are presented. Chapter 5 summarizes the main conclusions of the thesis.

CHAPTER 2

FRACTIONAL CALCULUS

In classical calculus, derivatives and integrals are the concepts in which most of the fundamentals rely on. Moreover, one naturally extend these concepts to higher dimensions, but always taking derivatives and integrals to integer orders. It is natural to ask the question what if the order is not restricted to an integer order? Is it possible to define integrals and derivatives to arbitrary orders and still being consistent with the classical calculus? Most authors agree that this questions was first addressed by L'Hopital on a letter to Leibniz dated September 30, 1695. He posed a question about what would be the result if $n = \frac{1}{2}$ in $\frac{D^n x}{Dx^n}$, this notation was used by Leibniz to refer the nth-derivative of the linear function $f(x) = x$. Leibniz reply that it would be "an apparent paradox, from which one day useful consequences will be drawn" [22].

Following this discussion, fractional calculus was studied on formal foundations by many great mathematicians: Liouville, Abel, Riemann, Grëunwald, Euler, Laplace, Lagrange, Fourier, Letnikov, Caputo, to name a few [11, 22, 29]. Most of them proposed and used their own notation and methodology, which is described chronologically in [23]. The term *fractional derivative* was first used by Lacroix in 1819 and the first formal application of it was made by Abel in 1823 in the formulation of the *tautochrone problem* [26]. The theory includes complex structures for the derivative and integral of arbitrary order and also left and right definitions (analogously to left and right derivatives). The most common definitions of fractional integrals are the Riemann-Liouville and Grëunwald-Letnikov definitions, this thesis focuses on the Riemann-Liouville fractional integral definition since most of the other definitions are largely variations of it. For example, Caputo reformulated the more classic definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [25].

In the 20th century numerous applications and physical manifestations of fractional calculus have been found: fractional differential equations represent in a more accurate way certain processes involving long-range dependencies, power laws, long-term memory effects, diffusion process in semi-infinite media and thermal systems [23, 25, 29].

This chapter is organized as follows. Section 2.1 provides some definitions, special functions and techniques which are necessary for understanding fractional calculus. Subsequently, in Section 2.2 the Riemann-Liouville fractional integral definition is presented, including some basic examples, properties and sufficient conditions for the continuity. In Section 2.3 the Riemann-Liouville and Caputo fractional derivatives are described. Finally, Section 2.4 presents fundamental properties that will be key in the study of Chapter 4, such as linearity, composition rules, the fractional derivative of the product and composition functions, and the Laplace transform in the fractional context.

2.1 PRELIMINARIES

Some special functions which are used in connection with fractional calculus are presented first [1, 15, 22, 23, 25, 29].

2.1.1 The gamma function

The Euler gamma function is connected to the fractional calculus by its very definition, namely

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t, \; z \in \mathbb{C}. \tag{2.1.1}$$

This integral converges in the right half of the complex plane ($\mathrm{Re}(z) > 0$). For values in the left half of the complex plane, the gamma function reduce to

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \tag{2.1.2}$$

after letting $\mathrm{e}^{-t} = \lim_{n\to\infty}(1 - t/n)^n$ in (2.1.1) and performing integration by parts $n-$times. A basic property of the gamma function is the reduced formula

$$\Gamma(z+1) = z\Gamma(z) \tag{2.1.3}$$

which can be proved by integration by parts. As a consequence when $z \in \mathbb{N}$

$$\Gamma(z+1) = z!$$

with (as usual) $0! = 1$. The gamma function can also be used to extend the formula for the binomial coefficient as

$$\binom{k}{r} = \frac{\Gamma(k+1)}{\Gamma(r+1)\Gamma(k-r+1)}, \tag{2.1.4}$$

where $k, r \in \mathbb{R}$. It can be seen, in a simple way, that the gamma function is a generalization of the factorial for all positive real numbers. The graph of $\Gamma(z)$ for real values of z is shown in Figure 3.

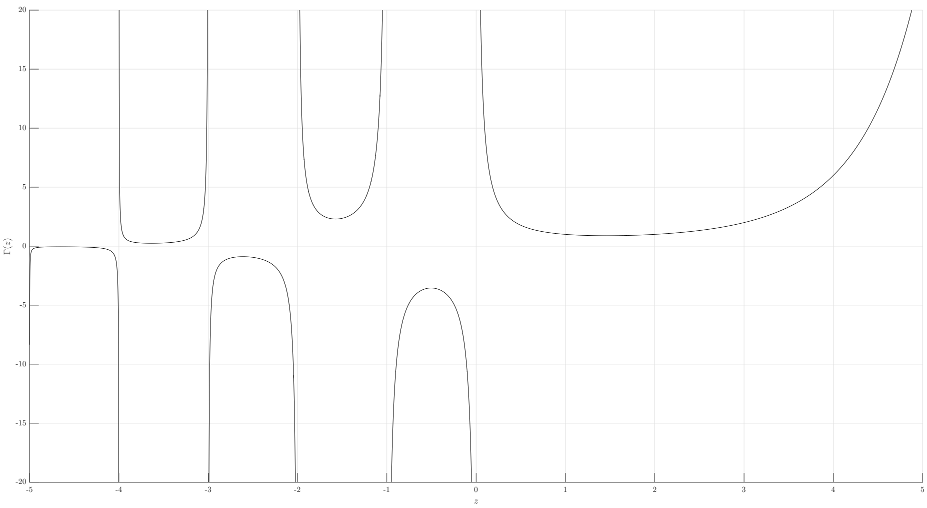

Fig. 3: The gamma function

2.1.2 The beta function

The beta function is defined by the Euler integral of the first kind

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \ \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{2.1.5}$$

This function is related to the gamma function by

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

which can be proved using the Laplace transform of the convolution product $h_{z,w}(t) = \int_0^t \tau^{z-1}(1-\tau)^{w-1}\mathrm{d}\tau$, noting that $h_{z,w}(1) = B(z,w)$. The beta function is a combination of multiples gamma functions. In certain cases it is more convenient to use as it has a similar structure to the fractional integrals and derivatives of particular polynomials.

2.1.3 The Mittag-Leffler function

The exponential function appears naturally in the solution of ordinary differential equations, analogously, its generalization given by Mittag-Leffler in 1903, plays a central role in fractional calculus.

Definition 2.1.1. The matrix **Mittag-Leffler function** is

$$\mathscr{E}_{\alpha,\beta}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k + \beta)},$$

where t is real-valued, $A \in \mathbb{R}^{n\times n}$, and α, β are positive real numbers.

This two parameter generalization was introduced by Agarwal in 1953 [21], in particular, $\mathscr{E}_{1,1}(At) = \mathrm{e}^{At}$ is the usual matrix exponential. The graph of $\mathscr{E}_{\alpha,1}(t)$ for different values of α is shown in Figure 4.

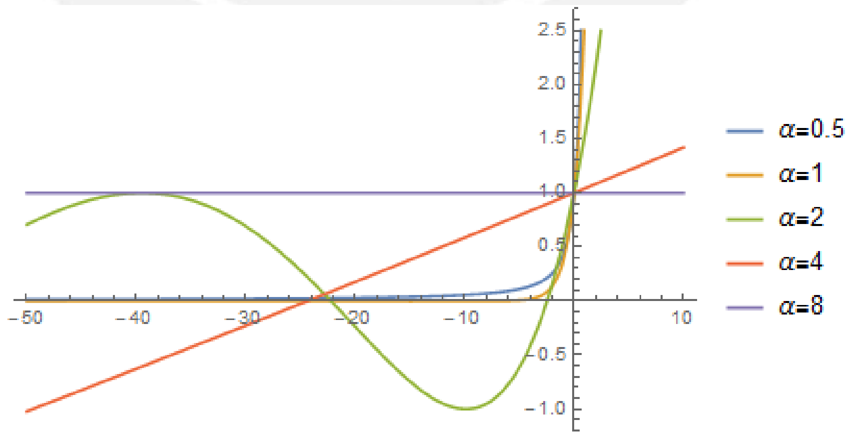

Fig. 4: The Mittag-Leffler function when $\beta = 1$ and $\alpha = 0.5, 1, 2, 4, 8.$

The next lemma gives the derivative of the Mittag-Leffler function.

Lemma 2.1.1. *Let $\alpha > 0$ and $A \in \mathbb{R}^{n\times n}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\alpha,1}(At^{\alpha}) = At^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}), , t \in \mathbb{R}.$$

Proof: By definition when $\beta = 1$

$$\mathscr{E}_{\alpha,1}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\alpha,1}(At^{\alpha}) = \frac{\alpha A t^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{2\alpha A^2 t^{2\alpha-1}}{\Gamma(2\alpha+1)} + \frac{3\alpha A^3 t^{3\alpha-1}}{\Gamma(3\alpha+1)} + \cdots .$$

Factoring and using (2.1.3), it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\alpha,1}(At^{\alpha}) = At^{\alpha-1}\left(\frac{\mathbf{I}}{\Gamma(\alpha)} + \frac{At^{\alpha}}{\Gamma(2\alpha)} + \frac{A^2t^{2\alpha}}{\Gamma(3\alpha)} + \cdots\right) = At^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}).$$

∎

2.1.4 The Laplace transform

The Laplace transform is a function transformation useful in solving linear ordinary differential equations with constant coefficients. In this thesis it would be use to derive formulas for fractional calculus and to solve fractional differential equations.

Definition 2.1.2. Let $f(t)$ be a complex-valued function in one real variable, then the **Laplace transform** of $f(t)$ is

$$F(s) = \mathscr{L}[f(t)] = \int_0^{\infty} \mathrm{e}^{-st} f(t)\mathrm{d}t.$$

The function $f(t)$ is called original and the function $F(s)$ is called the Laplace image of $f(t)$. An important property of this transformation is its linearity, given $f_k, \ \ k = 0, 1 \cdots$, which are complex-valued functions of one real variable, and complex numbers a_k, it follows that

$$\mathscr{L}\left[\sum_{k=0}^{\infty} a_k f_k(t)\right] = \sum_{k=0}^{\infty} a_k \mathscr{L}\left[f_k(t)\right] = \sum_{k=0}^{\infty} a_k F_k(s).$$

Other useful property is the image of convolution namely, given that f and g are complex-valued functions in one real variable, it follows that

$$\mathscr{L}[f(t) * g(t)] = \mathscr{L}[f(t)]\mathscr{L}[g(t)] = F(s)G(s), \tag{2.1.6}$$

where the convolutions is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau. \tag{2.1.7}$$

One final property that will play an important role later on, is the Laplace transform of a higher order derivative. Given f a complex-valued function of one real variable and $n \in \mathbb{N}$ then

$$\mathscr{L}[f^{(n)}(t)] = s^n\mathscr{L}[f(t)] - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$$

In particular when $n = 1$,

$$\mathscr{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = s\mathscr{L}[f(t)] - f(0) = sF(s) - f(0). \tag{2.1.8}$$

Example 2.1.1. Suppose $\alpha > -1$. Then by definition

$$\mathscr{L}[t^\alpha] = \int_0^\infty \mathrm{e}^{-st} t^\alpha \mathrm{d}t.$$

Letting $\tau = st$ gives

$$\mathscr{L}[t^\alpha] = \int_0^\infty \mathrm{e}^{-\tau}\left(\frac{\tau}{s}\right)^\alpha \frac{1}{s}\mathrm{d}\tau = \frac{1}{s^{\alpha+1}}\int_0^\infty \mathrm{e}^{-\tau}\tau^\alpha \mathrm{d}\tau.$$

Using Definition 2.1.1 yields

$$\mathscr{L}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

□

The next lemma is related to the Laplace transform of the Mittag-Leffler function.

Lemma 2.1.2. *Let $\alpha, \beta > 0$ and $A \in \mathbb{R}^{n\times n}$. Then*

$$\mathscr{L}[t^{\beta-1}\mathscr{E}_{\alpha,\beta}(At^{\alpha})] = (s^{\alpha}\mathbf{I} - A)^{-1}s^{\alpha-\beta}.$$

Proof:

$$\mathscr{L}[t^{\beta-1}\mathscr{E}_{\alpha,\beta}(At^{\alpha})] = \mathscr{L}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{A^k t^{\alpha k}}{\Gamma(\alpha k+\beta)}\right].$$

Using the linearity of the Laplace transform and the identity $\mathscr{L}[t^q] = \Gamma(q+1)/s^{q+1}$ gives

$$\begin{aligned}\mathscr{L}[t^{\beta-1}\mathscr{E}_{\alpha,\beta}(At^{\alpha})] &= \sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha k+\beta)}\mathscr{L}\left[t^{\alpha k+\beta-1}\right] = \sum_{k=0}^{\infty}\frac{A^k}{s^{\alpha k+\beta}}\\ &= \frac{1}{s^{\beta}}\sum_{k=0}^{\infty}\left(\frac{A}{s^{\alpha}}\right)^k = (s^{\alpha}\mathbf{I} - A)^{-1}s^{\alpha-\beta}.\end{aligned}$$

■

2.2 THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

In 1832, Liouville published several memoirs which gave two definitions, one concerning fractional derivative and the other addressing fractional integrals, however they were inconsistent with each other. Riemann also developed a fractional integration theory when he was a student, but the definition was not very clear and it caused a lot of confusion [27]. In 1869, N. Y. Sonin publish one of the first papers that consider the two approaches together in order to provide a robust definition for the fractional integral [30], however their formal structure was given in 1884 by H. Laurent [16]. The Riemann-Liouville fractional integral definition is based on the Cauchy's formula for repeated integration. The n-th integral of f based at 0 is given by

$$\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} f(\tau_n)\,\mathrm{d}\tau_n\cdots\mathrm{d}\tau_2\,\mathrm{d}\tau_1 = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}f(\tau)\,\mathrm{d}\tau \qquad (2.2.1)$$

for $n \in \mathbb{N}$ and f a Lebesgue integrable function defined in the interval $[0,t]$ and the Lebesgue integral will be used in the whole thesis, and if the Riemann integral of a function exists, both types of integrals correspond. Therefore, the analytic extension of such an integral of order $n \in \mathbb{N}$ to a real order $\alpha > 0$ is achieved by using the gamma Function instead of the factorial in (2.2.1) as follows.

Definition 2.2.1. The *Riemann-Liouville fractional integral* of order $\alpha > 0$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau,$$

for any $t \in [0, b]$ and f a Lebesgue integrable function defined in the interval $[0, t]$.

In addition $I^0 f(t) := f(t)$. Note that when $\alpha \in \mathbb{N}$ then the fractional integral is the well known integral.

Lemma 2.2.1. *The following relations hold when f, g are real-valued measurable functions on $[0, b]$ and $\alpha > 0$.*

1. $I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t) = I^{\alpha+\beta} f(t),\ \beta > 0.$
2. $I^{\alpha}(Af(t) + Bg(t)) = AI^{\alpha} f(t) + BI^{\alpha} g(t),\ A, B \in \mathbb{R}.$

Proof: The first relationship follows directly from the definition and using Dirichlet's formula, the second one follows from the linearity of the integrals. ∎

2.2.1 The right fractional integral

The concept given in Definition 2.2.1 is the so called *left sided fractional integral* because the integral is calculated at point t with the information of the function f evaluated at points to the left of it. For example, if t represents time, this definition makes sense since it uses the history of the function and future information (f evaluated in points beyond t) is not needed. In this thesis, t is considered to be the time, so only Definition 2.2.1 is needed. But in general the *right sided fractional integral* can be defined as follows

$$_{-}I_b^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau,$$

where $\alpha > 0$, $t \in [0, b]$ and f a Lebesgue integrable function defined in the interval $[t, b]$. A right sided version of Lemma 2.2.1 is easily shown to be true.

