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Gaussian Multiplicative Chaos

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Magíster en Matemáticas

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Resumen

Gaussian Multiplicative Chaos

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La teoría de Kolmogorov-Obukhov-Mandelbrot de disipación de energía en desarrollo de turbulencia se estableció para estudiar el comportamiento caótico de los fluidos. En ausencia de una base matemática rigurosa, Kahane introduce el caos gaussiano multiplicativo como un objeto aleatorio inspirado en la teoría del caos aditivo desarrollada por Wiener. En esta tesis desarrollamos teoría aleatoria en el espacio de medidas de Radon con el objetivo de definir rigurosamente el caos multiplicativo gaussiano. Seguimos el artículo de Kahane y debilitamos algunas condiciones para proporcionar una introducción accesible y autocontenida.

Palabras clave:

Medidas aleatorias
Procesos gaussianos
Caos gaussiano multiplicativo
Martingalas

Abstract

The Kolmogorov-Obukhov-Mandelbrot theory of energy dissipation in turbulence developed was established to study the chaotic behavior of fluids. In the absence of a rigorous mathematical basis, Kahane introduced the Gaussian multiplicative chaos as a random object inspired by the additive chaos theory developed by Wiener. In this thesis we developed random theory in the spaces of Radon measures in order to rigorously define Gaussian multiplicative chaos. We follow Kahane's paper and weaken some conditions to provide an accessible and self-contained introduction.

Key words and phrases: Random measures, Gaussian processes, Gaussian multiplicative chaos, martingales with discrete parameter.

2010 Mathematics Subject Classification: Primary 60G57; Secondary 60G15, 60G42.

. . . para mi familia.

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Introduction

The idea of Gaussian multiplicative chaos was introduced by Mandelbrot [Man72] in his work on the Komogorov-Obukhov's model of energy dissipation in turbulence. Despite its functionality, it could not be constructed mathematically until 1985 where Kahane, in his article *Sur le chaos multiplicative* [Kah85], provides the fundamental ideas to define it as a random object.

Gaussian multiplicative chaos appears naturally in several branches of mathematics, such as mathematical physics or mathematical finance. The reader is referred to [RV14] for some examples. Motivated by its wide applicability, this thesis explores the theory of multiplicative chaos in an introductory way by focusing on the first definition of Kahane. The goal is the definition and study of random measures of the form

$$M(dt) = \exp\left(X(t) - \frac{1}{2}\mathbb{E}[X^2(t)]\right)\sigma(dt) \tag{1}$$

where $(X(t))_{t\in T}$ is a centred Gaussian field and σ is a given Radon measure on a locally compact Polish metric space T. Gaussian multiplicative chaos is a random measure of this form that allows the Gaussian field to have infinity variance, i.e., $\mathbb{E}[X^2(t)] = \infty$. In this case it is not clear what meaning to give to (1) a priori because X is not longer a Gaussian field. We proceed in the text considering that the function $K(t,s) = \mathbb{E}[X(t)X(s)]$ is of the form

$$K(t,s) = \sum_{n=1}^{\infty} K_n.$$

For every $K_n(t,s)$, we find a Gaussian field $X_n(t)$ such that $K_n(t,s) = \mathbb{E}[X_n(t)X_n(s)]$ and then we use an approximation procedure to give meaning to (1). The text is organized as follows.

In Chapter 1, we present some basic definitions and results of probability theory in the context which we are interested in.

In Chapter 2, we introduce Gaussian random objects and some results about them. Here, we realize the importance of the covariance kernel. Its richness provides good properties to the Gaussian field.

In Chapter 3, we work with measures defined on locally compact Polish metric spaces. The set of Radon measures defined on this space is endowed with a Borel σ-algebra in order to define a random measure. Some facts about distribution and convergence of random measures are given.

In Chapter 4, Gaussian multiplicative chaos is defined as a random measure. It is obtained as a weak limit of random measures. We discuss some problems about the definition such as uniqueness and degeneracy. Finally, we briefly present an extension of the theory.

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2020

Chapter 1

Probability Theory

This chapter is devoted to the basic of probability theory that will be used in the text. For more detail and proofs, the reader is referred to [AL06], [Kal02], [Kle13], [Bog07] and [Ros06].

1.1 Stochastic Processes

1.1.1 Basic Notions

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of:

- the sample space Ω which is an arbitrary nonempty set,
- the σ-algebra $\mathcal{F}$ of subsets of Ω, which are called events, and
- the probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$.

The Borel σ-algebra on $\mathbb{R}^d$, $1 \leq d < \infty$, is denoted by $\mathcal{B}(\mathbb{R}^d)$. A measurable function

$$\begin{array}{rccc} X: & (\Omega, \mathcal{F}) & \to & (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \\ & \omega & \mapsto & X(\omega) \end{array}$$

is called random variable. If we want to specify that $d = 1$ or $d \geq 2$ we say that X is a real random variable or random vector, respectively. The expected value of the real random variable X is the integral with respect to the measure $\mathbb{P}$:

$$\int_\Omega X(\omega)\mathbb{P}(d\omega).$$

It is denoted by $\mathbb{E}[X]$ or $\mathbb{E}X$. This expectation exists if the the expectation of the positive part X^+ and the negative part X^- of X are not infinite at the same time. We are considering

$$x^+ = \max\{x, 0\}, \quad x^- = \min\{-x, 0\}.$$

Note that $\mathbb{E}X = \mathbb{E}X^+ - \mathbb{E}X^-$.

We say that a random variable is integrable if $\mathbb{E}|X| < \infty$. The expected value of the random vector $X = (X_1, \dots, X_d)$ is $\mathbb{E}[X] = (\mathbb{E}X_1, \dots, \mathbb{E}X_d)$ provided that each real random variable X_i, $i \in [\![1, d]\!]$, is integrable. Here $[\![1, d]\!] = [1, d] \cap \mathbb{Z}$. We say that a property is true $\mathbb{P}$ almost surely or simply a.s. if it is true outside a set $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$.

For every real $p \geq 1$, we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P})$, or simply L^p, the space of random variables X such that $|X|^p$ is integrable and where random variables which agree a.s. are identified. Since the measure $\mathbb{P}$ is finite, we have for $b < a$,

$$L^a(\Omega, \mathcal{F}, \mathbb{P}) \subset L^b(\Omega, \mathcal{F}, \mathbb{P}).$$

Let X be a random variable, the probability measure $\mathbb{P}_X(\cdot) := \mathbb{P}(X^{-1}(\cdot))$ defined on $\mathcal{B}(\mathbb{R}^d)$ is called the probability distribution (or simply distribution) of X. We write $X \overset{\mathbb{D}}{=} Y$ when the random variables X and Y have the same distribution.

Definition 1.1.1. The collection $\{X_\alpha : \alpha \in \mathcal{X}\}$ of real random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a stochastic process indexed by the set $\mathcal{X}$

We usually use the notation $(X_\alpha)_{\alpha \in \mathcal{X}}$ or $(X_\alpha)_\alpha$ if it is clear. When the index set is $\mathbb{N} = \{1, 2, \dots\}$, we can refer to it as a sequence of random variables. The family

$$\{\mathbb{P}_X : (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathcal{X}^k, X = (X_{\alpha_1}, \dots, X_{\alpha_k}), k \in \mathbb{N}\}$$

of distributions is said to be the family of finite dimensional distribution associated with the stochastic process $\{X_\alpha : \alpha \in \mathcal{X}\}$. Let $X = (X_\alpha)_{\alpha \in \mathcal{X}}$ and $Y = (Y_\alpha)_{\alpha \in \mathcal{X}}$ be stochastic processes, we say that they have the same distribution and it is denoted by $X \overset{\mathbb{D}}{=} Y$ if for every $k \in \mathbb{N}$ and $(\alpha_1, \dots, \alpha_k) \in \mathcal{X}^k$,

$$(X_{\alpha_1}, \dots, X_{\alpha_k}) \overset{\mathbb{D}}{=} (Y_{\alpha_1}, \dots, Y_{\alpha_k}).$$

1.1.2 Daniell-Kolmogorov Theorem

Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we can ask if there is a probability measure $(\Omega, \mathcal{F}, P)$ and a real random variable X on it such that its distribution is μ. The answer is affirmative, it is enough to consider $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ and the random variable X to be the map $X(\omega) = \omega$. Thus

$$P_X(A) = P(A) = \mu(A).$$

Analogously, let μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. The probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$ and the random vector $X(\omega) = \omega$ can be considered to obtain a random variable with distribution μ.

The Daniell-Kolmogorov theorem deal with this problem when we want to construct a stochastic process $\{X_\alpha : \alpha \in \mathcal{X}\}$ indexed by $\mathcal{X}$ on some probability space $(\Omega, \mathcal{F}, P)$ when a family of finite dimensional distributions is given. The version of the theorem we will use appears as Theorem 6.3.1 of [AL06].

Theorem 1.1.2. *Let $\mathcal{X}$ be a nonempty set. Let $Q_\mathcal{X} = \{\nu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} : (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{X}^k, k \in \mathbb{N}\}$ be a family of probability measures such that for each $k \in \mathbb{N}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{X}^k$,*

1. *$\nu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}$ is a probability measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$,*
2. *for all $B_1, B_2, \ldots, B_k \in \mathcal{B}(\mathbb{R})$, $2 \leq k < \infty$,*

$$\nu_{(\alpha_1,\ldots,\alpha_k)}(B_1 \times \cdots \times B_{k-1} \times \mathbb{R}) = \nu_{(\alpha_1,\ldots,\alpha_{k-1})}(B_1 \times \cdots \times B_{k-1}),$$

3. *and for any permutation $(i_1, i_2, \ldots, i_k)$ of $(1, 2, \cdots, k)$,*

$$\nu_{(\alpha_{i_1},\ldots,\alpha_{i_k})}(B_{i_1} \times \cdots \times B_{i_k}) = \nu_{(\alpha_1,\ldots,\alpha_k)}(B_1 \times \cdots \times B_k).$$

Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X_\mathcal{X} = \{X_\alpha : \alpha \in \mathcal{X}\}$ on $(\Omega, \mathcal{F}, P)$ such that $Q_\mathcal{X}$ is the family of finite dimensional distributions associated with $X_\mathcal{X}$.

Remark. In the proof of [AL06], $\Omega = \mathbb{R}^\mathcal{X}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})^{\otimes\mathcal{X}}$ and the stochastic process $X = \{X_\alpha : \alpha \in \mathcal{X}\}$ is a projection from $\mathbb{R}^\mathcal{X}$ to $\mathbb{R}$. Provided that the above conditions (usually called consistency conditions) hold, the probability measure P build on $(\mathbb{R}^\mathcal{X}, \mathcal{B}(\mathbb{R})^{\otimes\mathcal{X}})$ is unique.

1.1.3 Measure Theory

We recall some basic results of measure theory. In the remainder of the chapter we deal with random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 1.1.3 (Monotone convergence theorem)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive random variables and let $X = \lim_{n\to\infty} X_n$ a.s. Then*

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Theorem 1.1.4 (Dominated convergence theorem)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, let $X = \lim_{n\to\infty} X_n$ a.s. and $|X_n| \leq Z$ for every $n \in \mathbb{N}$ with $Z \in L^1$. Then*

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Theorem 1.1.5 (Differentiation under the integral sign I)**.** *Let I be an open interval of $\mathbb{R}$ and $f : I \times \Omega \to \mathbb{R}$ be a function such that*

(i) for every $t \in I$ fixed, the function $\omega \mapsto f(t, \omega)$ is in L^1 (then it is a stochastic process indexed by I);

(ii) for every $\omega \in \Omega$ fixed, the function $t \mapsto f(t, \omega)$ is differentiable on I;

(iii) there exists a random variable $Z \in L^1$ such that for every $t \in I$,

$$\left|\frac{\partial f}{\partial t}(t, \cdot)\right| \leq Z(\cdot) \ a.s.$$

Then the function

$$t \mapsto \mathbb{E}[f(t, \omega)]$$

is differentiable on I, $\frac{\partial}{\partial t} f(t, \cdot) \in L^1$ and

$$\frac{d}{dt}\mathbb{E}[f(t, \omega)] = \mathbb{E}\left[\frac{\partial}{\partial t} f(t, \omega)\right].$$

We can weaken the third hypotheses even further:

$(iii)'$ for any compact subset $K \subset I$, there exists a nonnegative random variable $Z_K \in L^1$ such that,

$$\sup_{t\in K}\left|\frac{\partial f}{\partial t}(t, \cdot)\right| \leq Z_K(\cdot) \ a.s.$$

Fubini and Tonnelli

If the index set $\mathcal{X}$ of a stochastic process $(X_\alpha)_{\alpha\in\mathcal{X}}$ is endowed with the σ-algebra $\mathcal{A}$ we can ask if the map

$$\begin{array}{rccc} X: & \mathcal{X}\times\Omega & \to & \mathbb{R} \\ & (\alpha,\omega) & \mapsto & X_\alpha(\omega) \end{array}$$

is measurable with respect to the product σ-algebra $\mathcal{A}\otimes\mathcal{F}$. In the affirmative case we say that the stochastic process is measurable. The following theorem due to Fubini and Tonnelli is used to interchange the order of integration of positive stochastic processes.

Theorem 1.1.6. *Let μ be a σ-finite measure on $(\mathcal{X},\mathcal{A})$ and $(X_\alpha)_{\alpha\in\mathcal{X}}$ a measurable stochastic process taking values in $\mathbb{R}_+$ then*

1. *The function*

$$\alpha\mapsto\int_{\mathcal{X}} X(\alpha,\omega)\mathbb{P}(d\omega)$$

is measurable with respect to $\mathcal{A}$ and

$$\omega\mapsto\int_{\Omega} X(\alpha,\omega)\mu(d\alpha)$$

is a random variable.

2. *We have*

$$\int_{\mathcal{X}\times\Omega} X(\alpha,\omega)d\mu\otimes\mathbb{P}=\mathbb{E}\left[\int_{\mathcal{X}} X(\alpha,\omega)\mu(d\alpha)\right]=\int_{\mathcal{X}}\mathbb{E}\left[X(\alpha,\omega)\right]\mu(d\alpha).$$

1.2 Weak Convergence

In this section we introduce convergence in distribution as a particular case of weak convergence. The latter will be useful to characterize probability distributions.

Definition 1.2.1. For any probability measures $\mu,\mu_1,\mu_2,\ldots$ on $\mathbb{R}^d$ we say that μ_n converges weakly to μ and it is denoted by $\mu_n\xrightarrow{w}\mu$ if

$$\int f d\mu_n\to\int f d\mu$$

for any bounded continuous function f on $\mathbb{R}^d$.

We say that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ converges to the random variable X in distribution if their distributions $\mathbb{P}_{X_n}$ converges weakly to the distribution $\mathbb{P}_X$ of X. More explicitly,

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$$

for any bounded continuous function f on $\mathbb{R}^d$. It is denoted by $X_n \xrightarrow{\mathbb{D}} X$.

In the study of the distributions of the random variables, the characteristic functions are a very useful and easy to apply tool.

Definition 1.2.2. The characteristic function of a probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is the function $\hat{\mu} : \mathbb{R}^d \to \mathbb{C}$ defined as

$$\hat{\mu}(\xi) = \int e^{i\xi\cdot x}\mu(dx), \quad \xi \in \mathbb{R}^d.$$

where $\cdot$ denotes the usual Euclidean product on $\mathbb{R}^d$ and $i = \sqrt{-1}$.

If X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $\mathbb{R}^d$, the characteristic function of X is the characteristic function of its probability distribution:

$$\phi_X(\xi) = \hat{\mathbb{P}}_X(\xi) = \int e^{i\xi\cdot x}\mathbb{P}_X(dx) = \mathbb{E}[\exp(i\xi \cdot X)], \quad \xi \in \mathbb{R}^d.$$

Example 1.2.3. A Dirac measure on $\mathbb{R}^d$ is the probability measure δ_x defined for a given $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$\delta_x(A) = 1_A(x) = \begin{cases} 0, & x \notin A; \\ 1, & x \in A. \end{cases}$$

If a random variable X has the Dirac measure δ_x as a distribution, then $X = x$ a.s. Its characteristic function is

$$\mathbb{E}[e^{i\xi\cdot X}] = e^{i\xi\cdot x}.$$

The following theorem is one of the most important tools in the text. It is used to characterize distributions and establish convergence in distribution. It appears as Theorem 5.3 of [Kal02].

