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Bases de Groebner y aplicaciones a la Conjetura del Jacobiano

Tesis para optar el grado académico de Maestra en Matemáticas Aplicadas que presenta:

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Informe de Similitud

Yo, Christian Valqui Haase, docente de la Escuela de Posgrado de la Pontificia Universidad Católica del Perú, asesor del trabajo de tesis titulado

Bases de Groebner y aplicaciones a la Conjetura del Jacobiano

De la autora Valeria Ramírez Arriola,

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Resumen

En esta tesis expositiva, exploraremos las Bases de Groebner y su aplicación a un sistema de ecuaciones polinómicas, particularmente en lo referente a la Conjetura Jacobiana. Esto se hará introduciendo algunos conceptos relacionados con el álgebra conmutativa y algunos conceptos relacionados con la geometría algebraica computacional.

Esta disertación está organizada de la siguiente manera. **En el Capítulo 1**, introduciremos los conceptos básicos relacionados con el álgebra conmutativa y la geometría algebraica básica. **En el Capítulo 2**, definiremos las bases de Groebner, y algunas propiedades relacionadas con ellas, discutiremos en detalle algunas propiedades relacionadas con ellas y discutiremos el algoritmo que está relacionado con ellas. **En el Capítulo 3**, discutiremos el Algoritmo de Buchberger y algunos refinamientos relacionados con el algoritmo. **En el Capítulo 4**, discutiremos la aplicación de los conceptos aprendidos a un sistema de ecuaciones polinómicas relacionadas con la Conjetura Jacobiana. Esta parte de la tesina tiene como base el trabajo de Valqui y Solorzano (2014) [VS14] que calculó el sistema de ecuaciones polinómicas y la base de Groebner del sistema para $n = 2$. Para ello se utilizó una fórmula recursiva para los números catalanes. En esta tesis, calcularemos el sistema de ecuaciones polinómicas, sus respectivas bases de Groebner y el conjunto de soluciones para $n = 3$. **En el Capítulo 5**, creamos una interfaz en Mathematica 13 que calcula el sistema de ecuaciones polinómicas, la base de Groebner, y clasifica los conjuntos de soluciones.

Abstract

In this expository thesis, we will explore the Groebner Bases and their application to a system of polynomial equations, particularly concerning the Jacobian Conjecture. This will be done by introducing some concepts related to commutative algebra and some concepts related to computational algebraic geometry.

This dissertation is organized as follows. **In Chapter 1**, we will introduce the basic concepts related to commutative algebra and basic algebraic geometry. **In Chapter 2**, we will define the Groebner basis, and some properties related to them, we will discuss in detail some properties related to them and discuss the algorithm that is related to it. **In Chapter 3**, we will discuss the Buchberger Algorithm and some refinements related to the algorithm. **In Chapter 4**, we will discuss applying the concepts learned to a system of polynomial equations related to the Jacobian Conjecture. This part of the dissertation has as its base the paper of Valqui and Solorzano (2014) [VS14] which computed the system of polynomial equations and Groebner basis of the system for $n = 2$. This was done by using a recursive formula for the Catalan numbers. In this dissertation, we will compute the system of polynomial equations, their respective Groebner basis, and the set of solutions for $n = 3$. **In Chapter 5**, we create an interface in Mathematica 13 that calculates the system of polynomial equations, the Groebner basis, and classifies the solution sets.

Contents

Chapter 1

Preliminaries

We will start by introducing some basic concepts in commutative algebra and algebraic geometry by introducing some well-known theorems and definitions.

1.1 Polynomial Rings and Ideals

We denote by $R[x_1, x_2, ..., x_n]$ the ring of polynomials with coefficients in the ring R. A subset $I \subset R[x_1, x_2, ..., x_n]$ is an ideal if it satisfies,

i) $0 \in I$,

ii) If $f, g \in I$, then $f + g \in I$,

iii) If $f \in I$ and $h \in R[x_1, x_2, ..., x_n]$ then $hf \in I$.