2.2.2 Examples

Example 2.2.1. Let $\alpha > 0$. By definition $I^\alpha \mathbb{1}(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{d}\tau$ where $\mathbb{1}$ is the unit step function whose value is 0 when $t < 0$ and 1 for $t > 0$. Using $\displaystyle\int_0^t (t-\tau)^{r-1}\,\mathrm{d}\tau = \frac{t^r}{r}$, $r \in \mathbb{R}$ and (2.1.3) above, gives

$$I^\alpha \mathbb{1}(t) = \frac{1}{\Gamma(\alpha)}\left(\frac{t^\alpha}{\alpha}\right) = \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

This implies that for any constant $c \in \mathbb{R}$,

$$I^\alpha c = \frac{c\, t^\alpha}{\Gamma(\alpha+1)}.$$

$\square$

Example 2.2.2. Let $\alpha > 0$ and $\beta > -1$. Then

$$\begin{aligned} I^\alpha t^\beta &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^\beta\,\mathrm{d}\tau = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \left(1-\frac{\tau}{t}\right)^{\alpha-1}\tau^\beta\,\mathrm{d}\tau \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^t \left(1-\frac{\tau}{t}\right)^{\alpha-1}\left(\frac{\tau}{t}\right)^\beta\,\mathrm{d}\tau = \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}\,\xi^\beta\,\mathrm{d}\xi. \end{aligned}$$

Using the equation (2.1.5) on the last integral above gives

$$I^\alpha t^\beta = \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1) = t^{\alpha+\beta}\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Analogously, for the right sided fractional integral, it is simple to show that

$${}_{-}I_b^\alpha (b-t)^\beta = (b-t)^{\alpha+\beta}\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

$\square$

Example 2.2.3. Let $\alpha > 1$ and f a Lebesgue integrable function on $[0,b]$. By definition $I^\alpha f(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau$. Using the equation (2.1.7) defining convolution gives

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t).$$

Finally, using $I^{\alpha-1}\mathbb{1}(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ obtained from Example 2.2.1, yields

$$I^{\alpha}f(t) = (I^{\alpha-1}\mathbb{1}(t)) * f(t).$$

□

Example 2.2.4. Let $\alpha > 0$. Then

$$\mathscr{E}_{\alpha,1}(t^{\alpha}) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} I^{\alpha k}\mathbb{1}(t),$$

since from Example 2.2.1 $I^{\alpha k}\mathbb{1}(t) = \dfrac{t^{\alpha k}}{\Gamma(\alpha k + 1)}$. □

2.2.3 Composition with the first-order derivative

In standard calculus, the Fundamental Theorem of Calculus establishes that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\tau)\,\mathrm{d}\tau = f(t),$$

where f is a continuous function defined on an open interval and $a \in \mathbb{R}$ is any point in that interval. Therefore, it is natural to seek a generalization of this concept for the first-order derivative of the Riemann-Liouville fractional integral, the next theorem replies partially this issue.

Theorem 2.2.1. *Let $\alpha > 0, n \in \mathbb{N}$ and f a continuous function on $[0, b]$. Then*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{\alpha+n} f(t) = I^{\alpha} f(t).$$

Proof: By Lemma 2.2.1 part 1, it follows that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{\alpha+n} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n} I^{\alpha} f(t).$$

Applying n times the Fundamental Theorem of Calculus, the theorem is proved. ■

Note that the case where the order of the fractional integral is more than 1 is implied in Theorem 2.2.1, in particular when $n = 1$, $\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha+1} f(t) = f(t)$. But it still remains unknown what the result is when the fractional integral has order less than 1, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(t)$. In order to address this case recall the Leibniz rule.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x + f(b(t),t)\frac{\mathrm{d}}{\mathrm{d}t}b(t) - f(a(t),t)\frac{\mathrm{d}}{\mathrm{d}t}a(t), \quad (2.2.2)$$

where $f(x,t)$ is a function such that the partial derivative of f with respect to t exists and is continuous. A similar result for the Riemann-Liouville fractional integral is shown in the next theorem.

Theorem 2.2.2. *Let $\alpha > 0$ and f a continuous function on $[0,b]$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(t) = I^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) + f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof: By definition, $I^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau$, so setting $\tau = t - x^{\beta}$ and $\beta = 1/\alpha$ gives

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_{t^{\alpha}}^{0} x^{\beta(\alpha-1)} f(t-x^{\beta})(-\beta x^{\beta-1})\,\mathrm{d}x = \frac{1}{\Gamma(\alpha+1)}\int_0^{t^{\alpha}} f(t-x^{\beta})\,\mathrm{d}x,$$

since $\mathrm{d}\tau = (-\beta x^{\beta-1})\mathrm{d}x$, $\beta\alpha = 1$ and using (2.1.3). Now, taking the first-order derivative of the expression above yields

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha+1)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^{\alpha}} f(t-x^{\beta})\,\mathrm{d}x\right).$$

Using (2.2.2) for the for differentiation under the integral sign above gives

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha+1)}\left(\int_0^{t^{\alpha}}\left(\frac{\partial}{\partial t} f(t-x^{\beta})\right)\mathrm{d}x + f(0)\Gamma(\alpha)t^{\alpha-1}\right).$$

Finally, using (2.1.3) and replacing τ,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\right)\mathrm{d}\tau + f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\\ &= I^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) + f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

■

The next lemma is also important.

Lemma 2.2.2. *Let* $\alpha > 0$ *and* f *a continuous function on* $[0, b]$*. Then*

$$I^{\alpha+1}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) = I^{\alpha} f(t) - f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof: Using $\alpha + 1$ instead of α in Theorem 2.2.2 gives

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha+1} f(t) = I^{\alpha+1}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) + f(0)\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

Applying Theorem 2.2.1 when $n = 1$ on the left side of the equality above gives

$$I^{\alpha} f(t) = I^{\alpha+1}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) + f(0)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

which completes the proof.

2.2.4 α-Continuity

Note that in Definition 2.2.1, the only property of the function $f(t)$ needed to ensure that $I^{\alpha} f(t)$ is well defined is its integrability, no other restrictions were imposed. However, in order to ensure continuity of the Riemann-Liouville fractional integral it is necessary to characterize the space of continuous functions in this context. Here it is introduced the concept of α-continuity when $0 \leq \alpha \leq 1$ [3, 29]

Definition 2.2.2. Let f be a Lebesgue measurable function defined on the interval $[0, b]$, $t_0 \in [0, b]$ and $0 \leq \alpha < 1$. The function f is called α*-continuous* in t_0 if, there exists $\lambda \in [0, 1-\alpha)$ for which $|t - t_0|^{\lambda} f(t)$ is a continuous function in t_0. In the case where $\alpha = 1$, the function is called 1-continuous in t_0 if it is continuous in t_0.

Definition 2.2.3. Let f be a Lebesgue measurable function. The function f is called α*-continuous* on $[0, b]$ if it is α-continuous for every point in $[0, b]$.

The space of all such functions described in Definition 2.2.3 is denoted by $C_{\alpha}[0, b]$. It can be proved by standard methods that $C_{\alpha}[0, b]$ is a linear space over $\mathbb{R}$. If $0 \leq \beta < \alpha \leq 1$ then $C_{\alpha}[0, b] \subset C_{\beta}[0, b]$. The following relations hold

$$C[0, b] \equiv C_1[0, b] \subset C_{\alpha}[0, b] \subset L[0, b],$$

where $C[0,b]$ and $L[0,b]$ denoted the space of all continuous and Lebesgue integrable functions over the interval $[0,b]$, respectively.

Definition 2.2.4. Let f be a function defined on the interval $[0,b]$ and $t_0 \in [0,b]$. The function f is called *integrable of order* α in t_0 if, $I^\alpha f(t_0)$ exists and is finite.

Definition 2.2.5. The function f is called *integrable of order* α on $[0,b]$ if it is integrable of order α for every point in $[0,b]$.

The space of all such a functions described in Definition 2.2.5 is denoted by $I_\alpha[0,b]$. It is easy to prove that $I_\alpha[0,b]$ is a linear space over $\mathbb{R}$, and therefore I^α can be seen as a linear operator. If $0 \leq \beta < \alpha$ then $I_\beta[0,b] \subset I_\alpha[0,b]$.

Lemma 2.2.3. *The following statements are true:*

1. *If $f \in L(0,b)$ and bounded on $[0,b]$. Then $f \in I_\alpha[0,b]$.*

2. *If $0 \leq \alpha \leq 1$ and $1-\alpha \leq \beta$, then $C_\alpha[0,b] \subset I_\beta[0,b]$.*

3. *If $f \in C[0,b]$, then $I^\alpha f \in C[0,b]$.*

Proof:

1. If $f \in L(0,b)$ and bounded on $[0,b]$. Then by Definition 2.2.1 $I^\alpha f(t_0)$ is well defined and finite for any $t_0 \in [0,b]$. Thus, by Definition 2.2.5, $f \in I_\alpha[0,b]$.

2. Consider a function f such that $f \in C_\alpha[0,b]$, then by Definition 2.2.3 there exists $\lambda \in [0, 1-\alpha)$ for which $|t-t_0|^\lambda f(t)$ is a continuous function in t_0. Moreover, it is possible to choose a λ such that $|t-t_0|^\lambda f(t)$ is bounded on $[0,b]$. Thus, $I^\beta f(t_0)$ is well defined and it is finite for any $t_0 \in [0,b]$ since $\lambda < 1-\alpha \leq \beta$. Therefore, by Definition 2.2.5, $f \in I_\beta[0,b]$ and the statement is proved.

3. The proof is quite long and beyond the scope of the present work. But it can be found in [29].

$\blacksquare$

In this thesis, all functions are assumed to be continuous, i.e., $f \in C[0,b]$, in order to ensure that the associated fractional integral, i.e., $I^\alpha f$ is well defined and continuous by Lemma 2.2.3 part 3.

2.3 FRACTIONAL DERIVATIVES

As shown in the previous section, the definition of the Riemann-Liouville fractional integral uses Cauchy's formula for integrating in an iterative manner. In a similar way, a first approach to defining a fractional derivative is to consider an iterative process. This approach was developed by Grëunwald-Letnikov. However in this thesis another approach is consider based on Definition 2.2.1 for the fractional integral. There are two approaches via this method, the first consists of perturbing the integer order by a fractional integral and then apply an integer number of derivatives. The second approach is simply to reverse the order of the operations, first apply an integer number of derivatives and then compute a fractional integral up to the required order. The following example will make more clear this procedure.

Example 2.3.1. Consider the function $f(t)$ and its first derivative $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$. Now, select an order $\alpha = 0.3$. Following the method mentioned before, there are two ways to obtain a fractional derivative of that order.

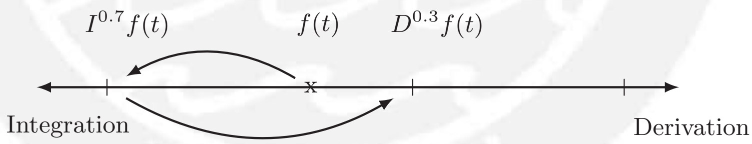

Fig. 5: Scheme for Riemann-Liouville fractional derivative

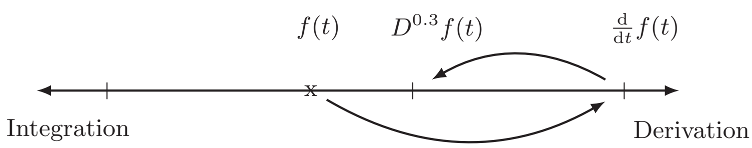

Fig. 6: Scheme for Caputo fractional derivative

The first one, shown in Figure 5, is obtained as

$$D^{0.3}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I^{0.7}f(t).$$

Clearly, the fractional integral is applied before the integer derivative, this approach corresponds to the *Riemann-Liouville fractional derivative*. On the other hand, the approach shown in Figure 6

$$D^{0.3}f(t) = I^{0.7}\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)$$

corresponds to the *Caputo fractional derivative*, where the fractional integral is applied after the integer derivative. □

2.3.1 Riemann-Liouville fractional derivative

Example 2.3.1 gives the intuition behind this approach, however the formal definition is given next.

Definition 2.3.1. The *Riemann-Liouville fractional derivative* of order $\alpha > 0$ is defined as

$$D^{\alpha}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}I^{n-\alpha}f(t).$$

for any $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$, $t \in [0,b]$ and f a Lebesgue integrable function defined in the interval $[0,t]$.

Note that when $\alpha \in \mathbb{N}$ then the fractional derivative is the standard notion of derivative. Also, $D^0 f(t) = f(t)$ since $I^0 f(t) = f(t)$.

Example 2.3.2. In particular, when $0 \leq \alpha \leq 1$ it follows that

$$D^{\alpha}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I^{1-\alpha}f(t).$$

□

Example 2.3.3. Let $0 \leq \alpha \leq 1$. Using Example 2.3.2, Example 2.2.1 and equation (2.1.3) gives

$$D^{\alpha}\mathbb{1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) = \frac{(1-\alpha)t^{-\alpha}}{\Gamma(2-\alpha)} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

This implies that for any constant $c \in \mathbb{R}$,

$$D^\alpha c = \frac{c\, t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Note that under the Riemann-Liouville fractional derivative approach, the derivative of a constant is not zero. This is the main difference as compared to the classical derivative, and this issue is pursued later in Chapter4. $\square$

Example 2.3.4. Let $\alpha > 0$ and $\beta > -1$. Then using Definition 2.3.1 and Example 2.2.2 gives

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \frac{\mathrm{d}}{\mathrm{d}t}\left(t^{n-\alpha+\beta}\right) = \frac{(n-\alpha+\beta)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} t^{n-\alpha+\beta-1},$$

where $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. Using (2.1.3) yields

$$D^\alpha t^\beta = \frac{t^{n-\alpha+\beta-1}\, \Gamma(\beta+1)}{\Gamma(\beta-\alpha+n)}.$$

In particular, when $0 \leq \alpha \leq 1$ it follows that

$$D^\alpha t^\beta = t^{\beta-\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}.$$

$\square$

It is natural to ask about the relationship between the Riemann-Liouville fractional integral and derivative, it is expected that Fundamental Theorem of Calculus should hold in some way. The next lemma addresses this issue.

Lemma 2.3.1. *Let $\alpha > 0$, then*

$$D^\alpha I^\alpha f(t) = f(t).$$

Proof: By definition

$$D^\alpha I^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}(I^\alpha f(t)),$$

where $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. Using Lemma 2.2.1 part 1 gives

$$D^\alpha I^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^n f(t)) = f(t),$$

since $n \in \mathbb{N}$. The Fundamental Theorem of Calculus was used in the last step. ■

It is interesting to note that Lemma 2.3.1 established that the Riemann-Liouville differentiation operator is a left inverse of the Riemann-Liouville integration operator of the same order. The inverse formula does not hold, i.e., $I^\alpha D^\alpha f(t) \neq f(t)$ in general the correct relationship is as follows.