Theorem 1.2.4. *(Uniqueness and continuity, Levy) For any probability measures $\mu, \mu_1, \mu_2, \ldots$ on $\mathbb{R}^d$, we have $\mu_n \xrightarrow{w} \mu$ if and only if $\hat{\mu}_n \to \hat{\mu}$ for every $\xi \in \mathbb{R}^d$.*

As a consequence, we can uniquely determine a probability measure μ on $\mathbb{R}^d$ with its characteristic function $\hat{\mu}$ using the theorem with $\mu_n = \nu$, i.e., if $\hat{\mu} = \hat{\nu}$ then $\mu = \nu$.

In the literature, characteristic functions also are called Fourier transform. The Fourier transform of a random variable X and a measure μ are ϕ_X and $\hat{\mu}$, respectively, both defined above. In this language, Theorem 1.2.4 tells us that the Fourier transform is injective for measures. If two random variables have the same Fourier transform then they have the same distribution.

For measures on $\mathbb{R}^d_+ = \{(x_1, \ldots, x_d) : x_k \geq 0,\ k \in [\![1, d]\!]\}$ (endowed with $\mathcal{B}(\mathbb{R}^d_+) = \mathcal{B}(\mathbb{R}^d) \cap \mathbb{R}^d_+$) we can consider the Laplace transform $\tilde{\mu}$, given by

$$\tilde{\mu}(\xi) = \int e^{-\xi \cdot x} \mu(dx), \quad \xi \in \mathbb{R}^d_+.$$

A corresponding statement of Theorem 1.2.4 holds for the Laplace transforms of measures on $\mathbb{R}^d_+$, then the Laplace transform is injective. Note that in this case $\mu_n \xrightarrow{w} \mu$ if and only if

$$\int f d\mu_n \to \int f d\mu$$

for any bounded continuous function f on $\mathbb{R}^d_+$. Thus, if $X_n \geq 0$ for every $n \in \mathbb{N}$ and $X \geq 0$, the sequence $(X_n)_n$ converges in distribution to X if and only if

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)],$$

for any bounded continuous function f on $\mathbb{R}_+$.

1.3 Independence

The notion of independence is an extremely useful tool in probability theory. It makes calculations much easier. We start with independence of events.

We say that the events $\{A_1, \ldots, A_n\} \subset \mathcal{F}$ are independent if

$$\mathbb{P}(A_{\lambda_1} \cap \cdots \cap A_{\lambda_k}) = \mathbb{P}(A_{\lambda_1}) \cdots \mathbb{P}(A_{\lambda_k})$$

for all $\{\lambda_1, \ldots, \lambda_k\} \subset [\![1, n]\!]$.

Definition 1.3.1. A collection of events $\{A_\lambda : \lambda \in \Lambda\} \subset \mathcal{F}$ is called independent if for every finite subcollection $\{\lambda_1, \ldots, \lambda_k\} \subset \Lambda$, $k \in \mathbb{N}$,

$$\mathbb{P}(A_{\lambda_1} \cap \cdots \cap A_{\lambda_k}) = \mathbb{P}(A_{\lambda_1})\mathbb{P}(A_{\lambda_2}) \cdots \mathbb{P}(A_{\lambda_k}).$$

More general, we can define independence for σ-algebras.

Definition 1.3.2. Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be sub-σ-algebras of $\mathcal{F}$. We say that they are independent if for every $A_n \in \mathcal{B}_n$, the collection of event $\{A_1, A_2, \dots\}$ is independent.

We say that the random variables $\{X_n : n \in \mathbb{N}\}$ are independent if the σ-algebras $\{\sigma(X_n) : n \in \mathbb{N}\}$ are independent, where $\sigma(X_n)$ is the σ-algebra generated by X_n, i.e.,

$$\sigma(X_n) = \{X_n^{-1}(A) : A \in \mathcal{B}(\mathbb{R}^d)\}.$$

Independence of real random variables can be characterized by Fourier transform.

Proposition 1.3.3. *Let $X_1, \dots, X_n$ be real random variables, the following are equivalent:*

(i) $X_1, \dots, X_n$ are independent.

(ii) For all bounded measurable function $f_i : \mathbb{R} \to \mathbb{R}$, $i \in [\![1, n]\!]$,

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i(X_i)\right] = \prod_{i=1}^{n} \mathbb{E}[f_i(X_i)].$$

(iii) The characteristic function of $X = (X_1, \dots, X_n)$ is

$$\phi_X(\xi_1, \dots, \xi_n) = \prod_{i=1}^{n} \phi_{X_i}(\xi_i).$$

As a consequence we have

Proposition 1.3.4. *If X and Y are independent real random variables then*

$$\phi_{X+Y}(\xi) = \phi_X(\xi)\phi_Y(\xi).$$

The following result is handy to verifies that σ-algebras are independent. We say that a nonempty family of subsets of Ω is a π-system if it is closed under finite intersections.

Proposition 1.3.5. *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be σ-algebras of $\mathcal{F}$. For every $n \in \mathbb{N}$, let $\mathcal{C}_n \subset \mathcal{B}_n$ be a π-system such that $\sigma(\mathcal{C}_n) = \mathcal{B}_n$. Suppose that for every $A_n \in \mathcal{C}_n$, the collection $\{A_n : n \in \mathbb{N}\}$ is independent. Then, the σ-algebras $\{\mathcal{B}_n : n \in \mathbb{N}\}$ are independent.*

Remark 1.3.6. The π-system

$$\mathcal{C} = \left\{ \bigcap_{i=1}^{k} X_{\alpha_i}^{-1}(B_{\alpha_i}) : (\alpha_1, \dots, \alpha_k) \in \mathcal{X}^k, B_{\alpha_i} \in \mathcal{B}(\mathbb{R}), k \in \mathbb{N} \right\}$$

generates the σ-algebra $\sigma(X_\alpha : \alpha \in \mathcal{X})$.

We say that the stochastic processes $\{X_\alpha^n : \alpha \in \mathcal{X}\}$, $n \in \mathbb{N}$, are independent if the σ-algebras $\sigma(X_\alpha^n : \alpha \in \mathcal{X})$ are independent. By Remark 1.3.6, it is satisfied if for every $k \in \mathbb{N}$ and $(\alpha_1, \dots, \alpha_k) \in \mathcal{X}^k$, the random vectors $\{X_n : n \in \mathbb{N}\}$ are independent, where $X_n = (X_{\alpha_1}^n, \dots, X_{\alpha_k}^n)$.

Given any sequence of sub-$\sigma-$algebras $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of $\mathcal{F}$, we may introduce the associated tail σ-algebra

$$\mathcal{T} = \bigcap_{n \in \mathbb{N}} \sigma(\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \dots),$$

where $\sigma(\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \dots)$ is the σ-algebra generated by $\cup_{k \geq n+1} \mathcal{F}_k$.

Theorem 1.3.7. *Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be an independent collection of σ-fields. Then if $A \in \mathcal{T}$ then $\mathbb{P}(A) = 1$ or 0.*

1.4 Martingales and Uniform Integrability

In this section we introduce two special stochastic processes with important limit properties. Martingales are hugely studied by its applications. In order to define them we need to introduce conditional expectation and filtration.

Definition 1.4.1 (Conditional expectation)**.** Let X be an integrable random variable and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. A random variable Y that is measurable with respect to $\mathcal{G}$ and

$$\mathbb{E}[X 1_A] = \mathbb{E}[Y 1_A]$$

for all $A \in \mathcal{G}$ is called the conditional expectation of X given $\mathcal{G}$ and is written as $\mathbb{E}[X|\mathcal{G}]$.

We have some properties from the definition. Let X_1 and X_2 be integrable random variables:

- If $\mathcal{G} = \mathcal{F}$, then

$$\mathbb{E}[X_1|\mathcal{G}] = X_1.$$

- Linearity. Let $a \in \mathbb{R}$, then

$$\mathbb{E}[aX_1 + X_2|\mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + \mathbb{E}[X_2|\mathcal{G}].$$

- If X_1 is measurable with respect to $\mathcal{G}$, then

$$\mathbb{E}[X_1|\mathcal{G}] = X_1.$$

- If X_1 is independent of X_2, then

$$\mathbb{E}[X_1|\sigma(X_2)] = \mathbb{E}[X_1].$$

Definition 1.4.2 (Filtration)**.** A collection $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of sub-σ-algebras of $\mathcal{F}$ is called a filtration if $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$.

Let $\{X_n : n \in \mathbb{N}\}$ be a stochastic process. The $\sigma-$algebras

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$$

form a filtration. It is called the natural filtration of the stochastic process.

Definition 1.4.3. We say that a stochastic process $\{X_n : n \in \mathbb{N}\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if

- X_n is integrable for every $n \in \mathbb{N}$,
- X_n is measurable with respect to $\mathcal{F}_n$ for every $n \in \mathbb{N}$ and
- for all $n \in \mathbb{N}$

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n.$$

Example 1.4.4 (Random walk)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent real random variables with the same distribution, namely

$$\mu(\pm 1) = \frac{1}{2}.$$

For every $n \in \mathbb{N}$, we set

$$S_n = \sum_{k=1}^{n} X_k.$$

The stochastic process $\{S_n : n \in \mathbb{N}\}$ is called random walk. We will prove that this process is a martingale with respect to the natural filtration. For every $n \in \mathbb{N}$,

- S_n is integrable:

$$\mathbb{E}[|S_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|X_n|] = n\mathbb{E}[|X_1|] = n,$$

- it is clear that S_n is measurable with respect to $\mathcal{F}_n$ for every $n \in \mathbb{N}$,

- and

$$\begin{aligned}\mathbb{E}[S_{n+1}|\sigma(X_1, ..., X_n)] &= \mathbb{E}[S_n + X_{n+1}|\sigma(X_1, ..., X_n)] \\ &= \mathbb{E}[S_n|\sigma(X_1, ..., X_n)] + \mathbb{E}[X_{n+1}|\sigma(X_1, ..., X_n)] \\ &= S_n + \mathbb{E}[X_{n+1}] \\ &= S_n.\end{aligned}$$

The following result is an example of the good properties that we have from martingales.

Theorem 1.4.5. *Let $(X_n)_{n\in\mathbb{N}}$ a nonnegative martingale with respect to $\mathcal{F}_n$ such that*

$$\mathbb{E}[X_1] < \infty.$$

Then $(X_n)_{n\in\mathbb{N}}$ converges a.s. to a finite limit X and $\mathbb{E}[X] < \infty$.

Uniform integrability is considered as a generalization of dominated convergence theorem because it allows us interchange limit and expectation.

Definition 1.4.6. A stochastic process $\{X_n : n \in \mathbb{N}\}$ is said to be uniformly integrable if

$$\lim_{K\to\infty} \left(\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n| 1_{\{|X_n|\geq K\}}] \right) = 0.$$

If we have a uniformly integrable sequence such that there is a random variable X such that

$$\lim_{n\to\infty} X_n = X \text{ a.s.}$$

Then

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Starting from the definition it is a bit difficult verifying the condition of uniform integrability. The next theorem give us a simple condition that implies this. It is most often applied (and we will do) with $p = 2$.

Theorem 1.4.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^p] < \infty$$

for some $p > 1$. Then the sequence is uniformly integrable.

Chapter 2

Gaussian Fields

Before introducing the reader to Gaussian Multiplicative Chaos, we need to mention some facts about Gaussian random objects. Most of them are related with the kernel: Existence, regularity of trajectories, independence and distributions. This chapter is based on [Jan97] and [LG16].

Throughout this chapter, we deal with random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Some of the probability spaces should be chosen appropriately in the existence results that follow.

2.1 Gaussian Random Variables

We say that a real random variable Z is standard Gaussian if its density with respect to Lebesgue measure is

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Its characteristic function is

$$\mathbb{E}[e^{i\xi Z}] = e^{-\xi^2/2}.$$

Using the fact that

$$\int_{-\infty}^{+\infty} e^{-x^2/2} dx = \sqrt{2\pi} \qquad \text{and} \qquad \int_0^{+\infty} x e^{-x^2/2} dx = 1$$

and integrating by parts we get that Z belongs to L^p, for any positive integer p. Since $L^a \subset L^b$ for $b < a$, then Z belongs to all spaces L^p, $1 \leq p < \infty$. Moreover, we have

$$\mathbb{E}Z = 0, \qquad var(Z) := \mathbb{E}[Z^2 - (\mathbb{E}[Z])^2] = 1.$$

Let $\sigma > 0$ and $m \in \mathbb{R}$, a real random variable X is said to be Gaussian with $\mathcal{N}(m, \sigma^2)$-distribution when it satisfies any of the following equivalent properties:

1. $X \overset{\mathbb{D}}{=} \sigma Z + m$, where Z is a standard Gaussian variable;

2. the distribution of X has density

$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right);$$

3. the characteristic function of X is

$$\mathbb{E}[e^{i\xi X}] = \exp\left(im\xi - \frac{\sigma^2}{2}\xi^2\right).$$

We have then

$$\mathbb{E}[X] = m, \qquad var(X) = \sigma^2.$$

The distribution $\mathcal{N}(m, \sigma^2)$ is also called normal distribution. By convention, when $\sigma = 0$ (degenerate case), $\mathcal{N}(m, 0)$ is simply the Dirac mass at m, Property 3 still holds in that case. The reason is that the sequence $(X_n)_{n\in\mathbb{N}}$ with distribution $\mathcal{N}(m, \sigma_n^2)$ converge in distribution to $X = m$ a.s. when $\sigma_n \to 0$. In fact,

$$\lim_{n\to\infty} \exp\left(im\xi - \frac{\sigma^2}{2}\xi^2\right) = \exp(im\xi),$$

so this follows from Theorem 1.2.4.

Proposition 2.1.1. *Suppose that X follows the $\mathcal{N}(m, \sigma^2)$-distribution, X' follows the $\mathcal{N}(m', \sigma'^2)$-distribution and they are independent. Then $X+X'$ follows the $\mathcal{N}(m + m', \sigma^2 + \sigma'^2)$-distribution.*

Proof. This is a straightforward consequence of Proposition 1.3.4:

$$\begin{aligned}\phi_{X+X'}(\xi) = \phi_X(\xi)\phi_{X'}(\xi) &= \mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{i\xi X'}] \\ &= \exp\left(i(m+m')\xi - \frac{\sigma^2 + \sigma'^2}{2}\xi^2\right).\end{aligned}$$

□

A random vector $X = (X_1, \ldots, X_d)$ is said to be Gaussian if for all $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, the variable

$$Z := v \cdot X = v^t X = v_1 X_1 + \cdots v_d X_d \tag{2.1}$$

is a Gaussian random variable.
Remark. When we use matrix algebra, the vectors are in $\mathbb{R}^{d\times 1}$

Example 2.1.2. If $X_1, \cdots, X_d$ are independent Gaussian variables, Proposition 2.1.1 shows that the random vector $X = (X_1, \ldots, X_d)$ is a Gaussian vector.

Example 2.1.3. If $X_1, \cdots, X_d$ are independent Gaussian variables, the random vector $X = (X_1, \ldots, X_d)$ is a Gaussian vector thanks to Proposition 2.1.1.