We define $\langle f_1, f_2, ...f_s \rangle$to be the ideal generated by $f_1, f_2, ..., f_s$ as follows.

$$\langle f_1, f_2, ...f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, h_2, ..., h_s \in R[x_1, x_2, ..., x_n] \right\}.$$

Lemma 1.1.1. *If $f_1, f_2, ..., f_s \in k[x_1, x_2, ..., x_n]$ then $\langle f_1, f_2, ..., f_n \rangle$ is an ideal of $k[x_1, x_2, ..., x_n]$.*

Proof. First, $0 \in \langle f_1, f_2, ..., f_n\rangle$ since $0 = \sum_{i=1}^{s} 0f_i$. Next suppose that $f = \sum_{i=1}^{s} r_i f_i$ and $g = \sum_{i=1}^{s} n_i f_i$, then

$$f + g = \sum_{i=1}^{s}(r_i + n_i)f_i \qquad \text{and}$$

$$h(x)f = \sum_{i=1}^{s}(hr_i)f_i, \quad h \in k[x_1, ..., x_n].$$

□

Let k be a field, and let $f_1, f_2, ..., f_s$ be polynomials in $k[x_1, x_2, ..., x_n]$. We define $V(\langle f_1, f_2, ..., f_s\rangle)$ to be the **affine variety** defined by $\langle f_1, f_2, ..., f_s\rangle$ as follows:

$$V(\langle f_1, f_2, ..., f_s\rangle) = \{(a_1, a_2, ..., a_k) \in k^n : f_i(a_1, a_2, ..., a_k) = 0, \quad \forall 1 \leq i \leq s\}.$$

We define the **variety** associated with I as

$$V(I) = \{a = (a_1, a_2, ..., a_k) \in k^n : f(a_1, a_2, ..., a_k) = 0, \quad \forall f \in I\}.$$

Let $V \in k^n$ be an affine variety, we define the **ideal of the variety** V as follows:

$$I(V) = \{f \in k[x_1, ..., x_n] : f(a_1, a_2, ..., a_n) = 0 \quad \forall (a_1, a_2, ..., a_n) \in V\}.$$

The **radical of an ideal** I, $rad(I) = \sqrt{I}$, is defined as follows:

$$\sqrt{I} = \{f \in k[x_1, x_2, ..., x_n], \exists r > 0, f^r \in I\}.$$

Definition 1.1.2. A ring is Noetherian if its ideals are finitely generated.

Theorem 1.1.3. *A ring R is Noetherian if and only if every ascending sequence of ideals $\{I_i\}_{i=1}^{\infty}$ of $\mathcal{R}$, is stationary.*

Proof. Assume R is Noetherian and consider an ascending sequence of ideals $\{I_n\}$ such that $\bigcup_{n=1}^{\infty} I_n = I \triangleleft R.$. Clearly I is an ideal, since if $x, y \in \bigcup_{n=1}^{\infty} I_n$ there are indices i, k such that $\quad x \in I_i \wedge \quad y \in I_k$. Then, if we denote the $max\{i,k\} = n$, we have that $x+y \in I_n \subset \bigcup_{n=1}^{\infty} I_n$. Futhermore, if $a \in \mathcal{A}$, such that $x_i \in I_i$, then $ax_i \in I_i \subset \bigcup_{n=1}^{\infty} I_n$. Therefore, $\bigcup_{n=1}^{\infty} I_n$ is an ideal. We have that I is finitely generated, i.e $I = \langle x_1, ..., x_m \rangle$. If $x_j \in I_{n_j}$, and $n = max(n_j)$, then $\langle x_1, ..., x_m \rangle \subset I_n$. Then $I_k = I_n \quad \forall k > n.$