Theorem 2.3.1. *Let* $0 \leq \alpha \leq 1$*, then*

$$f(t) = I^\alpha D^\alpha f(t) + I^{1-\alpha} f(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof: Using $I^{1-\alpha} f(t)$ instead of $f(t)$ in Theorem 2.2.2 gives

$$\frac{\mathrm{d}}{\mathrm{d}t} I^\alpha I^{1-\alpha} f(t) = I^\alpha \left(\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} f(t) \right) + I^{1-\alpha} f(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Applying Lemma 2.2.1 part 1, the Fundamental Theorem of Calculus and Definition 2.3.1 yields

$$f(t) = I^\alpha D^\alpha f(t) + I^{1-\alpha} f(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

■

2.3.2 Caputo fractional derivative

The next approach is based also on the Riemann-Liouville fractional integral.

Definition 2.3.2. The *Caputo fractional derivative* for any $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$, $t \in [0, b]$, and n times differentiable function f is defined by

$${}^C D^\alpha f(t) = I^{n-\alpha} \left(\frac{\mathrm{d}^n f}{\mathrm{d}t^n} \right) (t).$$

Note that when $\alpha \in \mathbb{N}$ then the fractional derivative is the standard derivative. In addition ${}^C D^0 f(t) = D^0 f(t) = f(t)$.

Example 2.3.5. When $0 \leq \alpha \leq 1$ it follows that

$${}^C D^\alpha f(t) = I^{1-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} f(t) \right).$$

□

Example 2.3.6. Let $0 \leq \alpha \leq 1$. Using Example 2.3.5 gives

$$^{C}D^{\alpha}\,\mathbb{1}(t) = I^{1-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{1}(t)\right) = 0.$$

This implies that for any constant $c \in \mathbb{R}$,

$$^{C}D^{\alpha}c = 0.$$

Unlike the first approach, the Caputo fractional derivative of a constant is zero. This property is one of the main differences with the Riemann-Liouville fractional derivative approach (see Example 2.3.3). □

Example 2.3.7. Let $\alpha > 0$. Then using Definition 2.3.2 and Example 2.2.2 gives

$$^{C}D^{\alpha}t^{\beta} = I^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}t^{\beta}\right) = \begin{cases} t^{\beta-\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} & : \beta \geq n \\ 0 & : \beta < n \text{ and } \beta \in \mathbb{N} \\ \text{not defined} & : \text{otherwise,} \end{cases}$$

where $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. Note that in Definition 2.3.2, the function $f(t)$ is required to be n times differentiable in order to ensure that the Caputo derivative is well defined. In this case, when β is non integer and $\beta < n$, it is obviously not n times differentiable, and therefore it is not possible to take the fractional derivative of order α. □

2.3.3 The right fractional derivative

The concept given in Definition in 2.3.1 is the so called *left sided Riemann-Liouville fractional derivative* because the left sided fractional integral is used. But in general the *right sided Riemann-Liouville fractional derivative* can be defined as follows

$$_{-}D_b^{\alpha}f(t) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(_{-}I_b^{n-\alpha}\right)f(t),$$

where $n-1<\alpha\leq n$ with $n\in\mathbb{N}$, $t\in[0,b]$ and f a Lebesgue integrable function defined in the interval $[0,t]$. Note that this definition is analogous to Definition 2.3.1 but using the right sided fractional integral introduced in Subsection 2.2.1.

2.4 PROPERTIES

In this section key properties to be used in Chapter 4 are presented: the linearity of fractional derivatives, some rules for the composition of fractional derivatives and integrals, the fractional version of the Leibniz rule (the derivative of the product of functions), the fractional version of the chain rule (the derivative of composite functions), and the use of the Laplace transform in the fractional setting. The final theorem will give the relationship between the Riemann-Liouville and Caputo fractional derivatives.

2.4.1 Linearity

Lemma 2.4.1. *Let $A,B\in\mathbb{R}$, f and g real-valued measurable functions on $[0,b]$ and $\alpha>0$, such that ${}^{C}D^{\alpha}f(t)$ and ${}^{C}D^{\alpha}g(t)$ are well defined. Then,*

1. $D^{\alpha}(Af(t)+Bg(t))=AD^{\alpha}f(t)+BD^{\alpha}g(t)$
2. ${}^{C}D^{\alpha}(Af(t)+Bg(t))=A\,{}^{C}D^{\alpha}f(t)+B\,{}^{C}D^{\alpha}g(t)$.

Proof: Both relationships follow from the linearity of the fractional integrals and integer derivatives. ■

2.4.2 Composition

In this subsection, the focus is on the composition of Riemann-Liouville fractional integral and derivatives. First, the integer derivative of the fractional derivative is analyzed.

Theorem 2.4.1. *Let $\alpha>0, m\in\mathbb{N}$ and f a continuous function on $[0,b]$. Then*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}D^{\alpha}f(t)=D^{\alpha+m}f(t).$$

Proof: By Definition 2.3.1, it follows that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}D^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}I^{n-\alpha}f(t)\right) = \frac{\mathrm{d}^{n+m}}{\mathrm{d}t^{n+m}}I^{n-\alpha}f(t),$$

where $n-1<\alpha\leq n$ with $n\in\mathbb{N}$. Letting $\beta=\alpha+m$ and $p=n+m$, note that $p-1<\beta\leq p$. Substituting above gives

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}D^\alpha f(t) = \frac{\mathrm{d}^p}{\mathrm{d}t^p}I^{p-\beta}f(t) = D^\beta f(t) = D^{\alpha+m}f(t).$$

■

Next, the fractional derivative of the fractional integral is analyzed.

Lemma 2.4.2. *Let* $\alpha,\beta>0$*, then*

$$D^\beta I^\alpha f(t) = I^{\alpha-\beta}f(t).$$

Proof: There is two cases: If $\alpha\geq\beta$, then using Lemma 2.2.1, part 1 it follows that

$$D^\beta I^\alpha f(t) = D^\beta(I^\beta I^{\alpha-\beta}f(t)).$$

Now, using Lemma 2.3.1 gives

$$D^\beta I^\alpha f(t) = I^{\alpha-\beta}f(t).$$

On the other hand, if $\alpha<\beta$, then by Definition 2.3.1

$$D^\beta I^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(I^{n-\beta}(I^\alpha f(t))\right),$$

where $n-1<\beta\leq n$ with $n\in\mathbb{N}$. Assume that $m-1<\beta-\alpha\leq m$, then $n\leq m$. Now, using Lemma 2.2.1, part 1 and the commutative property of the integral derivatives, it follows that

$$D^\beta I^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}I^{n-\beta+\alpha}f(t) = \left(\frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{\mathrm{d}^{n-m}}{\mathrm{d}t^{n-m}}\right)\left(I^{n-m}I^{m-\beta+\alpha}f(t)\right).$$

Thus, using Lemma 2.3.1 and Definition 2.3.1 gives

$$D^{\beta}I^{\alpha}f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}I^{m-(\beta-\alpha)}f(t) = D^{\beta-\alpha}f(t).$$

Both cases lead to the same conclusion, which proves the theorem. ■

Note that $\alpha = \beta$ in Theorem 2.4.2 leads to the same case as in Lemma 2.3.1.

Next, the fractional integral of the fractional derivative is analyzed when both orders are less than 1.

Theorem 2.4.2. *Let* $0 \leq \alpha, \beta \leq 1$*, then*

$$I^{\alpha}D^{\beta}f(t) = I^{\alpha-\beta}f(t) - I^{1-\beta}f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof: There are two cases: If $\alpha \geq \beta$, then using Lemma 2.2.1, part 1 and Theorem 2.3.1 gives

$$I^{\alpha}D^{\beta}f(t) = (I^{\alpha-\beta}I^{\beta})D^{\beta}f(t) = I^{\alpha-\beta}\left(f(t) - I^{1-\beta}f(0)\frac{t^{\beta-1}}{\Gamma(\beta)}\right).$$

Applying Lemma 2.2.1, part 2,

$$I^{\alpha}D^{\beta}f(t) = I^{\alpha-\beta}f(t) - \left(\frac{I^{1-\beta}f(0)}{\Gamma(\beta)}\right)I^{\alpha-\beta}t^{\beta-1}.$$

Then using Example 2.2.2, gives

$$I^{\alpha}D^{\beta}f(t) = I^{\alpha-\beta}f(t) - \left(\frac{I^{1-\beta}f(0)}{\Gamma(\beta)}\right)\left(\frac{\Gamma(\beta)t^{\alpha-1}}{\Gamma(\alpha)}\right) = I^{\alpha-\beta}f(t) - I^{1-\beta}f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

On the other hand, if $\alpha < \beta$, then $I^{\alpha}f(t) = D^{\beta-\alpha}I^{\beta}f(t)$ using Theorem 2.4.2 since $\beta - \alpha > 0$. Therefore, it follows that

$$I^{\alpha}D^{\beta}f(t) = D^{\beta-\alpha}I^{\beta}D^{\beta}f(t).$$

Using Theorem 2.3.1, Theorem 2.4.1, part 1 and Example 2.3.4 gives

$$\begin{aligned} I^\alpha D^\beta f(t) &= D^{\beta-\alpha}\left(f(t) - I^{1-\beta} f(0)\frac{t^{\beta-1}}{\Gamma(\beta)}\right) = D^{\beta-\alpha} f(t) - \left(\frac{I^{1-\beta} f(0)}{\Gamma(\beta)}\right) D^{\beta-\alpha} t^{\beta-1} \\ &= D^{\beta-\alpha} f(t) - \left(\frac{I^{1-\beta} f(0)}{\Gamma(\beta)}\right)\left(\frac{\Gamma(\beta) t^{\alpha-1}}{\Gamma(\alpha)}\right) = D^{\beta-\alpha} f(t) - I^{1-\beta} f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

Both cases lead to the same conclusion, which proves the theorem. ■

Note that $\alpha = \beta$ in Theorem 2.4.2 leads to the same case as in Theorem 2.3.1. The last case to be analyze is the composition of two fractional derivatives.

Theorem 2.4.3. *Let* $0 \leq \alpha, \beta \leq 1$*, then*

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t) - I^{1-\beta} f(0)\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

Proof: Using Example 2.3.2 for $D^\beta f(t)$ and Theorem 2.4.2, it follows that

$$D^\alpha D^\beta f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha}(D^\beta f(t))\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha-\beta} f(t) - I^{1-\beta} f(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right).$$

Since the integer derivative is linear, applying Definition 2.3.1 gives

$$\begin{aligned} D^\alpha D^\beta f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha-\beta} f(t) + \left(\frac{I^{1-\beta} f(0)}{\Gamma(1-\alpha)}\right)\frac{\mathrm{d}}{\mathrm{d}t} t^{-\alpha} = D^{\alpha+\beta} f(t) - \left(\frac{I^{1-\beta} f(0)}{\Gamma(1-\alpha)}\right)\alpha t^{-\alpha-1} \\ &= D^{\alpha+\beta} f(t) - I^{1-\beta} f(0)\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}, \end{aligned}$$

where (2.1.3) is used in the last step. ■

Note that in contrast to the Riemann-Liouville fractional integral which commutes (Lemma 2.2.1 part 1), in general, the Riemann-Liouville fractional derivative operator does not commute ($D^\alpha D^\beta f(t) \neq D^\beta D^\alpha f(t)$, for arbitrary $\alpha, \beta > 0$).

Lemma 2.4.3. *Let* $0 \leq \alpha, \beta \leq 1$*. If* $D^\alpha D^\beta f(t) = D^\beta D^\alpha f(t)$*, then* $\alpha = \beta$ *or* $I^{1-\alpha} f(0) = I^{1-\beta} f(0) = 0$*.*

Proof: Consider

$$D^\beta D^\alpha f(t) = D^{\alpha+\beta} f(t) - I^{1-\alpha} f(0)\frac{t^{-\beta-1}}{\Gamma(-\beta)}.$$

Comparing it with the expression given in Theorem 2.4.3, it follows immediately that if $D^\alpha f(t) = D^\beta f(t)$, then

$$I^{1-\beta} f(0) \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} = I^{1-\alpha} f(0) \frac{t^{-\beta-1}}{\Gamma(-\beta)}.$$

In order for the expression above to be true for any $t > 0$, the following must hold

$$I^{1-\alpha} f(0) = I^{1-\beta} f(0) = 0.$$

■

Example 2.4.1. Let $0 \leq \alpha, \beta \leq 1$. Then

$$D^\alpha D^\beta t^\beta = D^\alpha(\Gamma(\beta+1)) = \frac{\Gamma(\beta+1)t^{-\alpha}}{\Gamma(1-\alpha)}$$

since $D^\beta t^\beta = \Gamma(\beta+1)$ from Example 2.3.4 and using Example 2.3.3 in the last step. On the other hand,

$$D^\beta D^\alpha t^\beta = D^\beta \left(t^{\beta-\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} D^\beta t^{\beta-\alpha} = \frac{\Gamma(\beta+1)t^{-\alpha}}{\Gamma(1-\alpha)}$$

applying Example 2.3.4 two times. Clearly, $D^\alpha D^\beta t^\beta = D^\beta D^\alpha t^\beta$. This result agrees with Lemma 2.4.3 since

$$I^{1-\alpha} t^\beta |_{t=0} = t^{1-\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(2-\alpha+\beta)} \bigg|_{t=0} = 0$$

and

$$I^{1-\beta} t^\beta |_{t=0} = t\Gamma(\beta+1)|_{t=0} = 0,$$

using Example 2.2.2 in both cases. An important observation is that this condition is not equivalent to taking fractional derivatives. Observe that

$$D^\alpha t^\beta |_{t=0} = t^{-\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(1-\alpha+\beta)} \bigg|_{t=0} = 0$$

and

$$D^\beta t^\beta |_{t=0} = \Gamma(\beta+1)|_{t=0} = \Gamma(\beta+1) \neq 0,$$

using Example 2.3.4 in both cases. Therefore, $D^\alpha t^\beta|_{t=0} \neq D^\beta t^\beta|_{t=0}$. □

2.4.3 Fractional derivative of the product of two functions

The fractional version of the Leibniz Rule, the derivative of the product of functions, is given next.

Theorem 2.4.4. *Let $\alpha > 0$ and f and g be smooth functions on $[0, t]$. Then,*

$$D^\alpha(fg)(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^{\alpha-k} f(t)) \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} g(t) \right).$$

The proof is nontrivial and can be found in [25]. An important remark is that this fractional version of the Leibniz Rule has, in general, an infinite number of terms in the summation. Furthermore, it is known that this characteristic is intrinsic of derivatives of non-integer orders [31].

2.4.4 Fractional derivative of the composition of two functions

The fractional version of the chain rule, the derivative of the composition of two functions, is given next. The results in this subsection are largely based on [24], Another approach using the Faá di Bruno formula can be found in [25].