Note that if $X = (X_1, \ldots, X_d)$ is a Gaussian vector then X_i is a Gaussian variable for every $i \in [\![1, d]\!]$. The converse is not true.
Let X be a Gaussian vector with values in $\mathbb{R}^d$. We have for every $v \in \mathbb{R}^d$,

$$\mathbb{E}[Z] = \mathbb{E}[v \cdot X] = v \cdot m,$$

$$var(Z) = var(v \cdot X) = \sum_{j,k=1}^{d} v_j v_k \Sigma(j,k) = v^t \Sigma v,$$

where $m = (\mathbb{E}X_1, \ldots, \mathbb{E}X_d)$ and $\Sigma(j,k) = cov(X_j, X_k) := \mathbb{E}[X_j X_k]$. Note that $var(Z)$ is nonnegative, it follows that for every $v \in \mathbb{R}^d$.

$$v^t \Sigma v \geq 0.$$

In this case, we say that the matrix Σ is positive semi-definite. Since Z follows the $\mathcal{N}(v \cdot m, v^t \Sigma v)-$distribution, we can easily obtain the characteristic function of X:

$$\mathbb{E}[\exp(i\xi \cdot X)] = \exp(i\xi \cdot m - \frac{1}{2}\xi^t \Sigma \xi).$$

As a result of Proposition 1.2.4 we can characterize the distribution of any Gaussian vector with the vector m and the matrix Σ. We denote by $\mathcal{N}(m, \Sigma)$ the distribution of this random vector. In this case, the Gaussian variables do not always have density.

Proposition 2.1.4. *A random vector is Gaussian if and only if its characteristic function is*

$$\exp\left(i\xi \cdot m - \frac{1}{2}\xi^t \Sigma \xi\right),$$

where $m \in \mathbb{R}^d$ and Σ is a symmetric and positive semi-definite matrix. This Gaussian vector has a density if and only if the matrix Σ is nondegenerate. Its density is given by

$$\varphi(v) = (2\pi)^{-d/2}(\det(\Sigma))^{-1/2} \exp\left(-\frac{1}{2}(v-m)^t\Sigma^{-1}(v-m)\right), \quad v \in \mathbb{R}^d.$$

Remark 2.1.5. If a matrix Σ is symmetric and for all $v \in \mathbb{R}^d - \{\overline{0}\}$

$$v^t\Sigma v > 0,$$

where $\overline{0} = \{0, 0, \ldots, 0\}$, then it is nondegenerate because its eigenvalues are real and greater that zero. If a symmetric matrix Σ' is positive semi-definite and degenerate we can approximate it by nondegenerate matrices. In fact, we can consider $\Sigma_\epsilon = \Sigma' + \epsilon I$ where $\epsilon > 0$ and I is the identity matrix. They are symmetric and for every $v \neq \overline{0}$,

$$v^t\Sigma_\epsilon v = v^t(\Sigma' + \epsilon I)v = v^t\Sigma' v + \epsilon \|v\|^2 > 0.$$

Now, we introduce a very useful Gaussian integration by parts. Let X be a Gaussian variable with mean zero and variance σ^2, then its density is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Note that

$$\sigma^2\varphi'(x) = -\varphi(x)x.$$

Hence given a continuously differentiable function $F : \mathbb{R} \to \mathbb{R}$, we can integrate by parts

$$\begin{aligned}
\mathbb{E}[XF(X)] &= \int xF(x)\varphi(x)dx \\
&= -\sigma^2 \int F(x)\varphi'(x)dx \\
&= -\sigma^2 F(x)\varphi(x)\big|_{-\infty}^{+\infty} + \sigma^2 \int F'(x)\varphi(x)dx \\
&= \sigma^2\mathbb{E}[F'(X)]
\end{aligned}$$

Therefore,

$$\mathbb{E}[XF(X)] = \mathbb{E}[X^2]\mathbb{E}[F'(X)]. \tag{2.2}$$

These computations are possible if

1. $XF(X)$ and $F'(X)$ are integrable and

2. $\lim_{x\to\pm\infty} F(x)\varphi(x) = 0$.

The identity (2.2) can be generalized to d dimensions. In this case second condition is written as

$$\lim_{\|x\|\to\infty} F(x)\varphi(x) = 0.$$

Lemma 2.1.6. *Let $X = (X_1, \ldots, X_d)$ be a centred Gaussian vector (every Gaussian coordinate has mean zero) and let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous differentiable function satisfying the condition of computation 1 and 2. Then*

$$\mathbb{E}[X_i F(X)] = \mathbb{E}[X_i X] \cdot \mathbb{E}[\nabla F(X)] = \sum_{j=1}^{d} \mathbb{E}[X_i X_j]\mathbb{E}\left[\frac{\partial}{\partial x_j} F(X)\right].$$

for every $i \in [\![1, d]\!]$.

Proof. Assume first that X has nondegenerate covariance matrix Σ. Then, its density is

$$\varphi(v) = (2\pi)^{-d/2}(\det(\Sigma))^{-1/2} \exp\left(-\frac{1}{2} v^t \Sigma^{-1} v\right).$$

Note that

$$\nabla\varphi(v) = -\varphi(v)\Sigma^{-1}v.$$

Then $\Sigma\nabla\varphi(v) = -\varphi(v)v$ and integrating by parts

$$\begin{aligned}
\mathbb{E}[XF(X)] &= \int vF(v)\varphi(v)dv \\
&= -\int F(v)\Sigma\nabla\varphi(v)dv \\
&= \Sigma\int \nabla F(v)\varphi(v)dv \\
&= \Sigma\mathbb{E}[\nabla F(X)].
\end{aligned}$$

To see the general case, replace Σ by the nondegenerate $\Sigma + \epsilon_n I$ and take limit $\epsilon_n \downarrow 0$. We can consider an independent Gaussian vector Y with covariance matrix I (see Example 2.6.5) and $Y_n := \sqrt{\epsilon_n}Y$. Then $X + Y_n$ has $\Sigma + \epsilon_n I$ as a covariance matrix and

$$\mathbb{E}[(X + Y_n)F(X + Y_n)] = (\Sigma + \epsilon_n I)\,\mathbb{E}[\nabla F(X + Y_n)].$$

Since every component of Y_n converges to 0 a.s., we have by dominated convergence theorem,

$$\mathbb{E}[XF(X)] = \Sigma\mathbb{E}[\nabla F(X)].$$

□

As a consequence, with the same properties of F in Lemma 2.1.6, we have the following result.

Corollary 2.1.7. *Let $Z = (Y, X_1, \ldots, X_d)$ be a centred Gaussian vector and $X = (X_1, \ldots, X_d)$, then*

$$\mathbb{E}[YF(X)] = \mathbb{E}[YX] \cdot \mathbb{E}[\nabla F(X)] = \sum_{i=1}^{d} \mathbb{E}[YX_i]\mathbb{E}\left[\frac{\partial F}{\partial x_i}(X)\right],$$

provided that $YF(X)$ is integrable.

Proof. We consider the function $\tilde{F} : \mathbb{R}^{d+1} \to \mathbb{R}$ defined by

$$\tilde{F}(y, x_1, \ldots, x_d) = F(x_1, \ldots, x_d).$$

It follows from the previous lemma that,

$$\mathbb{E}[Z\tilde{F}(Z)] = \Sigma\mathbb{E}[\nabla\tilde{F}(Z)].$$

In the first coordinate of the vector we have

$$\begin{aligned}
\mathbb{E}[YF(X)] = \mathbb{E}[Y\tilde{F}(Z)] &= \mathbb{E}[Y^2]\mathbb{E}\left[\frac{\partial\tilde{F}}{\partial y}(Z)\right] + \sum_{i=1}^{d}\mathbb{E}[YX_i]\mathbb{E}\left[\frac{\partial\tilde{F}}{\partial x_i}(Z)\right] \\
&= \mathbb{E}[Y^2]\mathbb{E}\left[\frac{\partial F}{\partial y}(X)\right] + \sum_{i=1}^{d}\mathbb{E}[YX_i]\mathbb{E}\left[\frac{\partial F}{\partial x_i}(X)\right] \\
&= 0 + \sum_{i=1}^{d}\mathbb{E}[YX_i]\mathbb{E}\left[\frac{\partial F}{\partial x_i}(X)\right].
\end{aligned}$$

Then

$$\mathbb{E}[Y(F(X))] = \sum_{i=1}^{d}\mathbb{E}[YX_i]\mathbb{E}\left[\frac{\partial F}{\partial x_i}(X)\right].$$

□

2.2 Gaussian Fields

Given $\mathcal{X}$ an arbitrary set, we say that the stochastic process $X = (X(x))_{x\in\mathcal{X}}$ is a Gaussian field indexed by $\mathcal{X}$ if, for every $d \in \mathbb{N}$ and $(x_1, \dots, x_d) \in \mathcal{X}^d$, $(X(x_1), \dots, X(x_d))$ is a Gaussian vector. In the literature, Gaussian fields indexed by subsets of $\mathbb{R}$ are usually called Gaussian processes.

The important thing to know about Gaussian fields is that their distribution is completely determined by their mean function $m : x \to \mathbb{E}[X_x]$ and covariance (also called covariance kernel or simply kernel) $K(x, y) = cov(X(x), X(y))$.

Definition 2.2.1. A function $\Sigma : \mathcal{X}^2 \to \mathbb{R}$ is called positive definite if for any $d \in \mathbb{N}$ and $(v_1, \dots, v_d) \in \mathbb{R}^d$,

$$\sum_{j,k=1}^{d} v_j v_k \Sigma(x_j, x_k) \geq 0. \tag{2.3}$$

Since the left-hand side in (2.3) corresponds to the variance of the variable Z in (2.1) (with $X_j = X(x_j)$), the covariance of a Gaussian field is positive definite. Moreover, it is easy to see that the covariance is symmetric. The following reciprocal statement also holds:

Theorem 2.2.2. *(Existence of Gaussian Fields) Given $m : \mathcal{X} \to \mathbb{R}$ arbitrary, $\Sigma : \mathcal{X}^2 \to \mathbb{R}$ positive definite and symmetric, there exists a unique distribution P on $(\mathbb{R}^{\mathcal{X}}, \mathcal{B}(\mathbb{R})^{\otimes\mathcal{X}})$, under which the coordinate projections $(X(x))_{x\in\mathcal{X}}$ constitute a Gaussian field with mean function m and covariance K.*

Proof. This is a consequence of the Daniell-Kolmogorov theorem (1.1.2). We consider

$$Q_{\mathcal{X}} = \{\nu_x = \mathcal{N}(m_x, \Sigma_x) : d \geq 1, x = (x_1, \dots, x_d) \in \mathcal{X}^d\},$$

where $m_x = (m(x_1), \dots, m(x_d))$ and $\Sigma_x = (\Sigma(x_j, x_k))_{1\leq j,k\leq d}$. The three conditions have to be fulfilled:

1. ν_x is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for every $x \in \mathcal{X}^d$.

2. Let $A = A_1 \times \cdots \times A_d$ where $A_i \in \mathcal{B}(\mathbb{R})$ for every $i \in [\![1, d]\!]$, $x' = (x_1, \dots, x_{d+1}) \in \mathcal{X}^{d+1}$, $x = (x_1, \dots, x_d)$ and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the function that projects into the first d coordinates,

$$\pi(x') = x.$$

We denote by $\nu_{x'}\pi^{-1}$ the measure on $\mathbb{R}^d$ defined by

$$\nu_{x'}\pi^{-1}(A) := \nu_{x'}(A \times \mathbb{R}).$$

for every $A \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\begin{aligned}
\nu_{x'}(A \times \mathbb{R}) &= \int_{\mathbb{R}^{d+1}} 1_{A\times\mathbb{R}}(v)\nu_{x'}(dv) \\
&= \int_{\mathbb{R}^{d+1}} 1_A(\pi(v))\nu_{x'}(dv) \\
&= \int_{\mathbb{R}^d} 1_A(w)(\nu_{x'}\pi^{-1})(dw).
\end{aligned}$$

The last equality is obtained by the change of variable formula. Hence, it is enough to show that $\nu_{x'}\pi^{-1} = \nu_x$. Its characteristic function is

$$\begin{aligned}
\int_{\mathbb{R}^d} \exp(i\xi \cdot w)(\nu_{x'}\pi^{-1})(dw) &= \int_{\mathbb{R}^{d+1}} \exp(i\xi \cdot \pi(v))\nu_{x'}(dv) \\
&= \int_{\mathbb{R}^{d+1}} \exp(i\xi' \cdot v)\nu_{x'}(dv) \\
&= \exp\left(i\xi' \cdot m_{x'} - \frac{1}{2}(\xi')^t\Sigma_{x'}\xi'\right) \\
&= \exp\left(i\xi \cdot m_x - \frac{1}{2}\xi^t\Sigma_x\xi\right),
\end{aligned}$$

where $\xi' \in \pi^{-1}(\xi)$ and its last coordinate is 0.

3. It is analogous to the previous one. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a permutation of coordinates $g(v_1, \ldots, v_d) = (v_{j_1}, \ldots, v_{j_d})$ and $y = (x_{j_1}, \ldots, x_{j_d})$. Then

$$\begin{aligned}
\nu_y(B_{j_1} \times \cdots \times B_{j_d}) &= \int_{\mathbb{R}^d} 1_{B_{j_1}\times\cdots\times B_{j_d}}(v)\nu_y(dv) \\
&= \int_{\mathbb{R}^d} 1_{B_{j_1}\times\cdots\times B_{j_d}}(g(g^{-1}(v)))\nu_y(dv) \\
&= \int_{\mathbb{R}^d} 1_{B_1\times\cdots\times B_d}(g^{-1}(v))\nu_y(dv) \\
&= \int_{\mathbb{R}^d} 1_{B_1\times\cdots\times B_d}(w)(\nu_y g)(dw).
\end{aligned}$$

Hence, it is enough to show that $\nu_y g = \nu_x$, its characteristic function is

$$\begin{aligned}\int_{\mathbb{R}^d} \exp(i\xi \cdot w)(\nu_y g)(dw) &= \int_{\mathbb{R}^d} \exp(i\xi \cdot g^{-1}(v))\nu_y(dv) \\ &= \int_{\mathbb{R}^d} \exp(ig(\xi) \cdot v)\nu_y(dv) \\ &= \exp\left(ig(\xi) \cdot m_y - \frac{1}{2}(g(\xi))^t \Sigma_y g(\xi)\right) \\ &= \exp\left(i\xi \cdot m_x - \frac{1}{2}\xi^t \Sigma_x \xi\right).\end{aligned}$$

□

The beautiful thing about this is that it lets us construct Gaussian fields with any prescribed mean function and positive definite covariance kernel. Now, we are going to choose some special kernels. For the three examples $(x_1, \ldots, x_d) \in \mathcal{X}^d$ and $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$. These are zero-mean Gaussian fields also called centered.

Example 2.2.3. (Random plane). $\mathcal{X} = \mathbb{R}^d$, $m(x) = 0$ and $K(x, y) = x \cdot y$. This satisfies the covariance kernel conditions,

$$\sum_{i,j=1}^{d} v_i v_j x_i \cdot x_j = \left|\sum_{i=1}^{d} v_i x_i\right|^2 \geq 0.$$

Since $X(x)$ is Gaussian for every $x \in \mathbb{R}^d$, it follows $\mathcal{N}(0, \|x\|)$-distribution. If we consider $Z = (Z_1, \ldots, Z_d)$ with independent coordinates which follow $\mathcal{N}(0, 1)$-distribution, it is a Gaussian vector with distribution $\mathcal{N}(0, \mathrm{Id})$. Then for every $x \in \mathbb{R}^d$, the random variable $Z \cdot x$ is Gaussian with the same distribution of $X(x)$, i.e., $X(x) \overset{\mathbb{D}}{=} Z \cdot x$. For each sample we have a plane in $\mathbb{R}^{d+1}$ that passes through the origin and has $(Z_1, \ldots, Z_d, -1)$ as a normal vector.

Example 2.2.4. (Standard Brownian Motion). $\mathcal{X} = [0, \infty)$, $m(x) = 0$ and $K(x, y) = \min\{x, y\}$. This satisfies the covariance kernel conditions,

$$\sum_{i,j=1}^{d} v_i v_j \min\{x_i, x_j\} \geq \min_{i \in [\![1,d]\!]} \{x_i\} \left(\sum_{i=1}^{d} v_i\right)^2 \geq 0.$$

Brownian motion is by far the most known stochastic process for its many applications.