For the converse, we use induction on the finite number of generators of an ideal I. Let f_1 be a polynomial $\in I$. If $I = I_1 = \langle f_1 \rangle$, then I is finitely generated. Else there exists a polynomial f_2, such that $f_2 \in I - I_1$. If $I = I_2 = \langle f_1, f_2 \rangle$ then the proof is done. Else there is a polynomial f_3, such that $f_3 \in I - I_2$ If $I = I_3 = \langle f_1, f_2, f_3 \rangle$, then the proof is done, else we continue inductively with the same construction. Since the ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq ... \subsetneq I_k \subsetneq ...$ is stationary at some point, we must have $I = I_k$ and so I is finitely generated, otherwise we end up with a sequence of ideals that ascend infinitely, $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$, which contradicts that R is Noetherian. □

Theorem 1.1.4 (Hilbert Basis Theorem)**.** *If a ring R is Noetherian then $R[X]$, the ring of polynomials with coefficients in the ring, is Noetherian.*

Proof. By contradiction, assume that $I \triangleleft R[X]$ is not finitely generated. We can construct inductively an infinite sequence $\{f_i\}_{i=1}^{\infty}$ of polynomials such that $f_n \in I \setminus \langle f_1, ...f_{n-1} \rangle$ is of minimal degree. It is obvious that $gr(f_k) \geq gr(f_j) \quad \forall j$ such that $j \leq k$. The ideals generated by the leading coefficients (LC) of the polynomials define an ascending chain of ideals $J_i = \langle a_1, \dots, a_i \rangle \triangleleft R$. Since R is Noetherian, the sequence $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \langle a_1, a_2, a_3 \rangle ...$ is stationary, so there is an $k \geq 1$, such that $J_s = J_k, \forall s \geq k$

For $j = 1, ..., k+1$, we have

$$f_j = a_j x^{gr(f_j)} + \cdots . \quad \text{Note that} \quad a_{k+1} \in J_{k+1} = J_k, \text{and so} \quad a_{k+1} = \sum_{j=1}^{k} \lambda_j a_j .$$

We define $g \in \langle f_1, ..., f_n \rangle$ then

$$g = \sum_{j=1}^{k} \lambda_j f_j x^{gr(f_{k+1})-gr(f_j)} = \sum_{j=1}^{n} \lambda_j a_j x^{gr(f_{k+1})} + \cdots .$$

Then we have that, $f_{k+1} - g \in I \setminus \langle f_1, ..., f_k \rangle$ and $gr(f_{k+1} - g) < gr(f_{k+1})$, which contradicts the minimality of $gr(f_{k+1})$.

Then every ideal $I \in R[X]$, has to be finitely generated, $I = \langle f_0, f_1, ..., f_k \rangle$. This proves that $R[X]$ is Noetherian. □

Chapter 2

Hilbert's Nullstellensatz

In this section, we will discuss a fundamental theorem in algebraic geometry known as Hilbert's Nullstellensatz, which illustrates the connection between algebra and geometry. The explanation of this theorem is divided into two parts: the weak Hilbert Nullstellensatz and the strong Hilbert Nullstellensatz. Our main reference for this chapter is Allcock (2005) [All05].

2.1 Weak Nullstellensatz

In this subsection, we will prove the weak Hilbert Nullstellensatz, which says that an ideal J in $k[x_1, \ldots, x_n]$ contains the unity 1 (which means that it is the whole ring), if and only if $V(J) = \emptyset$.

We will prove it in three steps. First, we will introduce Zorn's Lemma, which guarantees the existence of a maximal ideal in any partially ordered set. This is crucial because it helps us show that all ideals are contained within a maximal ideal.

Next, we will prove the main technical result, Theorem 2.1.4 (see Proposition 7.9 of

[AM69]), which shows that if k is a field (not necessarily algebraically closed) and $\tilde{K}$ is a field extension such that $\tilde{K} = k[\alpha_1, \ldots, \alpha_n]$, then $\tilde{K}$ is algebraic over the field k. Finally, putting together these results, we obtain that all maximal ideals in $k[x_1, \ldots, x_n]$ are of the form

$$J = \langle x_1 - a_1, \ldots, x_n - a_n \rangle,$$

and, using this, we prove the Weak Nullstellensatz.