Theorem 2.4.5. *Let $\alpha > 0$ and f and g be smooth functions on $[0, b]$. Then,*

$$D^\alpha(f \circ g)(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{r=0}^{k} \binom{k}{r} (-g(t))^r D^\alpha g^{k-r}(t) \right) \frac{\mathrm{d}^k}{\mathrm{d}g^k} f(g(t)).$$

Proof: The claim follows from the chain rule for fractional calculus introduced by Osler [24], using equations (3.4)-(3.5) in [24] with $z = t$, $g(t) = t$ and $F(t, w) = 1$. ■

Note that similar to the identity in Theorem 2.4.4, the chain rule for fractional calculus has, in general, infinite number of terms in the summation.

Example 2.4.2. In the case where $\alpha = 1$ in Theorem 2.4.5, it follows that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f\circ g)(t) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r\frac{\mathrm{d}}{\mathrm{d}t}g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k}f(g(t))\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)f(g(t)) + \left(\frac{\mathrm{d}}{\mathrm{d}t}g(t) - g(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}}{\mathrm{d}g}f(g(t))\\
&+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}g^2(t) - 2g(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t) + g^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}^2}{\mathrm{d}g^2}f(g(t))\\
&+\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}g^3(t) - 3g(t)\frac{\mathrm{d}}{\mathrm{d}t}g^2(t) + 3g^2(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t) - g^3(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t)\right)\frac{\mathrm{d}^3}{\mathrm{d}g^3}f(g(t))\\
&+\cdots .
\end{aligned}
$$

Since for $k \in \mathbb{N}$ with $k > 1$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}g^k(t) = \sum_{r=1}^{k-1}\binom{k}{r}(-g(t))^r\frac{\mathrm{d}}{\mathrm{d}t}g^{k-r}(t),
$$

[8] and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}(t) = 0$ for $t > 0$, all the terms are zero except the one corresponding to $k = 1$. This term gives the usual chain rule. □

The next result follows directly as a consequence of Theorem 2.4.5 when the order of the fractional derivative is an integer. Also, this result is the well known formula that appears in [8].

Corollary 2.4.1. *If* $\alpha = N \in \mathbb{N}$ *then*

$$
\frac{\mathrm{d}^N}{\mathrm{d}t^N}(f\circ g)(t) = \sum_{k=0}^{N}\frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r\frac{\mathrm{d}^N}{\mathrm{d}t^N}g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k}f(g(t)).
$$

Another, interesting special case of Theorem 2.4.5 is when the function f is a polynomial. Let P_q denote the class of polynomials with real coefficients and having degree q.

Corollary 2.4.2. *Suppose* $f \in P_q$ *then*

$$
D^\alpha(f\circ g)(t) = \sum_{k=0}^{q}\frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}(-g(t))^r D^\alpha g^{k-r}(t)\right)\frac{\mathrm{d}^k}{\mathrm{d}g^k}f(g(t)).
$$

Proof: It follows directly as a consequence of Theorem 2.4.5 since $\frac{\mathrm{d}^k}{\mathrm{d}g^k}f(g(t)) = 0$

when $k > q$. ■

Example 2.4.3. In the case where $f \in P_1$ in Corollary 2.4.2, it follows that

$$D^\alpha(f \circ g)(t) = (D^\alpha \mathbb{1}(t))f(g(t)) + (D^\alpha g(t) - g(t)D^\alpha \mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}g}f(g(t)).$$

This result would be very useful in Chapter 4. □

2.4.5 Laplace Transform

In this subsection, the Laplace transform of the fractional integral and derivatives will be given. Also, some examples and a theorem about the relationship between the Riemann-Liouville and Caputo fractional derivatives is described. This will be useful in Chapter 4.

Theorem 2.4.6. *Let $\alpha > 0$. Then,*

$$\mathscr{L}[I^\alpha f(t)] = \frac{F(s)}{s^\alpha}.$$

Proof: Following Example 2.2.3,

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t).$$

Applying the Laplace transform and using property (2.1.6) gives

$$\mathscr{L}[I^\alpha f(t)] = \frac{1}{\Gamma(\alpha)}\mathscr{L}[t^{\alpha-1}]\mathscr{L}[f(t)].$$

Using Example 2.1.1 yields

$$\mathscr{L}[I^\alpha f(t)] = \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)}{s^\alpha}\right)F(s) = \frac{F(s)}{s^\alpha}.$$

■

Theorem 2.4.7. *Let $0 \leq \alpha \leq 1$. Then,*

$$\mathscr{L}[D^\alpha f(t)] = s^\alpha F(s) - I^{1-\alpha}f(0).$$

Proof: Applying the Laplace transform as in Example 2.3.2 gives

$$\mathscr{L}[D^{\alpha} f(t)] = \mathscr{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} f(t)\right].$$

Using property (2.1.8), it follows that

$$\mathscr{L}[D^{\alpha} f(t)] = s\mathscr{L}[I^{1-\alpha} f(t)] - I^{1-\alpha} f(0).$$

Also, from Definition 2.4.6

$$\mathscr{L}[D^{\alpha} f(t)] = s\left(\frac{F(s)}{s^{1-\alpha}}\right) - I^{1-\alpha} f(0) = s^{\alpha} F(s) - I^{1-\alpha} f(0).$$

■

Theorem 2.4.8. *Let* $0 \leq \alpha \leq 1$. *Then,*

$$\mathscr{L}[{}^{C}D^{\alpha} f(t)] = s^{\alpha} F(s) - s^{\alpha-1} f(0).$$

Proof: Applying the Laplace transform as in Example 2.3.5 gives

$$\mathscr{L}[{}^{C}D^{\alpha} f(t)] = \mathscr{L}\left[I^{1-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right)\right].$$

Using Theorem 2.4.6 gives

$$\mathscr{L}[{}^{C}D^{\alpha} f(t)] = \frac{1}{s^{1-\alpha}} \mathscr{L}\left[\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right].$$

Using property (2.1.8) above, one must concludes that

$$\mathscr{L}[{}^{C}D^{\alpha} f(t)] = \frac{sF(s) - f(0)}{s^{1-\alpha}} = s^{\alpha} F(s) - s^{\alpha-1} f(0).$$

■

The final theorem of this section gives the relationship between the Riemann-Liouville and Caputo fractional derivatives. A new proof, different from the classical one on [7], is presented here using the Laplace transform.

Theorem 2.4.9. *Let* $0 \leq \alpha \leq 1$. *Then,*

$$^{C}D^{\alpha} f(t) = D^{\alpha}(f(t) - f(0)).$$

Proof: Define $g(t) = f(t) - f(0)$. By Definition 2.4.7,

$$\mathscr{L}[D^{\alpha} g(t)] = s^{\alpha} G(s) - I^{1-\alpha} g(0). \tag{2.4.1}$$

On the other hand,

$$G(s) = \mathscr{L}[f(t) - f(0)] = F(s) - f(0)\mathscr{L}[1] = F(s) - \frac{f(0)}{s}, \tag{2.4.2}$$

since $\mathscr{L}[1] = 1/s$ using Example 2.1.1. Also,

$$I^{1-\alpha} g(t) = I^{1-\alpha} f(t) - I^{1-\alpha} f(0).$$

Thus, when $t = 0$,

$$I^{1-\alpha} g(0) = I^{1-\alpha} f(0) - I^{1-\alpha} f(0) = 0. \tag{2.4.3}$$

Substituting (2.4.2) and (2.4.3) in (2.4.1) gives

$$\mathscr{L}[D^{\alpha} g(t)] = s^{\alpha}\left(F(s) - \frac{f(0)}{s}\right) - 0 = s^{\alpha} F(s) - s^{\alpha-1} f(0).$$

Finally, from Definition 2.4.8, $\mathscr{L}[^{C}D^{\alpha} f(t)] = s^{\alpha} F(s) - s^{\alpha-1} f(0)$, thus

$$\mathscr{L}[D^{\alpha} g(t)] = \mathscr{L}[^{C}D^{\alpha} f(t)].$$

Taking the inverse Laplace transform on each side yields

$$^{C}D^{\alpha} f(t) = D^{\alpha}(f(t) - f(0)),$$

which completes the proof. ■

CHAPTER 3

FORMAL POWER SERIES AND FLIESS OPERATORS

This chapter presents some elements from the fundamental theory of formal power series. The treatment relies heavily on [2]. Formal power series appear naturally in the context of language theory; therefore, the terminology of this subject will be adopted. The definition of formal languages and formal power series are introduced first. Then, the solutions of a single-input single-output linear and bilinear time invariant system are exhibited from the power series point of view in terms of Fliess operators.

3.1 FORMAL LANGUAGES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words of length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. A *language* is any subset of X^*.

Definition 3.1.1. The **catenation product** is the associative mapping

$$\begin{aligned} \mathscr{C} \ : \ & X^* \times X^* \to X^* \\ & (\eta, \xi) \mapsto \eta\xi. \end{aligned}$$

Clearly, for any $\eta, \xi, \nu \in X^*$ it holds that

$$(\eta\xi)\nu = (\eta\xi)\nu.$$

Also, the empty word $\emptyset$ is the identity element for $\mathscr{C}$ since

$$\eta\emptyset = \emptyset\eta = \eta, \ \forall \eta \in X^*.$$

The triple $(X^*, \mathscr{C}, \emptyset)$ is a *free monoid* of X, the characteristic of *free* refers to the fact that there are no relationship between the letters. For example, the letter x_1 cannot represent the letter x_2 or the word x_2x_3.

Definition 3.1.2. Let $(M, \Box, e)$ and $(M', \Box', e')$ be two arbitrary monoids. A mapping $\rho : M \to M'$ is called a **morphism** if

$$\rho(\eta \Box \xi) = \rho(\eta) \Box' \rho(\xi), \ \forall \eta, \xi \in M,$$

where $\rho(e) = e'$. When ρ is bijective it is called an **isomorphism**.

Note that any mapping $\rho : X \to M'$ can be uniquely extended to a morphism $\rho : X^* \to M'$ by letting

$$\rho(x_{i_k} x_{i_{k-1}} \cdots x_{i_1}) = \rho(x_{i_k}) \Box' \rho(x_{i_{k-1}}) \Box' \cdots \Box' \rho(x_{i_1}).$$

If ρ is injective, i.e, $\rho(\eta) = \rho(\xi)$ implies $\eta = \xi, \ \forall \eta, \xi \in X^*$, then ρ is called a *coding* of X^*.

3.1.1 Formal power series

Given the alphabet $X = \{x_0, x_1, \ldots, x_m\}$ and a finite $\ell \in \mathbb{N}$, a *formal power series* in X is any mapping of the form

$$c : X^* \to \mathbb{R}^\ell.$$

The value of c for a specific word $\eta \in X^*$ is denoted by (c, η) and is called the *coefficient* of η in c. Typically, c is represented as the formal sum

$$c = \sum_{\eta \in X^*} (c, \eta) \eta.$$

The coefficient $(c, \emptyset)$ is referred to as the *constant term*. When the constant term is zero, c is called *proper*. The *support* of c is the language

$$\mathrm{supp}(c) = \{\eta : (c, \eta) \neq 0\}.$$

A series $\hat{c}$ is said to be a *subseries* of c if $\mathrm{supp}(\hat{c}) \subseteq \mathrm{supp}(c)$ and $(\hat{c}, \eta) = (c, \eta), \ \forall \eta \in \mathrm{supp}(\hat{c})$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. In addition, the set of all series with finite support is denoted by $\mathbb{R}^\ell \langle X \rangle$. Its elements are called *polynomials*.

The sets $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $\mathbb{R}^\ell\langle X\rangle$ have a considerable algebraic structure, each admits a vector space structure over $\mathbb{R}$. If $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, their *sum* is defined by

$$c + d = \sum_{\eta\in X^*}(c+d,\eta)\eta = \sum_{\eta\in X^*}((c,\eta)+(d,\eta))\eta,$$

and their *scalar multiplication* is given by

$$\alpha c = \sum_{\eta\in X^*}(\alpha c,\eta)\eta = \sum_{\eta\in X^*}\alpha(c,\eta)\eta, \forall\alpha\in\mathbb{R}.$$

The set of formal power series with these two operations forms an $\mathbb{R}$–vector space.

Definition 3.1.3. The **Cauchy product** of two series $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is

$$cd = \sum_{\eta\in X^*}(cd,\eta)\eta = \sum_{\eta\in X^*}\sum_{\xi\nu=\eta}(c,\xi)(d,\nu)\eta.$$

When $\ell = 1$, the $\mathbb{R}$–vector space $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ forms a ring, an $\mathbb{R}$–algebra and a module over the ring $\mathbb{R}^\ell\langle X\rangle$ using the Cauchy product.

3.2 FLIESS OPERATORS

As explained in Chapter 1, a series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ can be formally associated with an $m-$input, $\ell-$output operator Fliess operator

$$F_c[u](t) = \sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t,t_0),$$

when the inputs are measurable functions.

In order to solve an input-output linear or non linear system using formal power series, a Fliess operator representation is needed. This focuses mainly on two input-output time invariant systems, namely the linear and bilinear cases. When $m = 1$ and $\ell = 1$, the systems are called single-input single-output. A single-input single-output linear and bilinear time invariant system are shown in Figure 2 and 7, respectively. The following subsections explain how to obtain the known solutions for both systems obtained via iterated methods following the usual Peano-Baker formula [4, 28].