Example 2.2.5. (Square Exponential). $\mathcal{X} = \mathbb{R}^d$, $m(x) = 0$ and $K(x, y) = \exp(-\alpha|x - y|^2)$, where α is a positive real number. This process is heavily used in machine learning and regression (prediction). This satisfies the covariance kernel conditions,

$$\sum_{i,j=1}^{d} v_i v_j e^{-\alpha|x_i - x_j|^2} \geq \exp\left(\min_{i,j\in[\![1,d]\!]}\{-\alpha|x_i - x_j|^2\}\right)\left(\sum_{i=1}^{d} v_i\right)^2 \geq 0.$$

Note that if X is a centred Gaussian vector and $X \overset{\mathbb{D}}{=} Y$, then Y is a centred Gaussian vector with the same covariance kernel, it is an application of Theorem 1.2.4. It can be extended to Gaussian fields.

Lemma 2.2.6. *Let $X = (X(x))_{x\in\mathcal{X}}$ be a centred Gaussian process with covariance K and Y be a stochastic process with the same distribution, then Y is a centred Gaussian field with the same covariance kernel.*

Proof. Let $d \in \mathbb{N}$ and $(x_1, \ldots, x_d) \in \mathcal{X}^d$, then

$$(X(x_1), \ldots, X(x_d)) \overset{\mathbb{D}}{=} (Y(x_1), \ldots, Y(x_d)).$$

It follows that $(Y(x_1), \ldots, Y(x_d))$ is a Gaussian vector. By definition, Y is a Gaussian field. Since the kernel of Y and X coincide in $\{x_1, \ldots, x_d\}^2$, we can choose indexes appropriately and have

$$\mathbb{E}[Y(x)] = 0$$

and

$$\mathbb{E}[Y(x)Y(y)] = K(x, y)$$

for every $x, y \in \mathcal{X}$. This completes the proof of the lemma. □

2.3 Gaussian Spaces

For the sake of extensive study on Gaussian random objects, we add this section. Here we will see that centred Gaussian fields may also be viewed as a subset of a Hilbert space.

A Gaussian linear space is a real linear space of random variables which are centered Gaussian. This space is a linear subspace of L^2 with the norm

$$\|X\|_2 = \left(\int |X|^2 d\mathbb{P}\right)^{1/2}$$

and inner product

$$\langle X, Y \rangle = \int XY d\mathbb{P} = \mathbb{E}[XY]$$

of L^2 on it. Thus, if $X_1, \ldots, X_n$ belong to a Gaussian linear space G and $v_1, \ldots, v_n$ are arbitrary real numbers, $\sum_{i=1}^n v_i X_i \in G$, and then $\sum_{i=1}^n v_i X_i \in G$ is a centred Gaussian variable.

A Gaussian Hilbert space is a Gaussian linear space which is complete with to the norm $\|\cdot\|_2$ (see Definition 3.1.11), i.e., a closed subspace of L^2 consisting of centred Gaussian random variables. A Gaussian linear space can always be completed to a Gaussian Hilbert space.

Theorem 2.3.1. *If $G \subset L^2$ is a Gaussian linear space, then its closure $\overline{G}$ in L^2 is a Gaussian Hilbert space.*

Proof. Suppose that $X \in \overline{G}$. We have to show that X has centered normal distribution. There exists a sequence $X_n \in G$ such that $X_n \to X$ in L^2. Let $\sigma^2 = \|X\|_2^2 = \mathbb{E}[X^2]$ and $\sigma_n^2 = \|X_n\|_2^2$. Then

$$\sigma_n^2 = \mathbb{E}[X_n^2] \to \mathbb{E}[X^2] = \sigma^2$$

as $n \to \infty$. Convergence in L^2 imply convergence in distribution (see Remark 2.5.2). Then for every $\xi \in \mathbb{R}$,

$$\mathbb{E}[e^{i\xi X}] = \lim_{n\to\infty} \mathbb{E}[e^{i\xi X_n}] = \lim_{n\to\infty} \exp\left(-\frac{\sigma_n}{2}\xi^2\right) = \exp\left(-\frac{\sigma}{2}\xi^2\right).$$

The result then follows from Theorem 1.2.4. □

We give some important examples of Gaussian spaces.

Example 2.3.2. Let X be any nondegenerate, centred Gaussian variable. Then $\{vX : v \in \mathbb{R}\}$ is one-dimensional Gaussian Hilbert space.

Example 2.3.3. Let $X = (X_1, \ldots, X_d)$ be a centred Gaussian vector. Then their linear span

$$\left\{ \sum_{i=1}^d v_i X_i : v_i \in \mathbb{R} \right\}$$

is a finite-dimensional Gaussian Hilbert space.

Example 2.3.4. Let $Z_1, \ldots, Z_d$ be independent real Gaussian variables. Then their linear span

$$\left\{ \sum_{i=1}^{d} v_i Z_i : v_i \in \mathbb{R} \right\}$$

is a finite-dimensional Gaussian Hilbert space. In this case $\{Z_1, \ldots, Z_d\}$ is an orthogonal base.

Example 2.3.5. More generally, if $X = (X(x))_{x \in \mathcal{X}}$ is a centred Gaussian field, then the linear span of $\{X(x) : x \in \mathcal{X}\}$ is a Gaussian linear space, and by Theorem 2.3.1, the closed linear span of $\{X(x) : x \in \mathcal{X}\}$ in L^2 is a Gaussian Hilbert space.

2.4 Regularity of Trajectories

Definition 2.4.1. Let $(X(x))_{x \in \mathcal{X}}$ be a stochastic process. The sample paths of X are the mappings

$$\begin{array}{ccc} \mathcal{X} & \to & \mathbb{R} \\ x & \mapsto & X(x)(\omega) \end{array}$$

obtained by fixing $\omega \in \Omega$. The sample paths of X thus form a family of mappings from $\mathcal{X}$ into $\mathbb{R}$ indexed by $\omega \in \Omega$.

From now on the index set $\mathcal{X}$ will be a metric space (T, ρ). Thus, it makes sense to talk about continuity or measurability of sample paths. In this section, we will show that by making some small modifications we can obtain those properties.

Definition 2.4.2. Let $(X(t))_{t \in T}$ and $(\tilde{X}(t))_{t \in T}$ be two stochastic processes. We say that $\tilde{X}$ is a modification of X (also we can say that $\tilde{X}$ is a version of X) if

$$\forall t \in T, \quad \mathbb{P}\{X(t) = \tilde{X}(t)\} = 1.$$

Modifications conserve the distribution.

Proposition 2.4.3. *Let $\tilde{X}$ be a version of X, then $\tilde{X}$ and X has the same distribution.*

Proof. Let $d \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_d) \in \mathcal{X}^d$, consider $V = (X(x_1), \ldots, X(X_d))$ and $\tilde{V} = (\tilde{X}(x_1), \ldots, \tilde{X}(X_d))$. We have

$$\{V = \tilde{V}\} = \{X(x_1) = \tilde{X}(x_1), \ldots, X(x_d) = \tilde{X}(x_d)\} = \bigcap_{j=1}^{d} \{X(x_j) = \tilde{X}(x_j)\},$$

then $\mathbb{P}(V = \tilde{V}) = 1$. Now, for every $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{P}(V \in A) &= \mathbb{P}(V \in A, V = \tilde{V}) + \mathbb{P}(V \in A, V \neq \tilde{V}) \\ &= \mathbb{P}(\tilde{V} \in A, V = \tilde{V}) + \mathbb{P}(\tilde{V} \in A, V \neq \tilde{V}) \\ &= \mathbb{P}(\tilde{V} \in A).\end{aligned}$$

The equality $\mathbb{P}(\tilde{V} \in A, V \neq \tilde{V}) = \mathbb{P}(V \in A, V \neq \tilde{V}) = 0$ holds for every $A \in \mathcal{B}(\mathbb{R}^d)$. It follows that V and $\tilde{V}$ have the same distribution. □

The following result is a consequence of the previous proposition and Lemma 2.2.6.

Corollary 2.4.4. *Let X be a centred Gaussian field with covariance kernel K, then a modification $\tilde{X}$ is also a centred Gaussian field with covariance kernel K.*

Particularly, if X is a random plane or a standard Brownian motion, then $\tilde{X}$ is also a random plane or a standard Brownian motion, respectively. However, sample paths of $\tilde{X}$ might have quite different properties than the sample paths of X.

Example 2.4.5. Let $([0,1], \mathcal{B}([0,1), \lambda)$ be a probability space where λ is the Lebesgue measure on $[0,1]$ and $\mathcal{X} = [0,1]$. Let us consider D the diagonal of $[0,1] \times [0,1]$ and we define

$$X(x,\omega) = 0 \ \forall (x,\omega), \quad \tilde{X}(x,\omega) = 1_D(x,\omega).$$

We have $X(x,\omega) = 0$ for ever $x, \omega \in [0,1]$ and $\tilde{X}(x,\omega) = 0$ for $\omega \neq x$, 1 for $\omega = x$. We have therefore $X(x,\omega) = \tilde{X}(x,\omega)$ for all $\omega \neq x$ that it to say a.s. because $\lambda(D) = 0$. Then X and $\tilde{X}$ are version of the same process. However, the trajectories of X are continuous while those of $\tilde{X}$ are not.

The following theorem given by Kolmogorov and Chentsov provides a widely used criterion for establishing when a stochastic process has a version whose sample paths are continuous. It applies to processes indexed by parameters that takes values in Euclidean spaces.

Theorem 2.4.6. ***(Kolmogorov-Chentsov)*** *Let X be a stochastic process whose index set is a compact $D \subset \mathbb{R}^d$ such that exists $a, b, c > 0$ verifying for every $t, s \in D$:*

$$E[|X(t) - X(s)|^a] \leq c|t - s|^{1+b}.$$

Then exists a continuous version of X.

Example 2.4.7. Let us consider the Brownian motion $B = (B(x))_{x\in[0,1]}$. If $x < y$, the random variable $B(y) - B(x)$ is distributed according to $\mathcal{N}(0, y - x)$. In fact, $B(y) - B(x)$ is a centred Gaussian variable and

$$var(B(y) - B(x)) = y - 2(\min\{x, y\}) + x = y - x.$$

Thus $B(y) - B(x) \overset{\mathbb{D}}{=} \sqrt{y-x}Z$, where Z is a standard Gaussian variable. Consequently, for every $a > 0$,

$$E[|B(y) - B(x)|^a] = (y - x)^{a/2}E[|Z|^a] = C_a(y - x)^{a/2}$$

where $C_a = E[|Z|^a] < \infty$. Taking $a > 2$, we can apply Theorem 2.4.6 with $b = \frac{a}{2} - 1$. It follows that B has a modification which has continuous sample paths.

In Section 4.2 we will see, under more conditions on the metric space T, we can get a measurable modification of Gaussian fields with continuous kernel.

2.5 Convergence of Gaussian Processes

There are several different notions of continuity for stochastic processes; the three most important are continuity in probability, in L^2, and almost surely.

1. Continuity in probability:

   $$\lim_{s\to t} \mathbb{P}\{|X(t) - X(s)| > \epsilon\} = 0, \text{ for each } t \in T \text{ and each } \epsilon > 0.$$

   For t fixed we say that $X(s)$ converges in probability to $X(t)$. It is denoted by $X(s) \xrightarrow{\mathbb{P}} X(t)$.

2. Continuity in mean square, or L^2 continuity:

   $$\lim_{s\to t} E[|X(t) - X(s)|^2] = 0, \text{ for each } t \in T.$$

   For t fixed we say that $X(s)$ converges in L^2 to $X(t)$. It is denoted by $X(s) \xrightarrow{L^2} X(t)$.

3. Continuity with probability one, or almost sure

   $$\mathbb{P}\{\lim_{s\to t} |X(t) - X(s)| = 0, \text{ for all } t \in T\} = 1.$$

   For t fixed we say that $X(s)$ converges almost sure to $X(t)$. It is denoted by $X(s) \xrightarrow{a.s.} X(t)$.

The limit of functions $f : T \to \mathbb{R}$ can be obtained in a discrete way

$$\lim_{s\to t} f(t) = L \iff \text{For every } t_n \to t : \lim_{n\to\infty} f(t_n) = L.$$

As a consequence we obtain a discrete theory which is easier to study. The stochastic process (a sequence) $\{X_n : n \in \mathbb{N}\}$ converges to X

1. in probability if

$$\lim_{n\to\infty} \mathbb{P}\{|X_n - X| > \epsilon\} = 0, \quad \text{and each } \epsilon > 0.$$

   It is denoted by $X_n \xrightarrow{\mathbb{P}} X$;

2. in L^2 if

$$\lim_{n\to\infty} E[|X_n - X|^2] = 0, \ .$$

   It is denoted by $X_n \xrightarrow{L^2} X$;

3. with probability one, or almost sure if

$$\mathbb{P}\left\{\lim_{n\to\infty} |X_n - X| = 0\right\} = 1.$$

   It is denoted by $X_n \xrightarrow{a.s.} X$.

These modes of convergence are related

Proposition 2.5.1. *If a sequence $(X_n)_{n\in\mathbb{N}}$ converges a.s. or in L^2 to X, then it converges in probability. Conversely, if a sequence $(X_n)_{n\in\mathbb{N}}$ converges in probability to X, there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ which converges a.s. to the random variable X.*

Remark 2.5.2. Finally, we introduced convergence in distribution in Section 1.2, it is the weakest mode of convergence in the sense that convergence in probability implies convergence in distribution.

For Gaussian fields defined on metric spaces, continuity in probability and in L^2 are equivalent; moreover they can easily be characterized using the covariance kernel.

Theorem 2.5.3. *Let $(X(t))_{t\in T}$ be a centred Gaussian field defined on a metric space T. Then the following are equivalent.*

(i) $t \mapsto X(t)$ *is continuous in probability.*

(ii) $t \mapsto X(t)$ *is continuous in* L^2*.*

(iii) The covariance kernel $K(s,t)$ *is a continuous function on* $T \times T$*.*

Proof. $(i) \Rightarrow (ii)$ Let $t \in T$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in T such that $t_n \to t$. We want to prove

$$X(t_n) \xrightarrow{L^2} X(t).$$

By hypothesis

$$X(t_n) \xrightarrow{\mathbb{P}} X(t),$$

then $Y_n = X(t_n) - X(t) \xrightarrow{\mathbb{P}} 0$. The random variable Y_n is Gaussian and follows the $\mathcal{N}(0,\sigma_n)$-distribution. Since $Y_n \xrightarrow{\mathbb{D}} 0$, by Theorem 1.2.4

$$\exp\left(-\frac{\xi^2\sigma_n^2}{2}\right) = \phi_{Y_n}(\xi) \to \phi_0(\xi) = \exp\left(-\frac{\xi^2 0^2}{2}\right) = 1.$$

With $\xi = 1$ we have $\sigma_n \to 0$. Then

$$\mathbb{E}[|X(t_n) - X(t)|^2] = \mathbb{E}[Y_n^2] = \sigma_n^2 \to 0.$$

$(i) \Leftarrow (ii)$ It follows from Proposition 2.5.1.

$(ii) \Rightarrow (iii)$ By definition $K(t,s) = \mathbb{E}[X(t)X(s)]$, it is a inner product in L^2, then it is continuous in $L^2 \times L^2$, i.e., if $X_n \xrightarrow{L^2} X$ and $Y_n \xrightarrow{L^2} Y$ we have

$$\lim_{n\to\infty} \mathbb{E}[X_n Y_n] = \mathbb{E}[XY].$$

Let $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ be sequences in T such that $t_n \to t$ and $s_n \to s$. Since $X(t_n) \xrightarrow{L^2} X(t)$ and $Y(s_n) \xrightarrow{L^2} Y(s)$, we have

$$\lim_{n\to\infty} K(t_n, s_n) = \lim_n \mathbb{E}[X(t_n)Y(s_n)] = \mathbb{E}[X(t)Y(s)] = K(t,s).$$

$(ii) \Leftarrow (iii)$ Since

$$\mathbb{E}[|X(t) - X(s)|^2] = K(t,t) - 2K(t,s) + K(s,s),$$

it follows from continuity of $K(t,t)$ in $T \times T$. □

2.6 Independence of Gaussian Fields

Let $K(s,t)$ be a real function, positive definite and symmetric defined on $T \times T$. From now on we will call this functions as kernel, then by Theorem 2.2.2 there exists a probability space and a centred Gaussian field defined on it with covariance kernel K_1. If I have another kernel K_2 we can get a centred Gaussian field defined in another probability space, not necessarily the same. In this section we define a probability space where both Gaussian fields exist and they are independent. This idea will be extended for a countable collections of kernels.