Lemma 2.1.1 (Zorn's Lemma:). *If $(\mathcal{S}, \leq)$ is a partially ordered set, such that every chain has an upperbound, which is in S, then there exists $M \in \mathcal{S}$ such that M is maximal.*

Proposition 2.1.2. *Let A be a ring. For every ideal $I \triangleleft A$, such that $I \neq A$, there is a maximal ideal m such that $I \subset m$.*

Proof. Suppose that $A \neq \{0\}$. If there is an ideal $I \triangleleft A$, we define $\mathcal{A} = \{J \triangleleft A, I \subset J, J \neq A\}$ and let $\mathcal{C}$ be a chain of ideals $\mathcal{C} = \{J_i \triangleleft A : J_i \subset J_{i+1}, I \subset J_i, \forall i \in \Lambda\}$. We will show that if $\mathcal{C}$ is a chain in $\mathcal{A}$, then it has an upper bound.

Let,

$$\mathcal{Y} = \bigcup_{J_i \in \mathcal{C}} J_i$$

We will prove that $\mathcal{Y}$ is an ideal,

- If $x, y \in \mathcal{Y} \Rightarrow$ if there exists i,k, such that $x \in J_i \wedge \ y \in J_k \Rightarrow max\{i, k\} = n \Rightarrow x + y \in J_n \subset \mathcal{Y}$

- If $a \in \mathcal{A}, x_i \in J_i \Rightarrow ax_i \in J_i \subset \mathcal{Y}$

  $\mathcal{Y}$ is an ideal.

  It is straightforward that $\mathcal{Y} \neq A$ since if $1 \in \mathcal{Y} \Rightarrow 1 \in J_i$ for a particular i, then $J_i = A$, which is a contradiction.

Therefore, by Zorn's Lemma, there exists $m \in \mathcal{A}$ maximal in $\mathcal{A}$, such that $I \subset m$.

If $m \subset \tilde{J} \neq A \Rightarrow \tilde{J} \in \mathcal{A}$. Since m is maximal in $\mathcal{A} \Rightarrow m = \tilde{J}$, proving that m is maximal in the whole ring A. $\square$

We will use the following definitions and results: Let F and K be fields. If $F \subset K$ then K is a field extension of F. We say that $\alpha \in K$ is algebraic over F, if $\exists f(x) \in F[x]$, such that $f(\alpha) = 0$. If α is not algebraic over $F[x]$, then α is said to be transcendental over $F[x]$. If $dim_F(K) < \infty$ then K is a finite extension of F. The field k is a closed algebraic field if for any algebraic extension $\tilde{K}$ we have $k = \tilde{K}$. Let K be a field extension of F and let $a_1, a_2, ..., a_n \in K$. Then $F(a_1, a_2, ..., a_n) = \{\frac{f(a_1,a_2,...,a_n)}{g(a_1,a_2,...,a_n)} : f, g \in F[x_1, x_2, ..., x_n], g(a_1, a_2, ..., a_n) \neq 0\}$, so $F(a_1, a_2, ..., a_n)$ is the quotient field of $F[a_1, a_2, ..., a_n]$.

Theorem 2.1.3. *If $n = dim_F(K) < \infty$ then K is algebraic over F.*

Proof. If $\alpha \in K$, then $\{1, \alpha, \alpha^2, ..., \alpha^n\}$ is linearly dependent over F. Then $\exists a_i \in F$ $a_i \neq 0$ such that $a_0 + a_1\alpha + a_2\alpha^2 + ... + \alpha_n a^n = 0$. Set $f(x) = \sum_{i=0}^{n} a_i x^i$. Thus, α is algebraic over F. Therefore, K is algebraic over F. $\square$

Theorem 2.1.4. *If $k \subset \tilde{K}$ are both fields, such that $\tilde{K} = k[\alpha_1, \alpha_2, ..., \alpha_n]$, then $\tilde{K}$ is algebraic over k.*

This proof is based on Daniel Allcock's notes [All05].