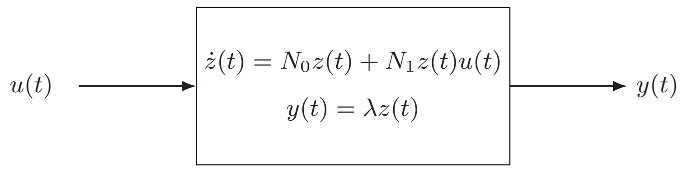

Fig. 7: Block diagram of single-input single-output bilinear time invariant system

3.2.1 On linear time invariant systems

As shown in Chapter 1, the first-order differential equation associated to this system is given in (1.1.2). Specifically the state equation is

$$\dot{z} = Az + Bu, \; z(0) = z_0, \tag{3.2.1}$$

where $A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nx1}$ and z_0 is any vector in $\mathbb{R}^n$. This equation can be written in integral form using the fundamental theorem of calculus, i.e, $\int_0^t \dot{z}(t)dt = z(t) - z(0)$, as

$$z(t) = z_0 + A \int_0^t z(\tau)d\tau + B \int_0^t u(\tau)d\tau. \tag{3.2.2}$$

Now substitute the expression above for $z(t)$ in the term $A \int_0^t z(\tau)d\tau$ of (3.2.2). This yields

$$\begin{aligned} z(t) &= z_0 + A \int_0^t \left(z_0 + A \int_0^\tau z(\tau_1)d\tau_1 + B \int_0^\tau u(\tau_1)d\tau_1 \right) d\tau + B \int_0^t u(\tau)d\tau \\ &= z_0 + Az_0 \int_0^t d\tau + A^2 \int_0^t \int_0^\tau z(\tau_1)d\tau_1 d\tau + AB \int_0^t \int_0^\tau u(\tau_1)d\tau_1 d\tau + B \int_0^t u(\tau)d\tau. \end{aligned}$$

Repeating the procedure in the term $A^2\int_0^t\int_0^\tau z(\tau_1)d\tau_1 d\tau$ above, gives

$$\begin{aligned}
z(t) &= z_0 + Az_0\int_0^t d\tau + A^2\int_0^t\int_0^\tau\left(z_0 + A\int_0^{\tau_1} z(\tau_2)d\tau_2 + B\int_0^{\tau_1} u(\tau_2)d\tau_2\right)d\tau_1 d\tau \\
&\quad + AB\int_0^t\int_0^\tau u(\tau_1)d\tau_1 d\tau + B\int_0^t u(\tau)d\tau \\
&= z_0 + Az_0\int_0^t d\tau + A^2 z_0\int_0^t\int_0^\tau d\tau_1 d\tau + A^3\int_0^t\int_0^{\tau_1}\int_0^\tau z(\tau_2)d\tau_2 d\tau_1 d\tau \\
&\quad + A^2 B\int_0^t\int_0^{\tau_1}\int_0^\tau u(\tau_2)d\tau_2 d\tau_1 d\tau + AB\int_0^t\int_0^\tau u(\tau_1)d\tau_1 d\tau + B\int_0^t u(\tau)d\tau.
\end{aligned}$$

Repeating the procedure agsin and arranging terms gives

$$\begin{aligned}
z(t) &= z_0 + Az_0\int_0^t 1d\tau + A^2 z_0\int_0^t\int_0^\tau 1d\tau_1 d\tau + A^3 z_0\int_0^t\int_0^{\tau_1}\int_0^\tau 1d\tau_2 d\tau_1 d\tau \\
&\quad + A^3 B\int_0^t\int_0^{\tau_2}\int_0^{\tau_1}\int_0^\tau u(\tau_3)d\tau_3 d\tau_2 d\tau_1 d\tau + A^2 B\int_0^t\int_0^{\tau_1}\int_0^\tau u(\tau_2)d\tau_2 d\tau_1 d\tau \\
&\quad + AB\int_0^t\int_0^\tau u(\tau_1)d\tau_1 d\tau + B\int_0^t u(\tau)d\tau + R_3(z(t)),
\end{aligned} \tag{3.2.3}$$

where

$$R_3(z(t)) = A^4\int_0^t\int_0^{\tau_2}\int_0^{\tau_1}\int_0^\tau z(\tau_3)d\tau_3 d\tau_2 d\tau_1 d\tau.$$

Recall, by definition, that for any $k \geq 0$

$$E_{x_0^k}[u](t,t_0) = \int_{t_0}^t \cdots \int_{t_0}^\tau 1d\tau_{k-1}\cdots d\tau,$$

since $u_0 = 1$. Also, for any $k \geq 0$

$$E_{x_0^k x_1}[u](t,t_0) = \int_{t_0}^t \cdots \int_{t_0}^\tau u_1(\tau_{k-1})d\tau_{k-1}\cdots d\tau.$$

Substituting the iterated integrals in (3.2.3) yields

$$\begin{aligned}
z(t) = z_0 E_\emptyset[u](t) &+ Az_0 E_{x_0}[u](t) + A^2 E_{x_0^2}[u](t) + A^3 E_{x_0^3}[u](t) \\
&+ A^3 B E_{x_0^3 x_1}[u](t) + A^2 B E_{x_0^2 x_1}[u](t) + AB E_{x_0 x_1}[u](t) + B E_{x_1}[u](t) \\
&+ R_3(z).
\end{aligned}$$

Continuing in this manner produces in the limit

$$z(t) = \sum_{k=0}^{\infty} A^k z_0 E_{x_0^k}[u](t) + \sum_{k=0}^{\infty} A^k B E_{x_0^k x_1}[u](t), \tag{3.2.4}$$

which can be viewed as the usual Peano-Baker formula [4, 28]. Equivalently, the series solution of (3.2.1) is

$$z(t) = F_{c_z}[u](t) = \sum_{\eta \in X^*} (c_z, \eta)\, E_\eta[u](t),$$

where $X = \{x_0, x_1\}$. Clearly, the only terms with coefficients different from zero in (3.2.4) are the ones related to the words ${x_0}^k$ and ${x_0}^k x_1$ for any $k \geq 0$. Therefore, the coefficients of the generating series for $y(t) = F_c[u](t) = Cz(t) = CF_{c_z}[u](t)$ are

$$(c, \eta) = \begin{cases} CA^k B & : \eta = {x_0}^k x_1,\ k \geq 0 \\ CA^k z_0 & : \eta = {x_0}^k,\ k \geq 0 \\ \quad 0 & : \text{otherwise.} \end{cases}$$

As a check, the explicit solution for the system (1.1.2) can be calculated from (3.2.4). The following two lemmas are needed in order to calculate it.

Lemma 3.2.1. *For any* $k \geq 0$*,*

$$E_{{x_0}^k}[u](t) = \frac{t^k}{k!}.$$

Proof: The prove is done after k repeated integrations from Examples 1.1.1 and 1.1.2 taking $t_0 = 0$. ■

Lemma 3.2.2. *For any* $k \geq 0$*,*

$$E_{x_0^k x_1}[u](t) = \int_0^t \frac{(t-\tau)^k}{k!} u(\tau) d\tau.$$

Proof: When $k = 0$, by definition $E_{x_1}[u](t) = \int_0^t u_1(\tau) d\tau$. Also, when $k = 1$,

$$E_{x_0 x_1}[u](t) = \int_0^t 1 E_{x_1}[u](\tau), d\tau = \int_0^t \int_0^\tau u_1(\tau_1) d\tau_1 d\tau = \int_0^t (t-\tau) u(\tau) d\tau,$$

using integration by parts. This set up can be extended inductively to produce

$$\begin{aligned} E_{x_0^2 x_1}[u](t) &= \int_0^t \int_0^{\tau_1} E_{x_1}[u](\tau_1) d\tau_1 d\tau \\ &= \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} u_1(\tau_2) d\tau_2 d\tau_1 d\tau = \int_0^t \frac{(t-\tau)^2}{2} u(\tau) d\tau, \end{aligned}$$

and after k repeated integrations the statement is proved. ■

Substituting (3.2.1) and (3.2.2) in (3.2.4), gives

$$\begin{aligned} z(t) &= \sum_{k=0}^{\infty} A^k z_0 \frac{t^k}{k!} + \sum_{k=0}^{\infty} A^k B \int_0^t \frac{(t-\tau)^k}{k!} u(\tau) d\tau \\ &= \left(\sum_{k=0}^{\infty} \frac{(At)^k}{k!} \right) z_0 + \int_0^t \left(\sum_{k=0}^{\infty} \frac{(A(t-\tau))^k}{k!} \right) Bu(\tau) d\tau \\ &= \mathrm{e}^{At} z_0 + \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau) d\tau. \end{aligned}$$

As expected this result is the same as the well known solution given in (1.1.3).

3.2.2 On bilinear time invariant systems

A bilinear time invariant state space realization is the first-order differential equation and output equation

$$\begin{aligned} \dot{z} &= N_0 z + N_1 z u, \ z(0) = z_0 \\ y &= \lambda z, \end{aligned} \tag{3.2.5}$$

where $N_0, N_1 \in \mathbb{R}^{nxn}, \lambda \in \mathbb{R}^{1xn}$ and z_0 is a any vector in $\mathbb{R}^n$. Note that the state equation can be written in integral form using the Fundamental Theorem of Calculus as

$$z(t) = z_0 + N_0 \int_0^t z(\tau) d\tau + N_1 \int_0^t z(\tau) u(\tau) d\tau. \tag{3.2.6}$$

The method applied next is similar to the one developed in Subsection 3.2.1, but in this case the presence of the bilinear term $z(t)u(t)$ generates a much more rich and complicated structure. Making the substitution of $z(t)$ in the corresponding two terms of (3.2.6), yields

$$\begin{aligned}
z(t) &= z_0 + N_0 \int_0^t \left(z_0 + N_0 \int_0^\tau z(\tau_1) d\tau_1 + N_1 \int_0^\tau z(\tau_1) u(\tau_1) d\tau_1 \right) d\tau \\
&\quad + N_1 \int_0^t \left(z_0 + N_0 \int_0^\tau z(\tau_1) d\tau_1 + N_1 \int_0^\tau z(\tau_1) u(\tau_1) d\tau_1 \right) u(\tau) d\tau \\
&= z_0 + N_0 z_0 \int_0^t d\tau + N_1 z_0 \int_0^t u(\tau) d\tau \\
&\quad + N_0^2 \int_0^t \int_0^\tau z(\tau_1) d\tau_1 d\tau + N_0 N_1 \int_0^t \int_0^\tau z(\tau_1) u(\tau_1) d\tau_1 d\tau \\
&\quad + N_1 N_0 \int_0^t u(\tau) \int_0^\tau z(\tau_1) d\tau_1 d\tau + N_1^2 \int_0^t u(\tau) \int_0^\tau z(\tau_1) u(\tau_1) d\tau_1 d\tau.
\end{aligned}$$

Repeating this procedure one more time, in order to see the structure of the terms, gives

$$\begin{aligned}
z(t) = z_0 &+ N_0 z_0 \int_0^t d\tau + N_1 z_0 \int_0^t u(\tau) d\tau + N_0^2 \int_0^t \int_0^\tau d\tau_1 d\tau + N_0 N_1 \int_0^t \int_0^\tau u(\tau_1) d\tau_1 d\tau \\
&+ N_1 N_0 \int_0^t u(\tau) \int_0^\tau d\tau_1 d\tau + N_1^2 \int_0^t u(\tau) \int_0^\tau u(\tau_1) d\tau_1 d\tau + R_2(z(t), u(t)),
\end{aligned} \tag{3.2.7}$$

where $R_2(z(t), u(t))$ contains all the integrals depending explicitly on $z(t)$ and $u(t)$. Using iterated integrals in (3.2.7) yields

$$\begin{aligned}
z(t) &= z_0 E_\emptyset[u](t) + N_0 z_0 E_{x_0}[u](t) + N_1 z_0 E_{x_1}[u](t) \\
&\quad + N_0^2 z_0 E_{x_0^2}[u](t) + N_0 N_1 z_0 E_{x_0 x_1}[u](t) + N_1 N_0 z_0 E_{x_1 x_0}[u](t) + N_1^2 z_0 E_{x_1^2}[u](t) \\
&\quad + R_2(z(t), u(t)).
\end{aligned}$$

Note that all the possible words over $X = \{x_0, x_1\}$ of length two or shorter appear above, specifically $\{\emptyset, x_0, x_1, x_0^2, x_0 x_1, x_1 x_0, x_1^2\}$. Also observe, that the coefficient N_0 is attached to the letter x_0 and N_1 to the letter x_1. For example, the coefficient of the word x_0^2 is N_0^2 and similarly the coefficient for $x_0 x_1$ is $N_0 N_1$. Continuing in this manner produces in the limit the usual Peano-Baker formula [4, 28], so the series solution of (3.2.5) is

$$z(t) = F_{c_z}[u](t) = \sum_{\eta \in X^*} (c_z, \eta)\, E_\eta[u](t),$$

where $X = \{x_0, x_1\}$. The coefficients of the generating series for $y(t) = F_c[u](t) = \lambda z(t) = \lambda F_{c_z}[u](t)$ are

$$(c, \eta) = \lambda N_{i_k} \cdots N_{i_1} z_0,\ \eta = x_{i_k} \cdots x_{i_1}, \tag{3.2.8}$$

where $i_1, \cdots, i_k \in \{0, 1\}$, $N_{i_1}, \cdots, N_{i_k} \in \{N_0, N_1\}$ and $x_{i_1}, \cdots, x_{i_k} \in X$.

CHAPTER 4

FRACTIONAL FLIESS OPERATORS

The concept of a fractional Fliess operator is presented as a generalization of the usual Fliess operator introduced in Chapter 1. The basic idea is that iterated Riemann-Liouville fractional integrals are used instead of the classical integrals. There are in fact two choices available for the definition, one that is compatible with the Riemann-Liouville fractional derivative and the other with the Caputo fractional derivative. Both approaches are pursued in the context of state space realizations for these operators. At this stage, no requirement is placed on the coefficients of the generating series, so the development is purely formal.

Let $X = \{x_0, x_1, \ldots, x_m\}$, $\mathfrak{p} \geq 1$ and $a < b$. For a Lebesgue measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[a, b]$. The first definition is the fractional version of the iterated integral $E_\eta[u](t, t_0)$ given in Section 1.1. Without loss of generality it is assumed that $t_0 = 0$, and thus it will be omitted from the notation.

Definition 4.0.1. Let $0 < \alpha \leq 1$ and $T \in \mathbb{R}$ be fixed. The **fractional iterated integral** for any $\eta = x_i\eta' \in X^*$, $x_i \in X$, $\eta' \in X^*$, is the mapping $E^\alpha_\eta : L^m_1[0, T] \to \mathbb{R}[0, T]$ defined by the recursion

$$E^\alpha_\eta[u](t) = E^\alpha_{x_i\eta'}[u](t) = I^\alpha[u_i(\tau)\, E^\alpha_{\eta'}[u](\tau)](t),$$

where $E^\alpha_\emptyset := 1$ and

$$u_0(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & : \text{Riemann-Liouville derivative approach} \\ 1 & : \text{Caputo derivative approach.} \end{cases}$$

Note that when $\alpha = 1$, the resulting iterated integrals are the usual continuous functions $E_\eta[u](t, 0)$. But when $0 < \alpha < 1$, the corresponding fractional iterated integrals will be continuous if, for example, the functions u_i, $i = 1, 2, \ldots, m$ are all continuous. Weaker sufficient conditions also exist, see Subsection 2.2.4.

Example 4.0.1. Let $0 < \alpha \leq 1$, thus the fractional version of Example 1.1.1 is

$$E_{x_0}^{\alpha}[u](t) = I^{\alpha}(u_0(t)) = \begin{cases} I^{\alpha}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right) & \text{: Riemann-Liouville derivative approach} \\ I^{\alpha}(1) & \text{: Caputo derivative approach.} \end{cases}$$

Using Example 2.2.2 and 2.2.1 yields

$$E_{x_0}^{\alpha}[u](t) = \begin{cases} \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} & \text{: Riemann-Liouville derivative approach} \\ \frac{t^{\alpha}}{\Gamma(\alpha+1)} & \text{: Caputo derivative approach.} \end{cases}$$

□

Example 4.0.2. Let $0 < \alpha \leq 1$, when the word is $\eta = x_0^2 = x_0x_0$, the fractional iterated integral associated with it is

$$\begin{aligned} E_{x_0^2}^{\alpha}[u](t) &= I^{\alpha}[u_0(\tau)E_{x_0}^{\alpha}[u](\tau)](t) = I^{\alpha}[u_0(\tau)I^{\alpha}(u_0(\tau))](t) \\ &= \begin{cases} I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}I^{\alpha}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)\right] & \text{: Riemann-Liouville derivative approach} \\ I^{\alpha}[1I^{\alpha}(1)] = I^{2\alpha}(1) & \text{: Caputo derivative approach.} \end{cases} \end{aligned}$$

Using Example 2.2.2, 2.2.1 and 4.0.1 yields

$$E_{x_0^2}^{\alpha}[u](t) = \begin{cases} I^{\alpha}\left[\frac{t^{3\alpha-2}}{\Gamma(\alpha)\Gamma(2\alpha)}\right] & \text{: Riemann-Liouville derivative approach} \\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} & \text{: Caputo derivative approach.} \end{cases}$$

□

Example 4.0.3. When the word is $\eta = x_0x_1$, the fractional iterated integral associated with it is

$$\begin{aligned} E_{x_0x_1}^{\alpha}[u](t) &= I^{\alpha}[u_0(\tau)E_{x_1}^{\alpha}[u](\tau)](t) = I^{\alpha}[u_0(\tau)I^{\alpha}(u_1(\tau))](t) \\ &= \begin{cases} I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}I^{\alpha}(u_1(t))\right] & \text{: Riemann-Liouville derivative approach} \\ I^{\alpha}[1I^{\alpha}(u_1(t))] = I^{2\alpha}(u_1(t)) & \text{: Caputo derivative approach.} \end{cases} \end{aligned}$$

□

Note that while the fractional iterated integral associated with the Riemann-Liouville derivative approach involves complicated operations even for the input u_0, the Caputo derivative approach has a structure more similar to the usual iterated integral. The fractional version of the input-output operator corresponding to c is given next.