Proposition 2.6.1. *Let K_1 and K_2 be kernels defined on $T \times T$ then there exists a probability space and independent centred Gaussian fields $(X(t))_{t\in T}$ and $(Y(t))_{t\in T}$ defined on it with covariance kernel K_1 and K_2, respectively.*

Proof. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be the spaces obtained by Theorem 2.2.2 and the Gaussian fields $X' = (X'(t))_{t\in T}$ and $Y' = (Y'(t))_{t\in T}$ with covariance function K_1 and K_2, respectively, and

$$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$$

be the product space and $\pi_i : \Omega_1 \times \Omega_2 \to \Omega_i$ the projections on the $i-$th coordinate, $i = 1, 2$. We claim that the process $X = (X(t))_{t\in T}$, where

$$X(t) = X'(t) \circ \pi_1,$$

defined on the product space is a Gaussian field with K_1 as covariance kernel. According to Lemma 2.2 we have to proof that X and X' have the same distribution. Let $d \in \mathbb{N}$ and $(t_1, \ldots, t_d) \in T^d$, then the characteristic function of $V = (X(t_1), \ldots, X(t_d))$ is

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} \exp\left(i\sum_{i=1}^{d} \xi_i X(t_i)(\omega_1,\omega_2)\right) d\mathbb{P}_1 \otimes \mathbb{P}_2 \\ = \int_{\Omega_1\times\Omega_2} \exp\left(i\sum_{i=1}^{d} \xi_i X'(t_i)(\omega_1)\right) d\mathbb{P}_1 \otimes \mathbb{P}_2 \\ = \int_{\Omega_1} \exp\left(i\sum_{i=1}^{d} \xi_i X'(t_i)(\omega_1)\right) d\mathbb{P}_1.\end{aligned}$$

The latter is the characteristic function of $V' = (X'(t_1), \ldots, X'(t_d))$. Analogously, we define $Y = (Y(t))_{t\in T}$. Let us see that they are independent. Let $W =$

$(Y(t_1), \dots, Y(t_d))$ and $W' = (Y'(t_1), \dots, Y'(t_d))$. Note that

$$V = V' \circ \pi_1 \quad \text{and} \quad W = W' \circ \pi_2.$$

Then for every $A, B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}
\mathbb{P}_1 \otimes \mathbb{P}_2(V \in A, W \in B) &= \mathbb{P}_1 \otimes \mathbb{P}_2((V')^{-1}(A) \times \Omega_2 \cap \Omega_1 \times (W')^{-1}(B)) \\
&= \mathbb{P}_1 \otimes \mathbb{P}_2((V')^{-1}(A) \times (W')^{-1}(B)) \\
&= \mathbb{P}_1(V' \in A)\mathbb{P}_2(W' \in B) \\
&= \mathbb{P}_1 \otimes \mathbb{P}_2(V \in A) \cdot \mathbb{P}_1 \otimes \mathbb{P}_2(W \in B).
\end{aligned}$$

□

The last equation is due to the equality of distributions.

Example 2.6.2. We can repeat the procedure in a finite case. Let Σ_1 and Σ_2 be symmetric and positive semi-definite matrices $\mathbb{R}^{d\times d}$. Then there exists a probability space and independent centred Gaussian vectors X and Y defined on it with covariance matrix Σ_1 and Σ_2, respectively. In this case is not necessary Daniell-Kolmogorov theorem.

Example 2.6.3. Let K_1 be a kernel and σ a positive number. Note that if we set $K_2(s,t) = \sigma$, it is a kernel. Thus, we can get a centred Gaussian field $(X(t))_{t\in T}$ and a Gaussian variable σZ where Z is a standard Gaussian variable such that for every $d \in \mathbb{N}$ and $(t_1, \dots, t_d) \in T^d$, the variables $(X(t_1), \dots, X(t_d))$ and Z are independent.

We repeat the procedure for a countable collection of kernels.

Theorem 2.6.4. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of kernels defined on $T \times T$, then there exists a probability space and independent centred Gaussian fields $(X_n(t))_{t\in T}$ defined on it with covariance kernel K_n, respectively for every $n \in \mathbb{N}$.*

Proof. For every $n \in \mathbb{N}$, let $(X'_n(t))_{t\in T}$ be a centred Gaussian field with covariance kernel K_n defined on $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$,

$$(\Omega, \mathcal{F}, \mathbb{P}) = \left(\prod_{n\in\mathbb{N}} \Omega_n, \bigotimes_{n\in\mathbb{N}} \mathcal{F}_n, \bigotimes_{n\in\mathbb{N}} \mathbb{P}_n \right),$$

the product space and $\pi_n(\omega) = \omega_n$ are projections. We consider the stochastic processes

$$\{X_n(t) = X'_n(t) \circ \pi_n, t \in T : n \in \mathbb{N}\}$$

defined on the product space. Let $d \in \mathbb{N}$ and $(t_1, \dots, t_d) \in T^d$, the characteristic function of $V_n = (X_n(t_1), \dots, X_n(t_d))$ is

$$\begin{aligned}\int_\Omega \exp\left(i \sum_{i=1}^d \xi_i X_n(t_i)(\omega)\right) d\mathbb{P} &= \int_\Omega \exp\left(i \sum_{i=1}^d \xi_i X'_n(t_i)(\omega_n)\right) d\mathbb{P} \\ &= \int_{\Omega_n} \exp\left(i \sum_{i=1}^d \xi_i X'_n(t_i)(\omega_n)\right) d\mathbb{P}_n.\end{aligned}$$

Then $(X_n(t))_{t\in T}$ and $(X'_n(t))_{t\in T}$ have the same distribution. To prove that the collection of stochastic processes $\{X_n(t) : n \in \mathbb{N}\}$ is independent we need to prove that the collection $\{V_n : n \in \mathbb{N}\}$ is independent (see Remark 1.3.6), i.e., that the collection of σ-algebras $\{V_n^{-1}(A_n) : n \in \mathbb{N}, A_n \in \mathcal{B}(\mathbb{R}^d)\}$ is independent. For a finite collection $\{n_1, \dots, n_k\} \subset \mathbb{N}$ we set

$$V'_{n_i} = (X'_{n_i}(t_1), \dots, X'_{n_i}(t_d)) = V_{n_i} \circ \pi_{n_i}$$

for $i \in [\![1, k]\!]$. Then,

$$\begin{aligned}\mathbb{P}(V_{n_1} \in A_{n_1}, \dots V_{n_k} \in A_{n_k}) &= \mathbb{P}((V'_{n_1})^{-1}(A_{n_1}) \times \cdots \times (V'_{n_k})^{-1}(A_{n_k})) \\ &= \mathbb{P}_{n_1}(V'_{n_1} \in A_{n_1}) \times \cdots \times \mathbb{P}_{n_k}(V'_{n_k} \in A_{n_k}) \\ &= \mathbb{P}(V_{n_1} \in A_{n_1}) \times \cdots \times \mathbb{P}(V_{n_k} \in A_{n_k}).\end{aligned}$$

In the first equation the product is in $\otimes_n \mathcal{F}_n$, complete the rest with Ω_n for $n \in \mathbb{N} \setminus \{n_1, \dots, n_k\}$.

□

Example 2.6.5. Again as Example 2.6.2. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of symmetric and positive semi-definite matrices $\mathbb{R}^{d\times d}$. Then there exists a probability space and independent centred Gaussian vectors $(X_n)_{n\in\mathbb{N}}$ defined on it with covariance matrix Σ_n, respectively for every $n \in \mathbb{N}$.

Chapter 3

Random Measures

A Gaussian Multiplicative Chaos is formally defined as a random measure. We develop random theory on the space of Radon measures defined on a Polish metric space, then we define random measures and some basic notions of them. This chapter is based on [DVJ08a] and [DVJ08b].

3.1 Measures on Metric Spaces

The subject of this section is measure theory on metric spaces. We denote the space by T and its metric by ρ. Let $\mathcal{B}(T)$ be the Borel $\sigma-$algebra, the one generated by the open sets; its elements are the Borel sets. Every measure on T in this text is a nonnegative, countably additive set function defined on $\mathcal{B}(T)$.

A metric space is called locally compact if for each point $x \in T$ it has a compact neighborhood, i.e., there exists an open set U and a compact K included in T, such that $x \in U \subset K$.

Let $E \in \mathcal{B}(T)$. The measure μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open}\}.$$

Similarly, μ is named inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\},$$

and locally finite if every point of T has a neighborhood U for which $\mu(U)$ is finite. This property can be characterized with compact subsets.

Lemma 3.1.1. *On locally compact metric spaces, locally finiteness is equivalent to finiteness on compact subsets.*

Proof. Let μ be a locally finite measure and $K \subset T$ compact. For every $x \in K$, we have an open set U_x such that $\mu(U_x) < \infty$. Since $\{U_x : x \in K\}$ is an open cover for K, there is a finite subcover $\{U_{x_i} : i \in [\![1, n]\!], x_i \in K\}$ such that

$$K \subset \bigcup_{i=1}^{n} U_{x_i},$$

then

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^{n} U_{x_i}\right) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty.$$

Conversely, for every $x \in T$, there is an open set U and a compact set K, such that $x \in U \subset K$. Since $\mu(K) < \infty$, then $\mu(U) < \infty$. □

If μ is outer regular and inner regular on all Borel sets, μ is called regular.

Definition 3.1.2. A measure μ on a locally compact metric space is called Radon measure if it is locally finite and regular.

A metric space is called separable if it contains a countable dense subset i.e. there exists a set $D = \{x_n \in T : n \in \mathbb{N}\}$ such that $\overline{D} = T$. When T is locally compact and separable there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of T such that $T = \bigcup_{n\in\mathbb{N}} K_n$. This space is σ-compact.

Definition 3.1.3. A metric space is said to be σ-compact if it is union of countable many compact subsets.

As a consequence, in locally compact separable metric spaces.

- It can be obtained a special increasing sequence of compacts which contain all compacts (see Proposition 3.1.8).
- A Radon measure is σ-finite.
- Every closed set F is σ-compact, i.e., it can be written as a countable union of compact sets. In fact,

$$F = \bigcup_{n\in\mathbb{N}} F \cap K_n.$$

- Every open set is σ-compact, i.e., it can be written as a countable union of compact sets. In fact, consider

$$F_n = \left\{ x \in T : \rho(x, F) < \frac{1}{n} \right\},$$

where $\rho(x, F) = \inf\{d(x, y) : y \in F\}$. Then

$$F = \bigcap_{n \in \mathbb{N}} F_n.$$

Note that F_n is an open set. Thus, by Morgan's law every open set is a countable union of σ-compact sets.

An important property of these spaces is the ease of finding regular measures. The next result is a special case of Theorem 2.18 of [Rud87] recalling here only what we need.

Theorem 3.1.4. *Let T be a locally compact separable metric space and μ a measure on T such that $\mu(K) < \infty$ for every compact set. Then μ is regular.*

Remark 3.1.5. Probability measures defined on Polish metric spaces (see Definition 3.1.13) are regular (see [Bil99]).

As a consequence of the previous theorem, a measure μ on T is a Radon measure if and only if μ is finite on compact sets.

Example 3.1.6. Dirac measure is a Radon measure on locally compact separable metric spaces. For every compact set K, we have $\delta_a(K) = 1$ or 0, for some a in the space. In general, finite measures in these spaces are Radon measures.

Example 3.1.7. Lebesgue measure λ on $\mathbb{R}^d$ is a Radon measure. Since every closed ball is compact, this space is locally compact. $\mathbb{R}^d$ is separable, and for every compact K

$$\lambda(K) < \infty.$$

Proposition 3.1.8. *Let T be a locally compact separable metric space. Then there exists an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of T such that $K_n \subset K_{n+1}^{\circ}$ for every $n \in \mathbb{N}$ and $T = \cup_{n \in \mathbb{N}} K_n$. Furthemore, every compact K of T is contained in a K_j for some $j \in \mathbb{N}$.*

Proof. Let $(K'_n)_{n\in\mathbb{N}}$ be the family of compact sets of T such that $\bigcup_{n\in\mathbb{N}} K'_n = T$. We define the desired family of compacts as follows. Let $K_0 = K'_0$. For each $x \in K_0$ let U_x be the open set obtained by definition of locally compact. Since $\{U_x : x \in K_0\}$ is an open cover for K_0, there is a finite subcover $\{U_{x_i} : i \in [\![1, m]\!]\}$. Set

$$U_0 = \overline{\bigcup_{i=0}^{m} U_{x_i}} = \bigcup_{i=0}^{m} \overline{U_{x_i}}.$$

Since every U_{x_i} has compact closure, U_0 is compact and

$$K_0 \subset \bigcup_{i=1}^{m} U_{x_i} \subset U_0^\circ.$$

Define $K_1 = K'_1 \cup U_0$. It follows that $K_0 \subset K_1^\circ$. We continue with the same procedure by induction. Let U_n be a compact set such that $K_n \subset U_n^\circ$ and define $K_{n+1} = K'_{n+1} \cup U_n$ for every $n \in \mathbb{N}$. Since $K'_n \subset K_n$, we have,

$$T = \bigcup_{n\in\mathbb{N}} K'_n = \bigcup_{n\in\mathbb{N}} K_n.$$

Suppose $K \subset T$ is compact, then $\{K_n^\circ : n \in \mathbb{N}\}$ is an open cover of K and there is a finite subcover $\{K_{n_i}^\circ : i \in [\![1, m]\!]\}$. Set $N = \max\{n_1, \ldots, n_m\}$ then

$$K \subset \bigcup_{i=1}^{m} K_{n_i}^\circ = K_N^\circ \subset K_N.$$

□

The next definition will be important to define random measures.

Definition 3.1.9. A measure μ on a metric space is called boundedly finite if $\mu(B) < \infty$ for all bounded measurable set $B \in \mathcal{B}(T)$.

A subset B of T is called bounded if it is included in a ball of finite radius. Thus, this definition only depends on the metric; if we change the metric, it is possible that a bounded set is not bounded in the new metric and vice versa as we will see in Example 3.1.10.

Note that if T is a locally compact separable metric space and μ is boundedly finite, then it is a Radon measure because compact sets are bounded, so we can use Theorem 3.1.4. However, the next example shows that the converse does not hold.

Exercise 3.1.10. We consider $\mathbb{R}$ with the metric $\rho(x,y) = \min\{|x-y|, 1\}$ which induces the same topology than the usual metric, but in this metric every measurable set is bounded, in particular $\mathbb{R}$. The Lebesgue measure λ is a Radon measure but $\lambda(\mathbb{R}) = \infty$.

Remark. If two metrics ρ_1 and ρ_2 on T generate the same topology we say that they are equivalent. Note that if there exists two positive constants α, β such that

$$\alpha\rho_1(x,y) \leq \rho_2(x,y) \leq \beta\rho_1(x,y),$$

then ρ_1 and ρ_2 are equivalent. The converse is not true.

Since measures on a metric space T are defined on $\mathcal{B}(T)$ that is generated by open sets, they ignore the interchange of equivalent metrics. Topological properties such as locally compactness or separability are preserved under this interchange. With this in mind, we can ask if there exists an equivalent metric of a locally compact separable metric space such that bounded sets are included in compact sets. The answer is affirmative, it is a consequence of the Theorem 2.61 of [HG61]. It says that there is an equivalent metric such that

$$\forall B \subset T \text{ bounded, } \overline{B} \text{ is compact,} \tag{3.1}$$

that is, a set is compact if and only if it is bounded and closed. Therefore, in this space, a measure μ is a Radon measure if and only if it is boundedly finite.

There is a property that is not preserved by interchange of equivalent metrics, completeness.

Definition 3.1.11. A metric space is said to be complete if every Cauchy sequence of points in T has a limit that is also an element of T.

The following proposition shows that a metric space with the property (3.1) is complete.