We first prove a special case: # $k = \infty$, $\beta \in k$ is transcendental over k and $k(\beta) = \tilde{K} = k[\alpha_1, ..., \alpha_n]$

We will show that, $\tilde{K} = k[\alpha_1, ..., \alpha_n] \subsetneq k(\beta)$. Note that,

$$\alpha_i \in k(\beta) = \left\{ \frac{f(\beta)}{g(\beta)} \quad , f, g \in k[x], g(\beta) \neq 0 \right\}.$$

Furthermore, all the elements of $\tilde{K}$, can be expressed as an algebraic combination of.

$$\alpha_1 = \frac{f_1(\beta)}{g_1(\beta)}, ..., \alpha_n = \frac{f_n(\beta)}{g_n(\beta)}$$

So, any polynomial $j \in \tilde{K} = k[\alpha_1, ..., \alpha_n]$ can be expressed as, $j = \frac{h(\beta)}{\prod_i^n g_i(\beta)^{m_i}}$.

We choose an element $c \in k$ such that $g_1(c) \neq 0, g_2(c) \neq 0, ..., g_n(c) \neq 0$.

We define $n(\beta) = \frac{f(\beta)}{g(\beta)} = \frac{1}{\beta - c} \in k(\beta)$. But $n(\beta) \notin k[x_1, ..., x_n]$, since $\beta - c$ is not a factor in none of the g_i, which proves $\tilde{K} = k[\alpha_1, ..., \alpha_n] \subsetneq k(\beta)$. This contradiction proves that $\tilde{K}$ cannot be generated by only one trascendental element β.

Now we prove the general case. Assume first that the degree of transcendence $(degtrasc)$ of $\tilde{K}$ over $k, degtrasc(\tilde{K}) = 1$, or equivalently, that there exists $\beta \in \tilde{K}$ (not algebraic), such that $k \subset k(\beta) \subset \tilde{K}$, and $\tilde{K}$ is an algebraic extension of $k(\beta)$. Consider the following chain of extensions:

$$k(\beta) \subset k(\beta)(\alpha_1) \subset k(\beta)(\alpha_1, \alpha_2) \subset ... \subset k(\beta)(\alpha_1, ..., \alpha_n) = \tilde{K}$$

Since each extension is algebraic, hence finite dimensional,this proves that $l = dim_{k(\beta)}\tilde{K} < \infty$.

Let $\{e_1, e_2, ..., e_l\}$ be the basis of $\tilde{K}$ over $k(\beta)$. As before we will arrive at the contradiction: $\tilde{K} = k[\alpha_1, \alpha_2, ..., \alpha_n] \subsetneq k(\beta)$.

All the elements of $\tilde{K}$, can be expressed as polynomials in the $\alpha_i' s$, with coefficients in k,

$$\alpha_0 = 1, \quad \alpha_i = \sum_{j=1}^{l} \lambda_{ij} e_j, \quad \lambda_{ij} = \frac{c_{ij}(\beta)}{d_{ij}(\beta)} \in k(\beta).$$

Moreover,

$$e_i e_j = \sum_{k=1}^{l} \gamma_{ijk} e_k, \quad \gamma_{ijk} = \frac{a_{ijk}(\beta)}{b_{ijk}(\beta)} \in k(\beta).$$

Hence every element a of $\tilde{K}$ is a rational function $a = \frac{f(\beta)}{g(\beta)}$, and $g(\beta)$ is a product of the $d_{ijk}(\beta)$ and the $b_{ij}(\beta)$. There exists an irreducible polynomial $t(\beta)$, which does not divide any of the $d_{ij}(\beta), b_{ijk}(\beta)$. Then $\frac{1}{t(\beta)} \notin \tilde{K} = k[\alpha_1, ..., \alpha_n]$, which contradicts $k(\beta) \subset \tilde{K}$. This proves the case when $degtras(k) = 1$.
Assume that the degree of transcendence $(degtrasc)$ of $\tilde{K}$ over k, $\quad degtrasc(\tilde{K}) > 1$. Consider the extension
$k \subset k(\alpha_1) \subset k(\alpha_1, \alpha_2) \subset ... \subset k(\alpha_1, \alpha_2, ..., \alpha_{n-1}) \subset k(\alpha_1, ..., \alpha_n) = k[\alpha_1, ..., \alpha_n] = \tilde{K}$. Since $\tilde{K}$ is trascendental over k, there exists

$$j = \max\{i, k(\alpha_1, \alpha_2, ..., \alpha_{i-1}) \subset k(\alpha_1, ..., \alpha_i) \quad \text{is trascendental}\}$$