Definition 4.0.2. For any $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $0 < \alpha \leq 1$ the **fractional Fliess operator** is defined formally by

$$F_c^{\alpha}[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_{\eta}^{\alpha}[u](t).$$

As expected, when $\alpha = 1$, the operator is the usual one given in (1.1.1).

Example 4.0.4. When the series is $c = 2x_0 + 3x_0x_1$, the corresponding Fliess operator is

$$\begin{aligned} F_c^{\alpha}[u](t) &= F_{2x_0+3x_0x_1}^{\alpha}[u](t) = 2E_{x_0}^{\alpha}[u](t) + 3E_{x_0x_1}^{\alpha}[u](t) \\ &= \begin{cases} 2\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + 3I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{\alpha}(u_1(t))\right] & : \text{Riemann-Liouville derivative approach} \\ 2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 3I^{2\alpha}(u_1(t)) & : \text{Caputo derivative approach.} \end{cases} \end{aligned}$$

□

4.1 ON FRACTIONAL LINEAR TIME INVARIANT SYSTEMS

The focus in this section will be on fractional linear time invariant systems. The objective is to generalize the theory in Subsection 3.2.1 for fractional systems, now using the tools introduced in previous chapters. The state equation related to this new input-output model will be of non integer order since Riemann-Liouville fractional integrals are used instead of classical integrals. When a fractional order is used instead of the first order in (3.2.1), the resultant derivative can be of two types: a Riemman-Liouville fractional derivative or a Caputo fractional derivative. The following subsections analyze each case.

4.1.1 Riemman-Liouville fractional derivative approach

A single-input single-output linear time invariant system with a fractional differential equation, where the derivative is the Riemann-Liouville fractional derivative is shown in Figure 8.

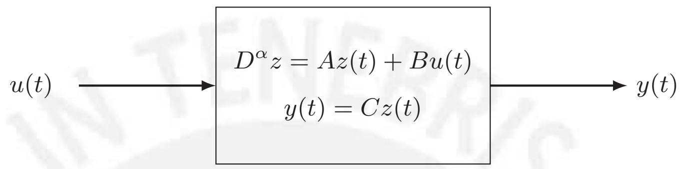

Fig. 8: Block diagram of single-input single-output Riemann-Liouville fractional linear time invariant system

As shown in Chapter 1, the fractional differential equation associated with this system is given in (1.1.7). Specifically, the state equation is

$$D^\alpha z = Az + Bu, \ z(0) = z_0, \tag{4.1.1}$$

where $0 \leq \alpha \leq 1$, $A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nx1}$ and z_0 is a any vector in $\mathbb{R}^n$. This equation can be written in integral form using the Fractional Fundamental Theorem of Calculus given in Theorem 2.3.1, namely,

$$I^\alpha(D^\alpha z(t)) = I^\alpha(Az(t) + Bu(t)) = AI^\alpha z(t) + BI^\alpha u(t).$$

Then,

$$z(t) - I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] = AI^\alpha z(t) + BI^\alpha u(t).$$

Rearranging terms gives

$$z(t) = I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^\alpha z(t) + BI^\alpha u(t). \tag{4.1.2}$$

Since integration and differentiation are defined componentwise, there is no loss of generality in the following analysis by assuming $n = 1$. For any $F \in P_1$, the differential chain rule in integral form given in Example 2.4.3 becomes

$$D^{\alpha}F(z(t)) = (D^{\alpha}\mathbb{1}(t))F(z(t)) + (D^{\alpha}z(t) - z(t)D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t)).$$

Replacing (4.1.1) gives

$$D^{\alpha}F(z(t)) = (D^{\alpha}\mathbb{1}(t))F(z(t)) + (Az(t) + Bu(t) - z(t)D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t)).$$

Taking the fractional integral at each side,

$$\begin{aligned} I^{\alpha}D^{\alpha}F(z(t)) &= I^{\alpha}((D^{\alpha}\mathbb{1}(t))F(z(t))) + A\, I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad + B\, I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Using Theorem 2.3.1 and rearranging gives

$$\begin{aligned} F(z(t)) &= I^{1-\alpha}F(z(0))\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + I^{\alpha}((D^{\alpha}\mathbb{1}(t))F(z(t))) + A\, I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad + B\, I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Now let $F(z(t)) = Az(t)$ above, then

$$\begin{aligned} Az(t) &= AI^{1-\alpha}z(0)I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^{\alpha}((D^{\alpha}\mathbb{1}(t))z(t)) + A^2I^{\alpha}z(t) \\ &\quad + ABI^{\alpha}u(t) - AI^{\alpha}\left((D^{\alpha}\mathbb{1}(t))z(t)\right), \end{aligned}$$

and cancelling two terms gives

$$Az(t) = AI^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + A^2I^{\alpha}z(t) + ABI^{\alpha}u(t).$$

Substitute the resulting equation into (4.1.2). This yields

$$z(t) = I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + AI^{1-\alpha}z(0)I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + BI^{\alpha}u(t) + ABI^{2\alpha}u(t) + A^2I^{2\alpha}z(t). \tag{4.1.3}$$

Repeating this procedure with $F(z(t)) = A^2 z(t)$ gives

$$A^2 z(t) = A^2 I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + A^4 I^{\alpha} z(t) + A^2 B I^{\alpha} u(t).$$

Substitute the resulting equation into (4.1.3). This yields

$$\begin{aligned} z(t) = I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + A I^{1-\alpha} z(0) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + A^2 I^{1-\alpha} z(0) I^{2\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ + B I^{\alpha} u(t) + A B I^{2\alpha} u(t) + A^2 B I^{3\alpha} u(t) + R_2(z(t)), \end{aligned}$$

where $R_2(z(t))$ contains all the integrals depending explicitly on $z(t)$. Continuing in this manner produces in the limit

$$z(t) = \sum_{k=0}^{\infty} A^k I^{1-\alpha} z(0) I^{\alpha k} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + \sum_{k=1}^{\infty} A^{k-1} B I^{\alpha k} u(t), \tag{4.1.4}$$

which can be viewed as a fractional form of the usual Peano-Baker formula [4, 28] given in (3.2.4) for linear time invariant systems. Equivalently, the series solution of (4.1.1) is

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\, E_{\eta}^{\alpha}[u](t), \tag{4.1.5}$$

where $u_0(t) := t^{\alpha-1}/\Gamma(\alpha)$ as indicated in Definition 4.0.1,$X = \{x_0, x_1\}$ and the coefficients of the generating series for $y(t) = Cz(t) = F_c^{\alpha}[u](t)$ are

$$(c, \eta) = \begin{cases} CA^k B & : \eta = {x_0}^k x_1,\ k \geq 0 \\ CA^k I^{1-\alpha}(z_0) & : \eta = {x_0}^k,\ k \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

The fractional iterated integral in (4.1.5) is

$$E_{\eta}^{\alpha}[u](t) = \begin{cases} I^{\alpha k} u(t) & : \eta = {x_0}^{k-1} x_1,\ k \geq 1 \\ I^{\alpha k} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] & : \eta = {x_0}^k,\ k \geq 0. \end{cases}$$

As a check, the explicit solution for the system (1.1.7) can be calculated from

(4.1.4). Observe that using Example 2.2.2 and 2.2.3 yields

$$z(t) = \sum_{k=0}^{\infty} A^k I^{1-\alpha} z(0) \left(\frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha)} \right) \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha k+\alpha)} \right) + \sum_{k=1}^{\infty} A^{k-1} B(I^{\alpha k-1} \mathbb{1}(t)) * u(t).$$

Thus,

$$z(t) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha k+\alpha)} t^{\alpha-1} I^{1-\alpha} z(0) + \sum_{k=1}^{\infty} A^{k-1} I^{\alpha k-1} \mathbb{1}(t) B * u(t).$$

Using Definition 2.1.1, Example 2.2.4 and Lemma 2.1.1 gives

$$z(t) = \mathscr{E}_{\alpha,\alpha}(At^\alpha) t^{\alpha-1} I^{1-\alpha} z(0) + (t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) B * u(t). \tag{4.1.6}$$

On the other hand, taking the Laplace transform of (4.1.1) and applying Theorem 2.4.7, then it follows that

$$s^\alpha \mathscr{L}[z(t)] - I^{1-\alpha} z(0) = A\mathscr{L}[z(t)] + B\mathscr{L}[u(t)],$$

and thus,

$$\mathscr{L}[z(t)] = (s^\alpha I - A)^{-1} I^{1-\alpha} z(0) + (s^\alpha I - A)^{-1} B \mathscr{L}[u(t)].$$

Using Lemma 2.1.2 gives

$$\mathscr{L}[z(t)] = \mathscr{L}[t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)] I^{1-\alpha} z(0) + \mathscr{L}[t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)] B \mathscr{L}[u(t)].$$

Taking the inverse Laplace transform and using (2.1.6) yields

$$z(t) = \mathscr{E}_{\alpha,\alpha}(At^\alpha) t^{\alpha-1} I^{1-\alpha} z(0) + (t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) B * u(t), \tag{4.1.7}$$

which is the same result obtained in (4.1.6) when the series solution (4.1.5) is used. That is, the generating series c can be written is terms of the function $t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^\alpha)$, which is a generalization of the matrix exponential e^{At} when $\alpha = 1$. Also note that in this Riemann-Liouville approach the coefficients are not explicit functions of $z(0)$ but rather $I^{1-\alpha} z(0)$. This *fractional* initial condition can be written as

$$I^{1-\alpha}z(t)\big|_{t=0} = \frac{z(t)\Gamma(\alpha)}{t^{\alpha-1}}\bigg|_{t=0}$$

[17, 29].

4.1.2 Caputo fractional derivative approach

A single-input single-output linear time invariant system with a fractional differential equation, where the derivative is the Caputo fractional derivative is shown in Figure 9.

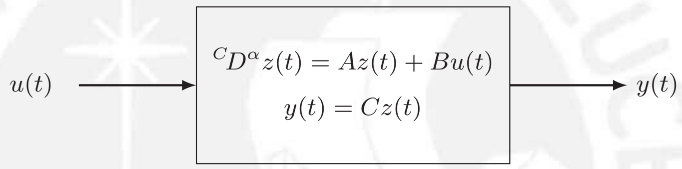

Fig. 9: Block diagram of single-input single-output Caputo fractional linear time invariant system

As shown in Chapter 1, the fractional differential equation associated to this system is given in (1.1.8). Specifically the state equation is

$${}^{C}D^{\alpha}z = Az + Bu, \; z(0) = z_0, \tag{4.1.8}$$

where $0 \leq \alpha \leq 1$, $A \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nx1}$ and z_0 is a any vector in $\mathbb{R}^n$. This equation can also be written in integral form. First, using Theorem 2.4.9, gives

$${}^{C}D^{\alpha}z = D^{\alpha}(z(t) - z(0)) = Az(t) + Bu(t). \tag{4.1.9}$$

Taking the fractional integral in each term yields

$$I^{\alpha}(D^{\alpha}(z(t) - z(0))) = I^{\alpha}(Az(t) + Bu(t)) = AI^{\alpha}z(t) + BI^{\alpha}u(t).$$

Using the Fractional Fundamental Theorem of Calculus given in Theorem 2.3.1, it follows that

$$(z(t)-z(0)) - I^{1-\alpha}(z(t)-z(0))\big|_{t=0}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] = AI^{\alpha}z(t) + BI^{\alpha}u(t).$$

Rearranging terms gives

$$z(t) = z(0) + AI^{\alpha}z(t) + BI^{\alpha}u(t), \tag{4.1.10}$$

since $I^{1-\alpha}(z(t)-z(0))\big|_{t=0} = 0$. Assume $n=1$ and use the same procedure as in the previous Section 4.1.1. For any $G \in P_1$, the differential chain rule in integral form given in Example 2.4.3 becomes

$$D^{\alpha}G(z(t)) = (D^{\alpha}\mathbb{1}(t))G(z(t)) + (D^{\alpha}z(t) - z(t)D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}G(z(t)).$$

Assume that $G(z(t)) = F(z(t)) - F(z(0))$ and note that $D^{\alpha}z(t) = Az(t) + Bu(t) + D^{\alpha}z(0)$ from (4.1.9). Substituting above gives

$$\begin{aligned} D^{\alpha}(F(z(t)) - F(z(0))) &= (D^{\alpha}\mathbb{1}(t))(F(z(t)) - F(z(0))) \\ &\quad + (Az(t) + Bu(t) - (z(t)-z(0))D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}(F(z(t)) - F(z(0))). \end{aligned}$$

Taking the fractional integral at each side,

$$\begin{aligned} I^{\alpha}D^{\alpha}F(z(t)) &= I^{\alpha}((D^{\alpha}\mathbb{1}(t))(F(z(t)) - F(z(0)))) \\ &\quad + A\,I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) + B\,I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))(z(t)-z(0))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right), \end{aligned}$$

since $\frac{\mathrm{d}}{\mathrm{d}z}F(z(0)) = 0$. Using Theorem 2.3.1 yields

$$\begin{aligned} F(z(t)) - F(z(0)) &= I^{1-\alpha}(F(z(t)) - F(z(0)))\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + I^{\alpha}((D^{\alpha}\mathbb{1}(t))F(z(t))) \\ &\quad + A\,I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) + B\,I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))(z(t)-z(0))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Since $I^{1-\alpha}(F(z(t)) - F(z(0)))|_{t=0} = 0$, rearranging gives

$$F(z(t)) = F(z(0)) + I^{\alpha}((D^{\alpha}\,\mathbb{1}(t))(F(z(t)) - F(z(0)))) + A\,I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right)$$
$$+ B\,I^{\alpha}\left(u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) - I^{\alpha}\left((D^{\alpha}\,\mathbb{1}(t))(z(t) - z(0))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right).$$