Proposition 3.1.12. *Let T be a metric space such that every bounded set has a compact closure then T is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in T, then it is bounded. Since $\overline{\{x_n : n \in \mathbb{N}\}}$ is bounded and closed, it is compact. Hence, the sequence has a convergence subsequence converging to $x_0 \in T$. For being Cauchy sequence, the whole sequence converges to x_0. □

Definition 3.1.13. A metric space separable and complete is called Polish

3.2 Random Measures

Random measures theory is a branch of modern probability of increasing interest in both theory and applications. Throughout the next sections of this chapter we deal with measures defined on a locally compact Polish metric space T with the property (3.1) and random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In order to define random measures we introduce the following family of measures

$$\mathcal{M} = \mathcal{M}(T) = \{\mu \text{ boundedly finite on } T\}$$

and the family of sets

$$\{A \in \mathcal{B}(T) : B \text{ is bounded}\}.$$

In $\mathcal{M}$ we introduce the $\sigma-$algebra $\mathcal{B}(\mathcal{M})$ generated by all evaluation maps π_A with $A \in \mathcal{A}$ which are defined by

$$\begin{array}{rccc} \pi_A : & \mathcal{M} & \to & \mathbb{R}_+ \\ & \mu & \mapsto & \mu(A). \end{array}$$

It is the smallest $\sigma-$algebra with respect to which every evaluation map is a measurable function. In $\mathbb{R}_+$ is considered $\mathcal{B}(\mathbb{R}_+)$.

Definition 3.2.1. A random measure M is a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$.

For the richness of the metric space we can make the definition more flexible. First, by Property (3.1),

$$\mathcal{M} = \{\mu \text{ Radon measure on } T\}.$$

Second, in the definition of $\mathcal{B}(\mathcal{M})$ we can consider the whole collection of evaluation maps $A \in \mathcal{B}(T)$.

Proposition 3.2.2. *The σ-algebra $\mathcal{B}(\mathcal{M})$ is the smallest with respect to which every evaluation map π_A, $A \in \mathcal{B}(T)$ is measurable, i.e.,*

$$\sigma(\pi_A : A \in \mathcal{B}(T)) = \sigma(\pi_A : A \in \mathcal{A}).$$

Proof. Let $A \in \mathcal{B}(T)$. We have to prove that the map

$$\begin{array}{rccc} \pi_A : & (\mathcal{M}, \mathcal{B}(\mathcal{M})) & \to & (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \\ & \mu & \mapsto & \mu(A) \end{array}$$

is measurable. We consider a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ included in A such that $\mu(A) = \lim_{n\to\infty} \mu(K_n)$. Then for every $a \in \mathbb{R}_+$,

$$\{\mu \in \mathcal{M} : \mu(A) \geq a\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \{\mu \in \mathcal{M} : \mu(K_n) > a - 1/n\}.$$

In other word,

$$\pi_A^{-1}([a,\infty)) = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \pi_{K_n}^{-1}\left((a - \frac{1}{n}, \infty)\right).$$

The proof follows by Remark 3.2.4. The other inclusion is trivial. □

Remark 3.2.3. In the other sense we can consider a smaller collection thanks to Theorem A.2.6.III of [DVJ08a]. The σ-algebra $\mathcal{B}(\mathcal{M})$ is the smallest σ-algebra with respect to which the evaluation map π_A is measurable for all sets in a semiring $\mathcal{S}$ of bounded sets generating $\mathcal{B}(T)$. Therefore

$$\sigma(\pi_A : A \in \mathcal{B}(T)) = \sigma(\pi_A : A \in \mathcal{A}) = \sigma(\pi_A : A \in \mathcal{S}).$$

A semiring $\mathcal{S}$ is a collection of subsets of T with the properties (i) $\mathcal{S}$ is a π-system (ii) every symmetric difference of sets of $\mathcal{S}$ can be represented as a finite union of disjoint sets in $\mathcal{S}$.

Remark 3.2.4. It is useful to note that the collection $\{\pi_A^{-1}(G) : G \in \mathcal{B}(\mathbb{R}_+), A \in \mathcal{R}\}$ generates the σ-algebra $\sigma(\pi_A : A \in \mathcal{S})$. In fact, this collection is contained in $\mathcal{M}$. Conversely, with the σ-algebra generated for it every evaluation map is measurable, then it contains $\mathcal{M}$. Particularly, it happens with $\mathcal{R} = \mathcal{A}$ or $\mathcal{B}(T)$.

In the notation of random measures, for every sample point $\omega \in \Omega$ we associate a particular realization that is a Radon measure on T; we denote it by $M(\cdot, \omega)$. For each fixed $A \in \mathcal{B}(T)$, $M(A) = M(A, \cdot)$ is a function mapping Ω into $\mathbb{R}^+$, and thus we can ask if it is a random variable. The following result shows that this is true.

Theorem 3.2.5. *Let M be a mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ and $\mathcal{S}$ a semiring of bounded Borel sets such that generating $\mathcal{B}(T)$. Then M is a random measure if and only if $M(A)$ is a random variable for every $A \in \mathcal{S}$.*

Proof. Because $M(A)(\omega) = M(A, \omega) = \pi_A(M(\cdot, \omega))$ as in Figure 3.1, we have for any $G \in \mathcal{B}(\mathbb{R}^+)$

$$M^{-1}(\pi_A^{-1}(G)) = (M(A))^{-1}(G).$$

When $M(A)$ is a random variable, $M^{-1}(\pi_A^{-1}(G)) \in \mathcal{F}$, it follows by Remark 3.2.4 that M is a random measure. Conversely, when M is a random measure, $M^{-1}(\pi_A^{-1}(G)) \in \mathcal{F}$, so then $M(A)$ is a random variable. □

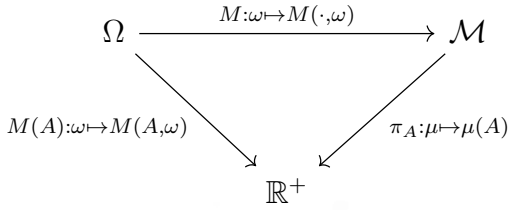

Figure 3.1: Random Measure M

Particularly we have the previous theorem with $\mathcal{S} = \mathcal{A}$ or $\mathcal{B}(T)$. Let us summarize the ideas. When M is a random measure, then

- for fixed $A \in \mathcal{B}(T)$, the mapping $\omega \mapsto M(\omega, A)$ is measurable from $(\Omega, \mathcal{F})$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and
- for every $\omega \in \Omega$, the mapping $A \mapsto M(\omega, A)$ is a Radon measure on $(T, \mathcal{B}(T))$.

We give some examples of random measures.

Example 3.2.6. The map $\delta : T \to \mathcal{M}$ defined by

$$\delta(\omega, A) = \delta_\omega(A) = 1_A(\omega), \quad \omega \in T, A \in \mathcal{B}(T),$$

is a random measure, where δ_ω are Dirac measures. In fact, the map is well defined since Dirac measures are totally finite and for every $A \in \mathcal{B}(T)$, we have a nonnegative random variable $1_A(\omega)$. We conclude with Theorem 3.2.5.

Example 3.2.7. Let $T = \mathbb{R}$ and $\{Z(x) : x \in \mathbb{R}\}$ be a Gaussian process with continuous paths. Then set

$$M(A, \omega) = \int_A Z^2(x, \omega) dx.$$

It is a random measure. For every $\omega \in \Omega$, it is clear that $M(A, \omega) \geq 0$ because $Z^2(x, \omega) \geq 0$. Moreover, $M(\emptyset, \omega) = 0$ and for a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of

elements of $\mathcal{B}(\mathbb{R})$:

$$\begin{aligned} M\left(\bigcup_{n\in\mathbb{N}} A_n, \omega\right) &= \int \sum_{n\in\mathbb{N}} 1_{A_n} Z^2(x,\omega)\sigma(dx) \\ &= \sum_{n\in\mathbb{N}} \int 1_{A_n} Z^2(x,\omega)\sigma(dx) \\ &= \sum_{n\in\mathbb{N}} M(A_n,\omega). \end{aligned}$$

We use monotone convergence theorem to interchange the integral with the sum. Since $x \mapsto Z^2(x,\omega)$ is continuous, it is bounded in every compact K, thus

$$M(K,\omega) = \int_K Z^2(x,\omega)dx < \infty.$$

Hence, by Theorem 3.1.4 it is a Radon measure. Now, we have to prove that for every element A of $\mathcal{S} = \{[a,b) : a \leq b \in \mathbb{R}\}$, $M(A)$ is a nonnegative random variable. The collection

$$\mathcal{S} = \{[a,b) : a \leq b \in \mathbb{R}\}$$

of subsets is a semiring on $\mathbb{R}$ and $\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})$. For every $n \in \mathbb{N}$, we can divide the set A into subintervals A_{ni} with lengths $1/n$ or less. Let

$$\mathcal{T}_n = \{A_{ni} : i \in [\![1, k_n]\!]\}$$

be the sequence of partitions and $t_{ni} \in A_{ni}$. Since Riemann integral and Lebesgue integral agree in continuous function, as $n \to \infty$

$$M_n(A) = \sum_{i=1}^{k_n} Z^2(t_{ni})\lambda(A_{ni}) \to \int_A Z^2(x)dx = M(A).$$

Therefore, $M(A)$ is a nonnegative random variable as the pointwise limit of nonnegative random variables. This completes the proof.

Given a random measure M,

$$\{M(A) : A \in \mathcal{B}(T)\}$$

is a family of nonnegative random variables. In the other direction, we can ask when a family of nonnegative random variables form a random measure. Since it is a measure we require for disjoint sets $A, B \in \mathcal{B}(T)$ that

$$M(A \cup B) = M(A) + M(B) \text{ a.s.} \tag{3.2}$$

and for all sequences of bounded Borel sets A_n such that $A_n \downarrow \emptyset$,

$$M(A_n) \to 0 \text{ a.s.} \tag{3.3}$$

The next result answers the question, it appears as Theorem 9.1.XV of [DVJ08b].

Theorem 3.2.8. *Let $\{M_A : A \in \mathcal{B}(T)\}$ be a family of nonnegative random variables indexed by the sets of $\mathcal{B}(T)$ and a.s. finite valued on bounded Borel sets. In order for there to exist a random measure $M(A, \omega)$ such that, for all $A \in \mathcal{B}(T)$*

$$M(A) = M_A \text{ a.s.},$$

it is necessary and sufficient that (3.2) *hold for all pairs A, B of disjoint elements of $\mathcal{A}$ and that* (3.3) *hold for all sequences $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ with $A_n \downarrow \emptyset$.*

In Chapter 4 we will use a stronger version of this theorem considering a countable collection that generates $\mathcal{B}(T)$, namely

$$\mathcal{R} = \left\{ \bigcup_{i=1}^{k} A_i : A_i \text{ are disjoint elements of } \mathcal{C}, i \in [\![1, k]\!], k \in \mathbb{N} \right\},$$

where

$$\mathcal{C} = \{\text{Closed balls with center } x_i \text{ and radius } q_i : x_i \in D, q_i \in \mathbb{Q}\}$$

and D is a countable dense subset of T. It is a consequence of Lemma 9.1.XIV of [DVJ08b].

Theorem 3.2.9. *Let $\{M_A : A \in \mathcal{R}\}$ be a family of nonnegative random variables indexed by the sets $A \in \mathcal{R}$. In order that, with probability 1, the $M_A(\omega)$ should admit an extension to a measure on $\sigma(\mathcal{R}) = \mathcal{B}(T)$, it is necessary and sufficient that* (3.2) *hold for all pairs A, B of disjoint elements of $\mathcal{R}$ and that* (3.3) *hold for all sequences $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{R}$ with $A_n \downarrow \emptyset$.*

Remark 3.2.10. Note that $\mathcal{R} \subset \mathcal{A}$, then if a family of nonnegative random variables indexed by the sets $\mathcal{A}$ satisfying the conditions (3.2) and (3.3) we can use the previous theorem.

3.3 Properties of Random Measures

The Gaussian Multiplicative Chaos is built as a weak limit of a sequence of random measures. Among the different modes of convergence of random measures we introduce one that is related with the metric space and weak convergence of random variables.

Definition 3.3.1. We say that the sequence of random measures $(M_n)_{n\in\mathbb{N}}$ defined on T converge weakly to the random measure M defined on T if for every finite family $\{A_1, \dots, A_k\}$ of bounded and measurable sets of $\mathcal{B}(T)$, i.e., $A_i \in \mathcal{A}$, $i \in [\![1, k]\!]$, the distribution of the random vector $(M_n(A_1), \dots, M_n(A_k))$ converge weakly in $\mathcal{B}(\mathbb{R}_+^k)$ to the distribution of $(M(A_1), \dots, M(A_k))$.

To obtain a random object well defined we need uniqueness in a distribution sense.

Definition 3.3.2. The distribution of a random measure M is the probability measure it induces on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. It is denoted by $\mathbb{P}_M$

Note that with $\pi_A^{-1}(G) \in \mathcal{B}(\mathcal{M})$, $A \in \mathcal{B}(T)$ and $G \in \mathcal{B}(\mathbb{R}_+)$ we have,

$$\begin{aligned}\mathbb{P}_M(\pi_A^{-1}(G)) &= \mathbb{P}(\omega \in \Omega : M(\cdot, \omega) \in \pi_A^{-1}(G)) \\ &= \mathbb{P}(\omega \in \Omega : M(A, \omega) \in G) \\ &= \mathbb{P}_{M(A)}(G).\end{aligned}$$

It gives us an idea that it is possible to characterize the distribution of a random measure with some family of elements of $\{M(A) : A \in \mathcal{B}(T)\}$.

Proposition 3.3.3. *The distribution of a random measure M is determined by the distribution of $(M(A_1), \dots, M(A_k))$ for all finite families of elements of $\mathcal{A}$.*

This is a consequence of the following lemma.

Lemma 3.3.4. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures on $(\Omega, \mathcal{F})$. Suppose that there exists a π-system $\mathcal{C} \subset \mathcal{F}$ such that $\sigma(\mathcal{C}) = \mathcal{F}$ and*

$$\forall A \in \mathcal{C}, \quad \mathbb{P}_1(A) = \mathbb{P}_2(A).$$

Then $\mathbb{P}_1 = \mathbb{P}_2$.

In Remark 3.2.4 we see a collection of subset of $\mathcal{M}$ that generates $\mathcal{B}(\mathcal{M})$ but it is not a π-system. We consider a slightly different collection which is a π-system and still generate $\mathcal{M}$:

$$\mathcal{C} = \left\{ \bigcap_{i=1}^{k} \pi_{A_i}^{-1}(G_i) : A_i \in \mathcal{A}, G_i \in \mathcal{B}(\mathbb{R}_+), k \in \mathbb{N}, i \in [\![1, k]\!] \right\}.$$

Proof of Proposition 3.3.3. Let M_1 and M_2 be random measures and their distributions $\mathbb{P}_{M_1}$ and $\mathbb{P}_{M_2}$, respectively. If

$$(M_1(A_1), \ldots, M_1(A_k)) \overset{\mathbb{D}}{=} (M_2(A_1), \ldots, M_2(A_k)),$$

then for $G_1, \ldots, G_k \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{P}(M_1(A_1) \in G_1, \ldots, M_1(A_k) \in G_k) = \mathbb{P}(M_2(A_1) \in G_1, \ldots, M_2(A_k) \in G_k).$$

Consequently, if $C = \cap_{i=1}^{k} \pi_{A_i}^{-1}(G_i)$

$$\begin{aligned} \mathbb{P}_{M_1}(C) &= \mathbb{P}(M_1 \in C) \\ &= \mathbb{P}(M_1(A_1) \in G_1, \ldots, M_1(A_k) \in G_k) \\ &= \mathbb{P}(M_2(A_1) \in G_1, \ldots, M_2(A_k) \in G_k) \\ &= \mathbb{P}_{M_2}(C). \end{aligned}$$

Chapter 4

Gaussian Multiplicative Chaos

In 1985, in his article Sur le chaos multiplicatif, Kahane introduced and developed the theory of Gaussian multiplicative chaos. Using a kernel σ-positive definite we will construct a random measure with an approximation procedure following the ideas of the article. We deal with the uniqueness and degeneracy of the object and finally we introduce an extension of the theory. This chapter is based on [Kah85] and [RV14].