Set $k' = k(\alpha_1, ..., \alpha_{j-1})$ and then $k' \subset \tilde{K}$ has $degtrasc(k') = 1$, and $k[\alpha_1, ..., \alpha_n] \subset k'[\alpha_1, ..., \alpha_n] = \tilde{K}$, so $\tilde{K}$ is finitely generated as a $k'-$algebra. This is impossible by the previous case, which concludes the proof.
□

Theorem 2.1.5. *If $J = \langle x_1 - a_1, x_2 - a_2, ..., x_n - a_n \rangle$ is an ideal in $A = k[x_1, ..., x_n]$ then J is maximal.*

Proof. Suppose there is a polynomial $p \in K = k[x_1, ..., x_n]$ such that p is not an element of J and define the ideal $G = \langle p, J \rangle$ such that,

$$J \subsetneq G \subset k[x_1, x_2, ..., x_n].$$

We will show that $G = k[x_1, x_2, ..., x_n]$. For certain c_α, we have

$$p(x_1 + a_1, x_2 + a_2, ..., x_n + a_n) = \sum_\alpha c_\alpha (x)^\alpha$$

But then $p(x_1, x_2, ..., x_n) = \sum_\alpha c_\alpha (x - a)^\alpha$, and so, we obtain $p(a_1, a_2, ..., a_n) = c_0 \neq 0$.
This affirmation is valid, since $p \notin J$ and $c_\alpha (x - a)^\alpha \in J$ if $\alpha \neq (0, 0, ..., 0)$.
The same argument shows that $p(x_1, ..., x_n) - p(a_1, a_2, ..., a_n) \in J$, which implies

$$c_0 = p(a_1, ..., a_n) \in \langle p, J \rangle.$$

Since $c_0 \in k$ and k is a field there exists c_0^{-1} such that, $1 = (c_0)(c_0^{-1})$.
Consequently $1 \in \langle p, J \rangle$, and so

$$G = \langle p, J \rangle = k[x_1, x_2, ..., x_n].$$

This shows that $J = \langle x_1 - a_1, ..., x_n - a_n \rangle$ is maximal.

□

Proposition 2.1.6. *Let k be algebraically closed and m an ideal in $k[x_1, ..., x_n]$. Then m is of the form $\langle x_1 - a_1, ..., x_n - a_n \rangle$ if and only if m is maximal.*

Proof. Consider the projection

$$k[x_1, ..., x_n] \xrightarrow{\pi} \frac{k[x_1, ..., x_n]}{m} = \tilde{K}.$$

Set $\alpha_i = \pi(x_i) = x_i + m \in \tilde{K}$. Then $\tilde{K} = k[\alpha_1, \ldots, \alpha_n]$ is a finitely generated k-algebra. By Theorem 2.1.4, $\tilde{K}$ is an algebraic extension of k. Since k is algebraically closed, $k \cong \tilde{K}$. So $\frac{k[x_1,...,x_n]}{m} \cong k$, and consider $a_i = \pi(x_i) \in k$. Then $\pi = ev_a$, where $a = (a_1, ..., a_n)$ and $m = ker(\pi) = \langle x_1 - a_1, ..., x_n - a_n \rangle$. In fact, clearly $\langle x_1 - a_1, ..., x_n - a_n \rangle \subset ker(ev_a)$, and since $\langle x_1 - a_1, ..., x_n - a_n \rangle$ is maximal, we have $m = \langle x_1 - a_1, ..., x_n - a_n \rangle$. □

Theorem 2.1.7 (Weak Nullstellensatz:). *An ideal $J \in k[x_1, ..., x_n]$ contains 1 if and only if $V(J) = \emptyset$.*

Proof. Clearly, if $1 \in J$, then $V(J) = \emptyset$. On the other hand, if J is a proper ideal, by Proposition 2.1.2 it is contained in a maximal ideal m, which by Proposition 2.1.6 is of the form:

$$m = \langle x_1 - a_1, x_2 - a_2, ..., x_n - a_n \rangle$$

But then $a = (a_1, ..., a_n) \in V(m) \subset V(J)$ and so $V(J) \neq \emptyset$, which concludes the proof.