Now let $F(z(t)) = Az(t)$ above, then

$$Az(t) = Az(0) + A^2 I^{\alpha} z(t) + ABI^{\alpha}u(t),$$

since $(F(z(t)) - F(z(0)))|_{t=0} = 0$ and $(z(t) - z(0))|_{t=0} = 0$. Substituting the result into (4.1.10) gives

$$z(t) = z(0) + Az(0)I^{\alpha}\,\mathbb{1}(t) + BI^{\alpha}u(t) + ABI^{2\alpha}u(t) + R_1(z(t)),$$

where $R_1(z(t))$ contains all the integrals depending explicitly on $z(t)$. Continuing in this way produces the Caputo analogue of (4.1.4), namely,

$$z(t) = \sum_{k=0}^{\infty} A^k z(0) I^{\alpha k}\,\mathbb{1}(t) + \sum_{k=1}^{\infty} A^{k-1} B I^{\alpha k} u(t). \tag{4.1.11}$$

The main differences in this case are the presence of $z(0)$ instead of $I^{1-\alpha}z(0)$ in the series coefficients and the factor of $\mathbb{1}(t)$ instead of $t^{\alpha-1}/\Gamma(\alpha)$ in the iterated integrals. The series solution of (4.1.8) is then

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\; E_{\eta}^{\alpha}[u](t), \tag{4.1.12}$$

where $u_0(t) := \mathbb{1}(t)$ as indicated in Definition 4.0.1 and $X = \{x_0, x_1\}$. The coefficients of the generating series for $y(t) = Cz(t) = F_c^{\alpha}[u](t)$ are

$$(c, \eta) = \begin{cases} CA^k B & : \eta = {x_0}^k x_1,\ k \geq 0 \\ CA^k z_0 & : \eta = {x_0}^k,\ k \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

The fractional iterated integral in (4.1.12) is

$$E_\eta^\alpha[u](t) = \begin{cases} I^{\alpha k} u(t) & : \eta = {x_0}^{k-1} x_1,\ k \geq 1 \\ I^{\alpha k}\, \mathbb{1}(t) & : \eta = {x_0}^{k},\ k \geq 0. \end{cases}$$

As a check, the explicit solution for the system (1.1.8) can be calculated from (4.1.11). First, recall that it was found in (4.1.6) that

$$\sum_{k=1}^{\infty} A^{k-1} B I^{\alpha k} u(t) = (t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) B * u(t).$$

Using Example 2.2.4 and the expression above in (4.1.11) yields

$$z(t) = \mathscr{E}_{\alpha,1}(At^\alpha) z_0 + C(t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) B * u(t). \tag{4.1.13}$$

On the other hand, taking the Laplace transform of (4.1.8) and applying Theorem 2.4.8 it follows that

$$s^\alpha \mathscr{L}[z(t)] - s^{\alpha-1} z(0) = A\mathscr{L}[z(t)] + B\mathscr{L}[u(t)],$$

and thus,

$$\mathscr{L}[z(t)] = (s^\alpha I - A)^{-1} s^{\alpha-1} z(0) + (s^\alpha I - A)^{-1} B \mathscr{L}[u(t)].$$

Using Lemma 2.1.2 gives

$$\mathscr{L}[z(t)] = \mathscr{L}[\mathscr{E}_{\alpha,1}(At^\alpha)] z(0) + \mathscr{L}[t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)] B \mathscr{L}[u(t)].$$

Taking the inverse Laplace transform and using (2.1.6) yields

$$z(t) = \mathscr{E}_{\alpha,1}(At^\alpha) z(0) + (t^{\alpha-1} \mathscr{E}_{\alpha,\alpha}(At^\alpha)) B * u(t), \tag{4.1.14}$$

which is the same result obtained in (4.1.13) when the series solution (4.1.12) is used.

4.2 ON FRACTIONAL BILINEAR TIME INVARIANT SYSTEMS

The focus in this section will be on fractional bilinear time invariant systems. The objective is to generalize the theory in Subsection 3.2.2 for fractional systems, now using the tools introduced in previous chapters. As in Section 4.1, two approaches will be presented: the Riemman-Liouville fractional derivative approach and the Caputo fractional derivative approach.

4.2.1 Riemman-Liouville fractional derivative approach

A single-input single-output linear time invariant system with a fractional differential equation, where the derivative is the Riemann-Liouville fractional derivative is used, is shown in Figure 10.

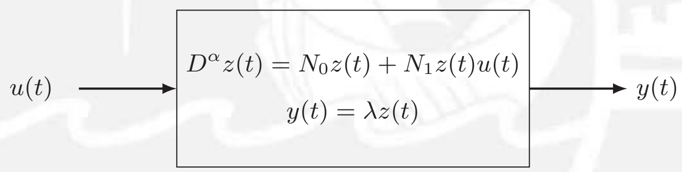

Fig. 10: Block diagram of single-input single-output Riemann-Liouville fractional bilinear time invariant system

A state-space representation for this system is

$$\begin{aligned} D^\alpha z &= N_0 z + N_1 z u,\ z(0) = z_0 \\ y &= \lambda z, \end{aligned} \tag{4.2.1}$$

where $0 \leq \alpha \leq 1$, $N_0, N_1 \in \mathbb{R}^{nxn}, \lambda \in \mathbb{R}^{1xn}$ and z_0 is a any vector in $\mathbb{R}^n$. Note that the state equation is

$$D^\alpha z = N_0 z + N_1 z u,\ z(0) = z_0. \tag{4.2.2}$$

This equation can be written in integral form using the Fractional Fundamental

Theorem of Calculus given in Theorem 2.3.1, namely,

$$I^{\alpha}(D^{\alpha}z(t)) = I^{\alpha}(N_0 z(t) + N_1 z(t)u(t)) = N_0 I^{\alpha}z(t) + N_1 I^{\alpha}(z(t)u(t)).$$

Then,

$$z(t) - I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] = N_0 I^{\alpha}z(t) + N_1 I^{\alpha}(z(t)u(t)).$$

Rearranging terms gives

$$z(t) = I^{1-\alpha}z(0)\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^{\alpha}z(t) + N_1 I^{\alpha}(z(t)u(t)). \tag{4.2.3}$$

Assuming $n = 1$, for any $F \in P_1$, the differential chain rule in integral form given in Example 2.4.3 becomes

$$D^{\alpha}F(z(t)) = (D^{\alpha}\mathbb{1}(t))F(z(t)) + (D^{\alpha}z(t) - z(t)D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t)).$$

Substituting (4.2.2) gives

$$D^{\alpha}F(z(t)) = (D^{\alpha}\mathbb{1}(t))F(z(t)) + (N_0 z(t) + Bz(t)u(t) - z(t)D^{\alpha}\mathbb{1}(t))\frac{\mathrm{d}}{\mathrm{d}z}F(z(t)).$$

Taking the fractional integral at each side,

$$\begin{aligned} I^{\alpha}D^{\alpha}F(z(t)) &= I^{\alpha}((D^{\alpha}\mathbb{1}(t))F(z(t))) + N_0\, I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad + N_1\, I^{\alpha}\left(z(t)u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Using Theorem 2.3.1 and rearranging gives

$$\begin{aligned} F(z(t)) &= I^{1-\alpha}F(z(0))\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + I^{\alpha}((D^{\alpha}\mathbb{1}(t))F(z(t))) + N_0\, I^{\alpha}\left(z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) \\ &\quad + N_1\, I^{\alpha}\left(z(t)u(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right) - I^{\alpha}\left((D^{\alpha}\mathbb{1}(t))z(t)\frac{\mathrm{d}}{\mathrm{d}z}F(z(t))\right). \end{aligned}$$

Now let $F(z(t))$ be replaced by $N_i z(t),\ i = 0, 1$ above, then

$$\begin{aligned} N_i z(t) &= N_i I^{1-\alpha}z(0)I^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_i I^{\alpha}((D^{\alpha}\mathbb{1}(t))z(t)) + N_0 N_i I^{\alpha}z(t) \\ &\quad + N_1 N_i I^{\alpha}(z(t)u(t)) - N_i I^{\alpha}((D^{\alpha}\mathbb{1}(t))z(t)), \end{aligned}$$

and cancelling two terms gives

$$N_0 z(t) = N_0 I^{1-\alpha} z(0) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0^2 I^{\alpha} z(t) + N_1 N_0 I^{\alpha}(z(t)u(t))$$

and

$$N_1 z(t) = N_1 I^{1-\alpha} z(0) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 N_1 I^{\alpha} z(t) + N_1^2 I^{\alpha}(z(t)u(t)).$$

Substituting both results into (4.2.3) yields

$$\begin{aligned} z(t) = {} & I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^{1-\alpha} z(0) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_1 I^{1-\alpha} z(0) I^{\alpha} \left[u(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_0^2 I^{2\alpha} z(t) + N_0 N_1 I^{2\alpha}(z(t)u(t)) + N_1 N_0 I^{\alpha}(I^{\alpha} z(t)u(t)) \\ & + N_1^2 I^{\alpha}(I^{\alpha}(z(t)u(t))u(t)). \end{aligned}$$

Repeating this procedure, let $F(z(t))$ be replaced by $N_i N_j z(t),\ \ i,j = 0,1$ and substitute back into the equation above. This gives

$$\begin{aligned} z(t) = {} & I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 I^{1-\alpha} z(0) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_1 I^{1-\alpha} z(0) I^{\alpha} \left[u(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_0^2 I^{1-\alpha} z(0) I^{2\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] + N_0 N_1 I^{1-\alpha} z(0) I^{2\alpha} \left[u(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ & + N_1^2 I^{1-\alpha} z(0) I^{\alpha}(u(t) I^{\alpha} u(t)) + N_1 N_0 I^{1-\alpha} z(0) I^{\alpha} \left[u(t) I^{\alpha} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\right] \\ & + R_2(z(t), u(t)), \end{aligned}$$

where $R_2(z(t), u(t))$ contains all the integrals depending explicitly on $z(t)$ and $u(t)$. Continuing in this manner produces in the limit a fractional form of the usual Peano-Baker formula [28], given in (3.2.2) for bilinear time invariant systems. So the series solution of (4.2.2) is

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\, E_{\eta}^{\alpha}[u](t), \tag{4.2.4}$$

where $u_0(t) := t^{\alpha-1}/\Gamma(\alpha)$ as indicated in Definition 4.0.1. The coefficients of the generating series for $y(t) = \lambda z(t) = F_c^{\alpha}[u](t)$ are

$$(c, \eta) = \lambda N_{i_k} \cdots N_{i_1} I^{1-\alpha} z(0), \tag{4.2.5}$$

where $i_1, \cdots, i_k \in \{0,1\}$ and $x_{i_1}, \cdots, x_{i_k} \in \{x_0, x_1\}$.

As a check, write (4.2.4) in the form

$$\begin{aligned} z(t) &= I^{1-\alpha} z(0) \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \\ &+ \sum_{i=0}^{1}\sum_{k=0}^{\infty}\sum_{i_0,\ldots,i_k=0}^{1} N_i N_{i_k}\cdots N_{i_1} I^{1-\alpha} z(0) I^{\alpha}[u_i(\tau) E^{\alpha}_{x_{i_k}\cdots x_{i_1}} u(\tau)](t), \end{aligned}$$

and take the Riemann-Liouville derivative of order α so that

$$\begin{aligned} D^{\alpha} z(t) &= \sum_{k=0}^{\infty}\sum_{i_0,\ldots,i_k=0}^{1} N_0 N_{i_k}\cdots N_{i_1} I^{1-\alpha} z(0) u_0(t) E^{\alpha}_{x_{i_k}\cdots x_{i_1}}[u](t) \\ &+ \sum_{k=0}^{\infty}\sum_{i_0,\ldots,i_k=0}^{1} N_1 N_{i_k}\cdots N_{i_1} I^{1-\alpha} z(0) u_1(t) E^{\alpha}_{x_{i_k}\cdots x_{i_1}}[u](t) \\ &= N_0 z(t) + N_1 z(t) u(t), \end{aligned}$$

which is consistent with (4.2.2). Also note that in this Riemann-Liouville approach, like in Section 4.1, the coefficients are not explicit functions of $z(0)$, but rather in terms of $I^{1-\alpha} z(0)$. This *fractional* initial condition can be written as

$$I^{1-\alpha} z(t)\Big|_{t=0} = \frac{z(t)\Gamma(\alpha)}{t^{\alpha-1}}\Bigg|_{t=0}$$

[17, 29].

4.2.2 Caputo fractional derivative approach

A single-input single-output bilinear time invariant system with a fractional differential equation, where the derivative is the Caputo fractional derivative is shown in Figure 11.

A state-space representation for this system is

$$\begin{aligned} {}^{C}D^{\alpha} z &= N_0 z + N_1 z u, \; z(0) = z_0 \\ y &= \lambda z, \end{aligned} \tag{4.2.6}$$

where $0 \le \alpha \le 1$, $N_0, N_1 \in \mathbb{R}^{nxn}, \lambda \in \mathbb{R}^{1xn}$ and z_0 is a any vector in $\mathbb{R}^n$. Note that

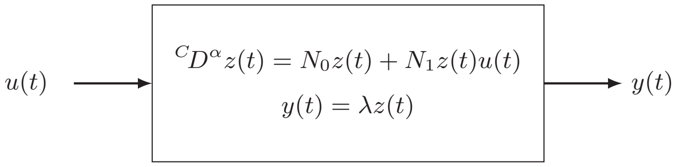

Fig. 11: Block diagram of single-input single-output Caputo fractional bilinear time invariant system

the state equation is

$$^{C}D^{\alpha}z = N_0 z + N_1 z u, \;\; z(0) = z_0. \tag{4.2.7}$$

This equation can be written in integral form. First, using Theorem 2.4.9, gives

$$^{C}D^{\alpha}z = D^{\alpha}(z(t) - z(0)) = N_0 z(t) + N_1 z(t)u(t). \tag{4.2.8}$$

Taking the fractional integral in each term yields

$$I^{\alpha}(D^{\alpha}(z(t) - z(0))) = I^{\alpha}(N_0 z(t) + N_1 z(t)u(t)) = N_0 I^{\alpha} z(t) + N_1 I^{\alpha}(z(t)u(t)).$$

Using the Fractional Fundamental Theorem of Calculus given in Theorem 2.3.1, it follows that

$$(z(t) - z(0)) - I^{1-\alpha}(z(t) - z(0))\big|_{t=0}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] = N_0 I^{\alpha} z(t) + N_1 I^{\alpha}(z(t)u(t)).$$

Rearranging terms gives

$$z(t) = z(0) + N_0 I^{\alpha} z(t) + N_1 I^{\alpha}(z(t)u(t)), \tag{4.2.9}$$

since $I^{1-\alpha}(z(t) - z(0))\big|_{t=0} = 0$. Assuming $n = 1$ and following the same procedure as in the previous subsections yields

$$F(z(t)) = F(z(0)) + I^\alpha((D^\alpha \mathbb{1}(t))(F(z(t)) - F(z(0)))) + N_0 I^\alpha \left(z(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t)) \right) \\ + N_1 I^\alpha \left(z(t) u(t) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t)) \right) - I^\alpha \left((D^\alpha \mathbb{1}(t))(z(t) - z(0)) \frac{\mathrm{d}}{\mathrm{d}z} F(z(t)) \right)$$

for any $F \in P_1$ using Example 2.4.3 and (4.2.9). Now let $F(z(t))$ be replaced by $N_i z(t),\ i = 0, 1$ and substitute back into (4.2.9). This yields

$$z(t) = z_0 + N_0 z_0 I^\alpha \mathbb{1}(t) + N_1 z_0 I^\alpha u(t) + R_1(z(t), u(t)),$$

where $R_1(z(t), u(t))$ contains all the integrals depending explicitly on $z(t)$ and $u(t)$. Continuing in this way produces the Caputo analogue of the result obtain in the Riemann-Liouville subsection, namely, the series solution of (4.2.7) is

$$z(t) = \sum_{\eta \in X^*} (c_z, \eta)\ E_\eta^\alpha[u](t), \tag{4.2.10}$$

where now $u_0(t) := \mathbb{1}(t)$ as indicated in Definition 4.0.1. The coefficients of the generating series for $y(t) = \lambda z(t) = F_c^\alpha[u](t)$ are

$$(c, \eta) = \lambda N_{i_k} \cdots N_{i_1} z_0 \tag{4.2.11}$$

for $\eta = x_{i_k} \cdots x_{i_1}$. As a check, take the Caputo derivative of order α of (4.2.10). A straightforward calculation analogous to the one for Riemann-Liouville case gives (4.2.7) as expected. Also, note that the coefficients in (4.2.11) coincide with the ones in the non-fractional case given in (3.2.8).