4.1 The Chaos

Chaotic phenomena are disordered and irregular dynamic behaviors found in many natural systems, have long been involved in physics, particularly in the study of fluid motion. In the latter, the Navier-Stokes equations have contributed to the complete description of fluid flow. Despite being widely applied in different branches of science, it has not yet been proven if this equation has a solution in three dimensions and if they are smooth. It belongs to the seven millennium problems for which the Clay Mathematics Institute has offered a US$ 1 million award for a correct solution. One of the key thing in this problem is about understanding turbulence: The chaotic random motion of particle in fluids. It is difficult to model and understand mathematically.

Around 1940, it is the Russian school that makes the essential mathematical contributions, studying the velocity flux of a turbulent flow as a random field (A.

N. Kolmogorov, A, Obukhoff, ...). In 1961 Kolmogorov and Obukhoff study the distribution of energy dissipation as a random measure introducing the concept of random multiplicative cascades. A rich geometrical and scaling perspective was subsequently advanced by Mandelbrot in 1974 (He defined this random measure as a limit of a martingale), and a complete mathematical treatment was initiated by Kahane and Peyrière in 1976.

With the aim of giving mathematical rigor to the Kolmogorov-Obukhov energy dissipation model introduced by Mandelbrot, Kahane developed the Gaussian multiplicative chaos in 1985.

4.2 The seminal work of Kahane in 1985

Throughout this chapter we deal with centred Gaussian fields indexed with (T, ρ), a Polish metric space locally compact with the property (3.1). As in the precede chapter, $\mathcal{M}(T)$ denotes the space of Radon measures defined on $\mathcal{B}(T)$.

4.2.1 Existence

In this section we provide the first definition of Gaussian multiplicative chaos due to Kahane in the article [Kah85]. Let $K(s,t)$ be a continuous kernel on $T \times T$. It is a kernel of a Gaussian field $X = (X(t))_{t\in T}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Theorem 2.2.2), then

$$K(t,s) = \mathbb{E}[X(t)X(s)].$$

Remark 4.2.1. We can get a measurable version of X if it is continuous in L^2, it is due to the Theorem 1 of the page 171 of [GS74]. The condition is satisfied because K is continuous on $T \times T$, we can use Theorem 2.5.3. This version is still a Gaussian field with the same covariance kernel (see Corollary 2.4.4).

Let $\sigma \in \mathcal{M}(T)$, then we are able to integrate $X(t,\omega)$ with respect to σ, $\mathbb{P}$ and $\sigma \otimes \mathbb{P}$. We associate to $X(t)$ the process P defined by

$$P(t) = \exp\left(X(t) - \frac{1}{2}\mathbb{E}[X^2(t)]\right) = \exp\left(X(t) - \frac{1}{2}K(t,t)\right).$$

Since the distribution of the process $X(t)$ depends only on $K(t,s)$, the same is true for the distribution of $P(t)$. Notice the normalization

$$\mathbb{E}P(t) = 1 \tag{4.1}$$

for every $t \in T$. It follows from the following lemma.

Lemma 4.2.2. *Let X be a centred Gaussian variable with variance σ and $v \in \mathbb{R}$, then $X - v$ is a Gaussian variable and*

$$\mathbb{E}\left[e^{X-v}\right] = \exp\left(\frac{\sigma^2}{2} - v\right).$$

Let $s, t \in T$, since $X(t)+X(s)$ is a centered Gaussian variable by the previous lemma,

$$EP(t)P(s) = \exp(K(t,s)).$$

With the measure σ and the process P we can built a random measure.

Theorem 4.2.3. *The mapping*

$$M(A,\omega) = \int_A P(t,\omega)\sigma(dt),$$

denoted by $P\sigma$, is a random measure.

Proof. For every $\omega \in \Omega$, $M(\cdot,\omega)$ is a measure on T (the proof is the same as Example 3.2.7). For every compact set K of T, we have by Fubini's theorem and (4.1),

$$\mathbb{E}M(K) = \mathbb{E}\int_K P(t)\sigma(dt) = \int_K \mathbb{E}P(t)d\sigma(t) = \sigma(K). \tag{4.2}$$

Then $M(K,\cdot)$ is finite a.s. Particularly, it is true to the compacts $(K_n)_{n\in\mathbb{N}}$ obtained in Proposition 3.1.8. Since every compact is contained in some K_j, $M(A,\cdot)$ is finite in compacts sets a.s. Hence by Theorem 3.1.4 it is a Radon measure a.s. Finally, by Fubini's theorem, $M(A)$ is a nonnegative random variable for every $A \in \mathcal{B}(T)$. We can conclude using Theorem 3.2.5. □

The random measure M depends only on $K(t,s)$ and σ where $K(t,t)$ is the variance of $X(t)$. The aim of this text is formally define a random measure as in Theorem 4.2.3 for a "Gaussian field" with infinite variance, i.e., $K(t,t)$ can be infinite. In [Kah85] Kahane constructed a theory relying on the notion of a σ-positive definite . A function $K : T \times T \to \mathbb{R}_+ \cup \{\infty\}$ is $\sigma-$positive definite if there exists a sequence $K_n : T \times T \to \mathbb{R}_+$ of continuous kernels such that

$$K(s,t) = \sum_{n=1}^{\infty} K_n(s,t).$$

We can consider independent Gaussian fields $(X_n(t))_{t\in T}$ with covariance K_n (see Theorem 2.6.4). It is considered an approximation procedure to get the random measure desired. For every $n \in \mathbb{N}$, we set

$$\Sigma_n(t,s) = (K_1 + K_2 + \cdots + K_n)(t,s).$$

It is a positive kernel, then the process

$$Y_n(t) = (X_1 + \cdots + X_n)(t)$$

is a Gaussian field with covariance Σ_n. We define

$$Q_n(t) = (P_1 P_2 \cdots P_n)(t) = \exp\left(Y_n(t) - \frac{1}{2}\Sigma_n(t,t)\right).$$

Given $\sigma \in \mathcal{M}(T)$, we have the random measure $M_n = Q_n\sigma$.

Theorem 4.2.4. *The sequence of random measures $(M_n)_{n\in\mathbb{N}}$ converge weakly in the space $\mathcal{M}(T)$ to a random measure M, which is called Gaussian multiplicative chaos.*

Proof. Let

$$\mathcal{F}_n = \sigma(X_i(t) : i \in [\![1,n]\!], t \in T)$$

and $A \in \mathcal{B}(T)$ be a bounded set. We will prove that the sequence $\{M_n(A) : n \in \mathbb{N}\}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

- Since the map

$$\begin{array}{rccc} Q_n : & (T\times\Omega) & \to & \mathbb{R}_+ \\ & (t,\omega) & \mapsto & 1_A(t)Q_n(t,\omega) \end{array}$$

  is measurable with the σ-algebra $\mathcal{B}(T)\otimes\mathcal{F}_n$, by Fubini's theorem 1.1.6 $M_n(A)$ is measurable with respect to $\mathcal{F}_n$ for every $n \in \mathbb{N}$.

- We deduce from (4.2) that $M_n(A)$ is integrable:

$$\mathbb{E}[M_n(A)] = \sigma(A) \le \sigma(\overline{A}) < \infty$$

  for every $n \in \mathbb{N}$.

- Finally, let $B \in \mathcal{F}_n$,

$$\mathbb{E}[M_{n+1}(K)1_B] = \mathbb{E}\left[\int_A (P_1\cdots P_n 1_B)P_{n+1}\sigma(dt)\right],$$

using Fubini and (4.1)

$$\begin{aligned}\mathbb{E}\left[\int_A (P_1 \cdots P_n 1_B) P_{n+1} \sigma(dt)\right] &= \int_A \mathbb{E}[P_1 \cdots P_n 1_B]\mathbb{E}[P_{n+1}]\sigma(dt) \\ &= \mathbb{E}\left[\int_A P_1 \cdots P_n 1_B \sigma(dt) 1_B\right] \\ &= \mathbb{E}\left[M_n(A)1_B\right].\end{aligned}$$

Therefore

$$\mathbb{E}[M_{n+1}(A)|\mathcal{F}_n] = M_n(A).$$

Since $\mathbb{E}[M_n(A)] = \sigma(A)$ for every $n \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}[M_n(A)] = \sigma(A) < \infty,$$

by Theorem (1.4.5) $M_n(A)$ converge a.s. to a random variable M_A. Moreover,

- for all disjoint and bounded A_1, A_2 in $\mathcal{B}(T)$,

$$\begin{aligned}M_{A_1\cup A_2} &= \lim_{n\to\infty} M_n(A_1 \cup A_2) \\ &= \lim_{n\to\infty} \int_{A_1\cup A_2} Q_n(t)\sigma(dt) \\ &= \lim_{n\to\infty} \int_{A_1} Q_n(t)\sigma(dt) + \int_{A_2} Q_n(t)\sigma(dt) \\ &= M_{A_1} + M_{A_2}.\end{aligned}$$

- for any bounded sequence $(A_n)_{n\in\mathbb{N}}$ such that $A_n \downarrow \emptyset$, we have by (4.2),

$$\mathbb{E}[M_{A_n}] = \sigma(A_n) \to 0$$

when $n \to \infty$. Thus $(M_{A_n})_n$ converges in L^1 to 0. On the other hand, the sequence $(M_{A_n})_n$ is decreasing, it follows that it converges a.s. to a random variable M_∞. We can extract a subsequence of $(M_{A_n})_n$ which converges to 0. Thus $M_\infty = 0$ a.s. Therefore

$$M_{A_k} \to 0 \ \ a.s.$$

Satisfied (3.2) and (3.3), Theorem 3.2.9 allows us to conclude that there is a random measure M such that for every bounded set B of T,

$$M(A) = M_A \ \ a.s.$$

Finally, the sequence of random measures M_n converges weakly to the random measure M. It follows of a.s. convergence of $(M_n(A_1), \dots, M_n(A_k))$ to $(M(A_1), \dots, M(A_k))$ for every finite family $\{A_1, \dots, A_k\}$ of bounded sets $A_i \in \mathcal{B}(T)$. □

Thus, for every $\sigma \in \mathcal{M}$ and a σ-positive definite kernel K, we have a random measure M. There are important questions about the construction of M related with the good definition what we will focus in the rest of the chapter.

4.2.2 Uniqueness

We can obtain other decomposition if we interchange the order of K_n in

$$K(s,t) = \sum_{n=1}^{\infty} K_n(s,t). \tag{4.3}$$

As a consequence, there are different nonnegative and positive definite kernels $(K'_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} K'_n = K$. For the sake of well definition of the Gaussian multiplicative chaos, we need that the limit of the random measures obtained with the kernels K'_n to be the same in distribution.

Theorem 4.2.5. *The distribution of the Gaussian multiplicative chaos does not depend on the sequence $(K_n)_{n\in\mathbb{N}}$ of kernels used in the decomposition (4.3) of K.*

In order to give a proof we need the following lemma.

Lemma 4.2.6. *Let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be two independent centred Gaussian vectors such that*

$$\forall i, j \in [\![1, n]\!],\ \mathbb{E}[X_i X_j] \leq \mathbb{E}[Y_i Y_j].$$

Then for all combination of nonnegative numbers $p_1, \dots, p_n$ and for $F(x) = e^{-\lambda x}$, $x \in \mathbb{R}_+$ and $\lambda > 0$,

$$\mathbb{E}\left[F\left(\sum_{i=1}^{n} p_i e^{X_i - \frac{1}{2}\mathbb{E}[X_i^2]}\right)\right] \leq \mathbb{E}\left[F\left(\sum_{i=1}^{n} p_i e^{Y_i - \frac{1}{2}\mathbb{E}[Y_i^2]}\right)\right].$$

Proof. We define

$$\varphi(t) = \mathbb{E}\left[F\left(\sum_{i=1}^{n} p_i e^{Z_i(t) - \frac{1}{2}\mathbb{E}[Z_i(t)^2]}\right)\right],$$

with $Z_i(t) = \sqrt{t}X_i + \sqrt{1-t}Y_i$, $t \in [0,1]$, $i \in [\![1,n]\!]$. It is a deterministic interpolation. The aim of the proof is show that φ is decreasing on $[0,1]$. Thus

$$\varphi(1) \leq \varphi(0).$$

We consider

$$f(t) = F\left(W_n(t)\right),$$

where

$$W_n(t) = \sum_{i=1}^{n} p_i \exp\left(Z_i(t) - \frac{1}{2}\mathbb{E}[Z_i(t)^2]\right).$$

Note that $\mathbb{E}[Z_i] = 0$ and

$$\mathbb{E}[Z_i^2] = t\mathbb{E}[X_i^2] + (1-t)\mathbb{E}[Y_i^2].$$

In order to derivative under the expectation we satisfy the conditions of Theorem 1.1.5:

(i) For every $t \in (0,1)$ fixed, $f(t,\omega)$ is a integrable random variable, because F is continuous and bounded.

(ii) For every ω fixed, $t \mapsto f(t,\omega)$ is differentiable and

$$f'(t,\omega) = \left(\sum_{i=1}^{n} p_i \exp\left(Z_i(t) - \frac{\mathbb{E}[Z_i(t)^2]}{2}\right)\left(\frac{X_i}{2\sqrt{t}} - \frac{Y_i}{2\sqrt{1-t}} - \frac{\mathbb{E}[X_i^2] - \mathbb{E}[Y_i^2]}{2}\right)\right) \times F'(W_n(t)).$$

(iii) Let K be a compact subset of $(0,1)$. For every $i \in [\![1,n]\!]$,

$$\sup_{t\in K}\left|\exp\left(Z_i(t) - \frac{\mathbb{E}[Z_i(t)^2]}{2}\right)\right| \leq \exp(X_i + Y_i),$$

$$\sup_{t\in K}\left|\left(\frac{X_i}{2\sqrt{t}} - \frac{Y_i}{2\sqrt{1-t}} - \frac{\mathbb{E}[X_i^2] - \mathbb{E}[Y_i^2]}{2}\right)\right| \leq |aX_i + bY_i + c|.$$

Since the bounds are in L^1 as well as their product and F is bounded, we can conclude that for every $w \in \Omega$,

$$\sup_{t\in K}\left|\frac{d}{dt}f(t,\omega)\right| \leq Z(\omega)$$

for some $Z \in L^1$.

Therefore, we can take the derivative under the expectation,

$$\begin{aligned}\varphi'(t) = &\mathbb{E}\left[\left(\sum_{i=1}^{n}\frac{p_i}{2}\exp\left(Z_i(t)-\frac{\mathbb{E}[Z_i(t)^2]}{2}\right)\left(\frac{X_i}{\sqrt{t}}-\frac{Y_i}{\sqrt{1-t}}-\mathbb{E}[X_i^2]+\mathbb{E}[Y_i^2]\right)\right)\right.\\ &\left.\times F'(W_n(t))\right]\\ =&\sum_{i=1}^{n}\frac{p_i}{2}\mathbb{E}\left[\left(\frac{X_i}{\sqrt{t}}-\frac{Y_i}{\sqrt{1-t}}\right)\exp\left(Z_i(t)-\frac{\mathbb{E}[Z_i(t)^2]}{2}\right)F'(W_n(t)\right]\\ &-\sum_{i=1}^{n}\frac{p_i}{2}\mathbb{E}\left[\left(\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2]\right)\exp\left(Z_i(t)-\frac{\mathbb{E}[Z_i(t)^2]}{2}\right)F'(W_n(t)\right]\end{aligned} \tag{4.4}$$

Considering $X=\left(\frac{X_i}{\sqrt{t}}-\frac{Y_i}{\sqrt{1-t}}\right)$, $Y=\left(Z_1-\frac{1}{2}\mathbb{E}[Z_1^2],\dots,Z_n-\frac{1}{2}\mathbb{E}[Z_n^2]\right)$ and

$$\tilde{F}(x_1,\dots,x_n)=e^{x_i}F'\left(\sum_{j=1}^{n}p_ie^{x_i}\right),$$

we can apply Corollary 2.1.7. Note that

$$\nabla\tilde{F}(x_1,\dots,x_n)=e^{x_i}F''\left(\sum_{j=1}^{n}p_ie^{x_i}\right)(p_1e^{x_1},\dots,1+p_ie^{x_i},\dots,p_ne^{x_n}).$$

Then

$$\begin{aligned}\mathbb{E}[XF(Y)] =&\mathbb{E}[XY]\cdot\mathbb{E}[\nabla F(Y)]\\ =&\left(\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2]\right)\mathbb{E}\left[\exp\left(Z_i(t)-\frac{\mathbb{E}[Z_i^2(t)}{2}\right)F'(W_n(t))\right]\\ &+\sum_{j=1}^{n}p_j\left(\mathbb{E}[X_iX_j]-\mathbb{E}[Y_iY_j]\right)\\ &\times\mathbb{E}\left[\exp\left(Z_i(t)+Z_j(t)-\frac{\mathbb{E}[Z_i^2(t)]}{2}-\frac{\mathbb{E}[Z_i^2(t)]}{2}F''(W_n(t))\right)\right],\end{aligned}$$

where we have used the following fact,

$$\mathbb{E}\left[\left(\frac{X_i}{\sqrt{t}}-\frac{Y_i}{\sqrt{1-t}}\right)Z_j\right]=\mathbb{E}[X_iX_j]-\mathbb{E}[Y_iY_j].$$

We use the last equation in (4.4) and we get

$$\begin{aligned}\varphi'(t) =&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j\left(\mathbb{E}[X_iX_j]-\mathbb{E}[Y_iY_j]\right)\\ &\times\mathbb{E}\left[\exp\left(Z_i(t)+Z_j(t)-\frac{\mathbb{E}[Z_i^2(t)]}{2}-\frac{\mathbb{E}[Z_i^2(t)]}{2}\right)F''(W_n(t))\right].\end{aligned}$$

Since F is convex in $\mathbb{R}_+$, $F''(x) \geq 0$, and by hypothesis we have

$$\varphi'(t) \leq 0$$

for every $t \in (0,1)$. It can be proved by dominated convergence theorem that the function φ is continuous on $[0,1]$. Therefore φ is decreasing on $[0,1]$. □

It can be extended in some sense to Gaussian fields

Corollary 4.2.7. *Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be independent Gaussian fields with covariance K_X and K_Y respectively such that*

$$K_X(t,s) \leq K_Y(t,s)$$

for every $t, s \in T$. Then for $F(x) = e^{-\lambda x}$, $x \in \mathbb{R}_+$ and $\lambda > 0$,

$$\mathbb{E}\left[F\left(\int_A \exp\left(X(t) - \frac{\mathbb{E}[X^2(t)]}{2}\right)\sigma(dt)\right)\right] \leq \mathbb{E}\left[F\left(\int_A \exp\left(Y(t) - \frac{\mathbb{E}[Y^2(t)]}{2}\right)\sigma(dt)\right)\right],$$

where $A \in \mathcal{B}(T)$ is bounded and $\sigma \in \mathcal{M}(T)$.

Proof. We prove the case when the Gaussian fields has continuous paths, the general case is proved in [Kah85]. Let K be a compact set and D a countable dense subset of K. Thus, for every $n \in \mathbb{N}$, we have k_n elements of D such that

$$K \subset \bigcup_{i=1}^{k_n} B_i^n.$$

where $B_i^n = B(t_i^n, 1/n)$ is the ball with center t_i^n and radius n^{-1} and $t_i^n \in D$ for $i \in [\![1, k_n]\!]$. By continuity of $t \mapsto X(t,\omega)$ we have

$$X(t) = \lim_{n\to\infty} \sum_{i=1}^{k_n} 1_{B_i^n}(t) X(t_i^n).$$

We can do the same to $P(t)$ and by dominated convergence theorem

$$\int_K P(t)\sigma(dt) = \lim_{n\to\infty} \sum_{i=1}^{k_n} \sigma(B_i^n) P(t_i^n).$$

With the same argument

$$\mathbb{E}\left[F\left(\int_K P(t)\sigma(dt)\right)\right] = \lim_{n\to\infty} \mathbb{E}\left[F\left(\sum_{i=1}^{k_n} \sigma(B_i^n)P(t_i^n)\right)\right].$$

Lemma 4.2.6 is used to conclude. The same procedure works to bounded sets because its closure is compact. □

Proof of Proposition 4.2.5. Let $(K_n)_{n\in\mathbb{N}}$ and $(K'_n)_{n\in\mathbb{N}}$ be two decomposition of K with associated independent Gaussian fields sequences $(X_n)_{n\in\mathbb{N}}$ and $(X'_n)_{n\in\mathbb{N}}$ and associated random measures $(M_n)_{n\in\mathbb{N}}$ and $(M'_n)_{n\in\mathbb{N}}$, respectively. Both sequences

$$\left(\sum_{k=1}^{n} K_k\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\sum_{k=1}^{n} K'_n\right)_{n\in\mathbb{N}}$$

converge pointwise toward K in a nondecreasing way. Therefore, if we choose a compact set $K \subset T$ then, for each $p \in \mathbb{N}$ the sequence

$$\left(\left(\sum_{k=1}^{n} K'_k - \sum_{k=1}^{p} K_k\right)^{-}\right)_{n\in\mathbb{N}}.$$

converges pointwise to 0 in a nonincreasing way, where f^- the negative part of f, i.e.,

$$(f(t))^- = \max\{-f(t), 0\}.$$

By Dini's Theorem (see A.0.1) we have uniform convergence, i.e., there exists a $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$\left|\left(\sum_{k=1}^{n} K'_k(t,s) - \sum_{k=1}^{p} K_k(t,s)\right)^{-}\right| < \epsilon,$$

for every $(t,s) \in K \times K$. Since $0 \leq f(t)^- \leq f(t)$, we have

$$\sum_{k=1}^{p} K_k(t,s) \leq \epsilon + \sum_{k=1}^{n} K'_k(t,s).$$

We can consider a standard Gaussian variable Z independent of $(X'_n)_n$ for every $n \in \mathbb{N}$ such that the random variable $\sum_{k=1}^{n} X'_k + \sqrt{\epsilon}Z$ has $\epsilon + \sum_{k=1}^{n} K'_k(t,s)$ as a covariance function. We can apply Lemma 4.2.7

$$\begin{aligned}
\mathbb{E}\left[F\left(M_p(K)\right)\right] &\leq \mathbb{E}\left[F\left(\int_K \exp\left(Y_n(t) + \sqrt{\epsilon}Z - \frac{\mathbb{E}[Y_n^2(t)]}{2} - \frac{\mathbb{E}[Z^2]}{2}\right)\sigma(dt)\right)\right] \\
&= \mathbb{E}\left[F\left(e^{\sqrt{\epsilon}Z - \frac{\epsilon}{2}}\int_K \exp\left(Y_n(t) - \frac{\mathbb{E}[Y_n^2(t)]}{2}\right)\sigma(dt)\right)\right] \\
&= \mathbb{E}\left[F\left(e^{\sqrt{\epsilon}Z}M'_n(K)\right)\right].
\end{aligned}$$

Since the sequences of random measures $(M_n)_{n\in\mathbb{N}}$ and $(M'_n)_{n\in\mathbb{N}}$ converge weakly to M and F is continuous and bounded,

$$\mathbb{E}[F(M(K))] = \lim_{p\to\infty} \mathbb{E}[F(M_p(K))]$$

and

$$\mathbb{E}[F(e^{\sqrt{\epsilon}Z}M'(K))] = \lim_{n\to\infty} \mathbb{E}\left[F\left(e^{\sqrt{\epsilon}Z}M'_n(K)\right)\right].$$

Therefore,

$$\mathbb{E}[F(M(K))] \leq \mathbb{E}\left[F\left(e^{\sqrt{\epsilon}Z}M'(K)\right)\right].$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we can use dominated convergence theorem to get

$$\mathbb{E}[F(M(K))] \leq \mathbb{E}[F(M'(K))].$$

The opposite inequality is proved analogously. Thus

$$\mathbb{E}[F(M(K))] = \mathbb{E}[F(M'(K))].$$

Since $F(x) = e^{-\lambda x}$, by Theorem 1.2.4 we have $M(K) \overset{\mathbb{D}}{=} M'(K)$. If we consider the measure

$$\mu(dt) = \sum_{i=1}^{n} \lambda_i 1_{B_i}(t)\sigma(dt),$$

where $B_i \subset T$ are bounded sets and we follow the same procedure we have

$$\mathbb{E}\left[F\left(\sum_{i=1}^{n}\lambda_i M(A_i)\right)\right] = \mathbb{E}\left[F\left(\sum_{i=1}^{n}\lambda_i M'(B_i)\right)\right].$$

By Theorem 1.2.4,

$$(M(B_1),\ldots,M(B_n)) \overset{\mathbb{D}}{=} (M'(B_1),\ldots,M'(B_n)).$$

Hence M and M' have the same distribution.

4.2.3 Degeneracy

Let A be a bounded set, then

$$M(A) = \lim_{n\to\infty}\int_A Q_n(t)\sigma(dt).$$

We are focus in the event

$$\{\omega \in \Omega : M(A)(\omega) = 0\}.$$

For fixed m we can consider for $n \geq m$

$$\begin{aligned} M_n(A) &= \int_A \exp\left(Y_n(t) - \frac{\mathbb{E}[Y_n^2(t)]}{2}\right)\sigma(dt) \\ &= \int_A \exp\left(Y_m(t) - \frac{\mathbb{E}[Y_m^2(t)]}{2}\right)\exp\left(\sum_{k=m+1}^{n} X_k(t) - \frac{\mathbb{E}[X_k^2(t)]}{2}\right)\sigma(dt). \end{aligned}$$

Since

$$\exp\left(Y_n(t) - \frac{\mathbb{E}[Y_n^2(t)]}{2}\right) > 0,$$

that $\{M(A) = 0\}$ happens depends on

$$\exp\left(\sum_{k=m+1}^{n} X_k(t) - \frac{\mathbb{E}[X_k^2(t)]}{2}\right)\sigma(dt).$$

Thus, $\{M(A) = 0\} \in \sigma(X_{m+1}, X_{m+2}, \dots)$ for every $m \in \mathbb{N}$. It follows that it is in the tail of the independent family of σ-algebras $\mathcal{B}_k = \sigma(X_k(t) : t \in T)$, $k \in \mathbb{N}$. Hence, by Theorem 1.3.7

$$\mathbb{P}(M(A) = 0) \in \{0, 1\}.$$

So it is possible that M identically vanishes. It is called degenerate case. It is not so straightforward to check if the Gaussian multiplicative chaos is degenerate or not, but there is an easy condition to avoid $\mathbb{P}(M(A) = 0) = 1$.

Proposition 4.2.8. *Let A be a bounded set such that $\sigma(A) > 0$. Then $\mathbb{P}(M(A) = 0) = 0$ if*

$$\mathbb{E}[M(A)] = \sigma(A).$$

This condition is satisfied if $M_n(A)$ is a martingale bounded in L^p, for some $p > 1$, i.e.,

$$\sup \mathbb{E}[M_n^p(A)] < \infty,$$

then it is uniformly integrable (see Theorem 1.4.7). Thus

$$\mathbb{E}[M(A)] = \lim_{n\to\infty} \mathbb{E}[M_n(A)] = \sigma(A).$$

We give an example in the next section.

4.3 Model of Mandelbrot

In [Man72], Mandelbrot introduced the "limit-lognormal" model, the Kahane's theory is developed to give meaning to it. We define $\ln_+(x)$ by

$$\ln_+(x) = \max\{\ln(x), 0\}.$$

The "limit-lognormal" model deals with a kernel K defined on $\mathbb{R}^d$ and given by

$$K(x, y) = \log_+\left(\frac{R}{|x - y|}\right), \tag{4.5}$$

where $R > 0$. We can obtain a Gaussian multiplicative chaos from this kernel.

Lemma 4.3.1. *The function K of the form (4.5) is of σ-positive type.*

For easy we can work with

$$f(x) := K(x+z,z) = ln_+\left(\frac{R}{|x|}\right).$$

Proof. It can be written by straightforward calculations as

$$\ln_+\left(\frac{R}{|x|}\right) = \int_0^\infty (t-|x|)_+ \nu_R(dt),$$

where

$$\nu_R(dt) = 1_{[0,R)}\frac{dt}{t^2} + \frac{\delta(dt)}{R}.$$

For all $r>0$, we have

$$\ln_+\left(\frac{R}{|x|}\right) = \frac{1}{r}\ln_+\left(\frac{R^r}{|x|^r}\right) = \frac{1}{r}\int_0^\infty (t-|x|^r)_+ \nu_{R^r}(dt).$$

Therefore we have to look the $r>0$ such that $(t-|x|^r)_+$ is positive definite: It is known as the Kuttner-Golubov problem (see [Gne01]). We can write

$$\ln_+\left(\frac{T}{|x|}\right) = \sum_{n=1}^\infty K_n$$

with

$$K_n(x) = \int_{\frac{1}{n}}^{\frac{1}{n-1}} (t-|x|^r)_+ \nu_{R^r}(dt)$$

with $r=1$ in dimension 1 and $r=1/2$ in dimension 2. In dimension 3 it is an open question whether it is of σ-positive definite. Note that, using the notation of Section 2, we have,

$$\begin{aligned}
\mathbb{E}[M_n^2(A)] &= \int\int \mathbb{E}[Q_n(x)Q_n(y)]\sigma(dx)\sigma(dy) \\
&= \int\int \exp\left(\Sigma_n\right)\sigma(dx)\sigma(dy) \\
&\leq \int\int \exp\left(K(x,y)\right)\sigma(dx)\sigma(dy) \\
&\leq \int\int \frac{R}{|x-y|}\sigma(dx)\sigma(dy).
\end{aligned}$$

If we consider $d=2$ and σ the Lebesgue measure on $\mathbb{R}^d$ it follows that the martingale is bounded in L^2. Then the Gaussian multiplicative chaos associated is nondegenerate.

□

Conclusions

1. **The result and applications.** The Gaussian multiplicative chaos is a random measure well defined on a Polish metric space. It is understood as the limit of random measures obtained from Gaussian fields and a Radon measure. In this text we present this procedure rigorously and detailed for being an introduction to the subject. We developed the Kahane's seminal paper [Kah85] with modern ideas of [RV14]. The latter provides several applications of the subject in Economics: Volatility of a financial asset, Physics: Liouville Quantum Gravity and KPZ, and mainly the Kolmogorov-Obhukov model in turbulence (we present a brief introduction in Section 4.3) which is where the Gaussian multiplicative chaos topic originates.

2. **Extensions of the theory.** In the light of natural applications of Kahane's theory, we note that it is difficult to use. Mainly because proving that a kernel is σ-positive definite is not an easy task. One way to facilitate this would be to develop a theory that allows obtaining a Gaussian multiplicative chaos with positive definite kernels, which appear more naturally.

   Our theory does not address the following:

   - Is a kernel σ-positive definite a positive definite? A kernel K of σ-positive definite is nonnegative and positive definite. The reciprocal is still an open question.
   - What happens if you work with another way of approximation? For example, convolutions.

   The second point is developed in [RV10] in Euclidean spaces. For σ-positive definite kernels the random measure obtained is the same as Kahane. The procedure consists of using a sequence of continuous functions that converge to the Dirac delta function. Thus, the convolution of the kernel with this

sequence gives us a sequence of differentiable kernels that smooth down their singularities. The Gaussian multiplicative chaos is obtained as the weak limit of the random measures of these kernels. It gives us a theory that works with positive definite functions and Gaussian fields with continuous paths. The theory is therefore much easier to use.

Appendix A

Analysis

Theorem A.0.1. *(Dini's Theorem) Let X be a compact topological space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions from X to $\mathbb{R}$. Suppose that:*

- *The sequence $(f_n)_{n\in\mathbb{N}}$ is nondecreasing, i.e., for every $x \in X$, the sequence of real numbers $(f_n(x))_{n\in\mathbb{N}}$ is nondecreasing.*
- *The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function f from X to $\mathbb{R}$.*

Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

In the previous theorem, we can replace the hypothesis "nondecreasing" by "nonincreasing".

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