The only difference between this Caputo fractional case and the regular case is in terms of the iterative integrals, this motivates the final theorem of this thesis. It is a generalization of a Volterra series [12] for a fractional bilinear system.

Theorem 4.2.1. *The solution $y(t) = \lambda z(t) = F_c^\alpha[u](t)$ of a bilinear system in the Caputo sense* (4.2.6) *can be written in terms of the matrix Mittag-Leffler function as*

$$y(t) = w_0(t) + \sum_{k=1}^{\infty} \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} w_k(t, \tau_k, \ldots, \tau_1) u(\tau_k) \cdots u(\tau_1)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k,$$

where

$$\begin{aligned} w_0(t) &= \lambda \mathscr{E}_{\alpha,1}(N_0 t^\alpha) z_0, \\ w_k(t,\tau_k,\ldots,\tau_1) &= \lambda (t-\tau_k)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_k)^\alpha) N_1 (\tau_k-\tau_{k-1})^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(\tau_k-\tau_{k-1})^\alpha) \\ &\quad \cdots N_1(\tau_2-\tau_1)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(\tau_2-\tau_1)^\alpha) N_1 \mathscr{E}_{\alpha,1}(N_0 {\tau_1}^\alpha) z_0,\ k\geq 1. \end{aligned}$$

Proof: Since each word $\eta = x_{i_k}\cdots x_{i_1}\in X^*$ can be rewritten uniquely in the form ${x_0}^{n_k}x_1{x_0}^{n_{k-1}}x_1\cdots {x_0}^{n_1}x_1{x_0}^{n_0}$, it follows from (4.2.10) and (4.2.11) that

$$y(t) = \sum_{k=0}^{\infty}\sum_{n_0,\ldots,n_k=0}^{\infty} \lambda {N_0}^{n_k}N_1{N_0}^{n_{k-1}}\cdots N_1{N_0}^{n_0}z_0 E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t).$$

The claim is that $y=\sum_{k\geq 0} y_k$, where

$$\begin{aligned} y_k(t) &:= \sum_{n_0,\ldots,n_k=0}^{\infty} \lambda {N_0}^{n_k}N_1{N_0}^{n_{k-1}}\cdots N_1{N_0}^{n_0}z_0 E^\alpha_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t) \\ &= \sum_{n_0,\ldots,n_k=0}^{\infty} \int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_2} w_k(t,\tau_k,\ldots,\tau_1)u(\tau_k)\cdots u(\tau_1)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_k \end{aligned}$$

for $k>0$ and $y_0 := \sum_{n_0\geq 0}\lambda N_0^{n_0}z_0\, E^\alpha_{x_0^{n_0}}[u] = w_0$. The expressions for y_k are proved by induction on k. When $k=0$ observe that

$$y_0(t) = \sum_{n_0=0}^{\infty}\lambda {N_0}^{n_0}z_0\, I^{\alpha n_0}\,\mathbb{1}(t),$$

and using Example 2.2.4 yields

$$y_0(t) = \lambda\mathscr{E}_{\alpha,1}(N_0t^\alpha)z_0 = w_0(t).$$

When $k=1$, it follows via Example 2.2.3 that

$$\begin{aligned} y_1(t) &= \sum_{n_0,n_1=0}^{\infty}\lambda {N_0}^{n_1}N_1{N_0}^{n_0}z_0\, I^{\alpha(n_1+1)}(u(t)I^{\alpha n_0}\,\mathbb{1}(t)) \\ &= \sum_{n_0=0}^{\infty}(\lambda t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0t^\alpha)N_1{N_0}^{n_0}z_0) * (u(t)I^{\alpha n_0}\,\mathbb{1}(t)). \end{aligned}$$

Applying next Example 2.2.4 yields

$$\begin{aligned} y_1(t) &= \int_0^t \lambda(t-\tau)^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau)^\alpha)N_1\mathscr{E}_{\alpha,1}(N_0{\tau_1}^\alpha)z_0u(\tau)\,\mathrm{d}\tau \\ &= \int_0^t w_1(t,\tau_1)u(\tau_1)\,\mathrm{d}\tau_1. \end{aligned}$$

Now suppose the identity in question holds for all terms up to some fixed $k \geq 0$. Then

$$\begin{aligned} y_{k+1}(t) = \sum_{n_0,\ldots,n_{k+1}=0}^{\infty} & \lambda {N_0}^{n_{k+1}} N_1 {N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0 I^{\alpha(n_{k+1}+1)} \\ & (u(t)E^{\alpha}_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)), \end{aligned}$$

and Example 2.2.3 gives

$$\begin{aligned} y_{k+1}(t) &= \sum_{n_0,\ldots,n_k=0}^{\infty} \lambda t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0t^\alpha)N_1{N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0 * \\ &\qquad (u(t)E^{\alpha}_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)) \\ &= \sum_{n_0,\ldots,n_k=0}^{\infty} \int_0^t \lambda(t-\tau_{k+1})^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_{k+1})^\alpha)N_1{N_0}^{n_k}\cdots N_1{N_0}^{n_0}z_0u(\tau_{k+1}) \\ &\qquad E^{\alpha}_{{x_0}^{n_k}x_1{x_0}^{n_{k-1}}\cdots x_1{x_0}^{n_0}}[u](t)\,\mathrm{d}\tau_{k+1}. \end{aligned}$$

Finally, from the induction hypothesis, it follows that

$$\begin{aligned} y_{k+1}(t) &= \int_0^t\int_0^{\tau_{k+1}}\cdots\int_0^{\tau_2}\lambda(t-\tau_{k+1})^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(N_0(t-\tau_{k+1})^\alpha)\cdots N_1(\tau_2-\tau_1)^{\alpha-1} \\ &\quad \mathscr{E}_{\alpha,\alpha}(N_0(\tau_2-\tau_1)^\alpha)N_1\mathscr{E}_{\alpha,1}(N_0{\tau_1}^\alpha)z_0u(\tau_{k+1})\cdots u(\tau_1)\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{k+1} \\ &= \int_0^t\int_0^{\tau_{k+1}}\cdots\int_0^{\tau_2} w_{k+1}(t,\tau_{k+1},\ldots,\tau_1)u(\tau_{k+1})\cdots u(\tau_1)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{k+1}. \end{aligned}$$

Thus, the expression for y_k holds for all $k \geq 0$, and the theorem is proved. ■

In the case of a bilinear system corresponding to the Riemann-Liouville approach, an analogous Volterra series representation can be directly computed using Theorem 2.4.9. Recall also in the non-fractional case that there is a dichotomy between causal Volterra series induced by initialized state space realizations [12] and a potentially noncausal variety whose kernel functions are often derived from measurements and represented in terms of their multivariable Laplace transforms [28]. Theorem 4.2.1 above corresponds to the former in this fractional setting, while the fractional generalization of the latter has appeared in [19, 20].

CHAPTER 5

CONCLUSIONS

The fractional extension of Fliess operators in two specific cases: using Riemann-Liouville fractional derivatives and Caputo fractional derivatives was defined and a characterization of a fractional extension of iterated integrals using Riemann-Liouville fractional integrals was used.

A general solution for a fractional linear and bilinear time invariant system via formal power series in terms of Fliess operators was created using this fractional extension. One solution for each approach, using Riemann-Liouville fractional derivatives and Caputo fractional derivatives.

In regards to the linear time invariant system, comparing the explicit solutions for both systems in (4.1.7) and (4.1.14), it is evident that both approaches produce the same impulse response, i.e, the part of $y(t)$ only considering the input $u(t) = \mathbb{1}(t)$, then $y(t) = C(t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}(At^{\alpha}))B$. While the zero-input state responses, i.e, the part of $y(t)$ considering the input $u(t) = 0$, differ in the way they depend on the initial conditions. Namely, $\mathscr{E}_{\alpha,\alpha}(At^{\alpha})t^{\alpha-1}I^{1-\alpha}z(0)$ in the Riemann-Liouville case versus $\mathscr{E}_{\alpha,1}(At^{\alpha})z(0)$ in the Caputo case. In the case of bilinear time invariant system, the main differences between the Caputo and Riemann-Liouville cases are the presence of $z(0)$ instead of $I^{1-\alpha}z(0)$ in the series coefficients and the factor of $\mathbb{1}(t)$ instead of $t^{\alpha-1}/\Gamma(\alpha)$ in the iterated integrals.

BIBLIOGRAPHY

[1] D. Baleanu, K. Diethelm, E. Scalas, and J. Trujillo, *Fractional Calculus Models and Numerical Methods*. Boston, MA: World Scientific, 2012.

[2] J. Berstel and C. Reutenauer, *Rational Series and Their Languages*. Berlin: Springer–Verlag, 1988.

[3] B. Bonilla, J. J. Trujillo, and M.Rivero, "Fractional order continuity and some properties about integrability and differentiability of real functions," *J. Math. Anal. Appl*, vol. 231, pp. 205–212, 1999.

[4] L. A. D. Espinosa, "Interconnections of nonlinear systems driven by l_2-itô stochastic processes," Ph.D. dissertation, Old Dominion University, 2009.

[5] M. Fliess, "Fonctionnelles causales non linéaires et indéterminées non commutatives," *Bull. Soc. Math. France*, vol. 109, pp. 3–40, 1981.

[6] ——, "Réalisation locale des syst'emes non linéaires, alg'ebres de lie filtrées transitives et séries génératrices non commutatives," *Invent. Math.*, vol. 71, pp. 521–537, 1983.

[7] R. Gorenflo and F. Mainardi, "Essentials of fractional calculus," 2000.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. New York: Academic Press, 2007.

[9] W. S. Gray and L. A. D. Espinosa, "A faá di bruno hopf algebra for analytic nonlinear feedback control systems," in *Faá di Bruno Hopf Algebras, Dyson-Schwinger Equations, and Lie-Butcher Series*, K. Ebrahimi-Fard and E. F. Fauvet, Eds. Zürich, Switzerland: IRMA Lectures in Mathematics and Theoretical Physics, European Mathematical Society, 2015, vol. 21, pp. 149–217.

[10] W. S. Gray and Y. Wang, "Fliess operators on l_p spaces: Convergence and continuity," *Systems and Control Letters*, vol. 46, pp. 67–74, 2002.

[11] R. Hilfer, *Applications of fractional calculus in physics*. River Edge, NJ: World Scientific Publishing Co., Inc., 2000.

[12] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer–Verlag, 1995.

[13] S. Kazem, "Exact solution of some linear fractional differential equations by laplace transform," *Int. J. Nonlinear Sci*, vol. 16, pp. 3–11, 2013.

[14] A. A. Kilbas, M. Rivero, L. Rodríguez-Germá, and J. J. Trujillo, "α-analytic solutions of some linear fractional differential equations with variable coefficients," *Appl. Math. Comput*, vol. 187, pp. 239–249, 2007.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier, 2006.

[16] H. Laurent, "Sur le calcul des dé rivé es a indicies quelconques," *Nouv. Annales de Mathematiques*, vol. 3, pp. 240–252, 1884.

[17] C. Li, D. Qian, and Y. Chen, "On riemann-liouville and caputo derivatives," *Discrete Dyn. Nat. Soc*, vol. 2011, pp. 1–15, 2011.

[18] P. Lino, G. Maione, and F. Saponaro, "Fractional-order modeling of high-pressure fluid-dynamic flows: An automotive application," *IFAC-PapersOnLine*, vol. 48, pp. 382 – 387, 2015.

[19] A. Maachou, "Application of fractional volterra series for the identification of thermal diffusion in an armco iron sample subject to large temperature variations," in *Proc. 18th IFAC World Congress*, Milano, Italy, 2010, pp. 5621–5626.

[20] A. Maachou, R. Malti, P. Melchior, J.-L. Battaglia, A. Oustaloup, and B. Hay, "Thermal system identification for large temperature variations using fractional volterra series," in *Proc. 4th IFAC Workshop on Fractional Differentiation and its Applications*, 2010.

[21] A. Mathai and H. Haubold, *Special Functions for Applied Scientists*. Springer, 2008.

[22] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: John Wiley & Sons Inc., 1993.

[23] K. Oldham and J. Spanier, *The Fractional Calculus: Integrations and Differentiations of Arbitrary Order*. New York: Academic Press, 1974.

[24] T. Osler, “The fractional derivative of a composite function,” *SIAM J. Math. Anal*, vol. 1, pp. 288–293, 1970.

[25] I. Podlubny, *Fractional Differential Equations*. San Diego, CA: Academic Press, 1999.

[26] B. Ross, “Fractional calculus and its applications,” in *Proceedings of the International Conference Held at the University of New Haven, June 1974*. Springer– Verlag, 1975.

[27] ——, “The development of fractional calculus 1695-1900,” *Historia Mathematica*, vol. 4, pp. 75–89, 1977.

[28] W. J. Rugh, *Nonlinear System Theory: The Volterra/Wiener Approach*. Baltimore, MD: The Johns Hopkins University Press, 1981.

[29] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. Yverdon, Switzerland: Gordon and Breach Science, 1993.

[30] N. Y. Sonin, “On differentiation with arbitrary index,” *Moscow Matem. Sbornik*, vol. 6, pp. 1–38, 1983.

[31] V. E. Tarasov, “No violation of the leibniz rule. no fractional derivative,” *Commun Nonlinear Sci Numer Simulat*, vol. 18, pp. 2945–2948, 2013.

[32] I. M. Winter-Arboleda, W. S. Gray, and L. A. D. Espinosa, “Fractional fliess operators: Two approaches,” in *Proc. 49th Conf. on Information Sciences and Systems*, Baltimore, M. D., 2015.

[33] ——, “On global convergence of fractional fliess operators with applications to bilinear systems,” in *Proc. 51st Conf. on Information Sciences and Systems*, Baltimore, M. D., 2017.