□

2.2 Strong Nullstellensatz

Theorem 2.2.1. *Let k be an algebraically closed field and let $J \triangleleft k[x_1, ..., x_n]$ be an ideal. Then*

$$I(V(J)) = Rad(J).$$

Proof. By Theorem 1.1.4 $k[x_1, ..., x_n]$ is Noetherian, and so $J = \langle f_1, f_2, ..., f_s \rangle$. Let $g \in k[x_1, x_2, ..., x_n]$ such that $g \in Rad(J)$. Then, there exists $m > 0$ s.t. $g^m \in J$,i.e.,$g^m = \sum_{i=1}^{s} h_i(x) f_i(x)$, for some $h_i \in k[x_1, x_2, ..., x_n]$

Let $a = (a_1, a_2, ..., a_n) \in V(J)$, $(g(a))^m = g^m(a_1, a_2, ..., a_n) = \sum_{i=1}^{s} h_i f_i(a_1, a_2, ..., a_n) = 0$. Then, $g(a) = 0$. Since $a \in V(J)$ is arbitrary, this implies that $g \in I(V(J))$. This proves that $Rad(J) \subset I(V(J))$.

For the other inclusion, let $g \in I(V(J))$.

We define the ideal $\tilde{J} = \langle f_1, f_2, ..., f_r, x_{n+1}g - 1\rangle \lhd k[x_1, x_2, ..., x_{n+1}]$. We want to prove that $V(\tilde{J}) = \emptyset$. Assume by contradiction that exists $c = (c_1, ..., c_{n+1}) \in V(\tilde{J})$. Then $f_i(c_1, ..., c_n) = 0$, for $i = 1, ..., r$, hence $(c_1, ..., c_n) \in V(J)$ and so, since $g \in I(V(J))$, we have $g(c_1, ..., c_n) = 0$. Hence,

$$(x_{n+1}g - 1)(c) = c_{n+1}g(c_1, ..., c_n) - 1 = -1 \neq 0.$$

This is a contradiction, and hence,

$$V(\langle f_1, f_2, ..., f_r, x_{n+1}g - 1\rangle) = \emptyset.$$

Then, by the **Weak Nullstellensatz**, J is all the ring $k[x_1, x_2, x_3, ..., x_{n+1}]$ and so $1 \in \tilde{J}$. Then, there exists $a_i \in k[x_1, ..., x_{n+1}], i = 1, ..., r$ and $B \in k[x_1, ..., x_{n+1}]$, such that

$$1 = \sum a_i(x_1, ..., x_{n+1})f_i + B(x_1, x_2, ..., x_{n+1})(x_{n+1}g - 1).$$

We define $y := 1/x_{n+1}$. Let $m = \max_i(gr(a_i))$, and $d = gr(B)$. If we fix $r = max(m, d+1)$, then $\bar{a}_i(x_1, ..., y) = a_i(x_1, ..., \frac{1}{y})y^r \in k[x_1, x_2, ..., x_n, y]$ and
$\bar{B}(x_1, .., x_n, y) = y^r B(x_1, ..., x_n, \frac{1}{y}) \in k[x_1, .., x_n, y]$. It follows that

$$y^r = \sum a_i\Big(x_1, ..., \frac{1}{y}\Big)f_i y^r + y^r B\Big(x_1, x_2, ..., \frac{1}{y}\Big)\Big(\frac{g}{y} - 1\Big).$$

$$y^r = \sum \bar{a}_i(x_1, ..., y)f_i(x_1, ..., x_n) + \bar{B}(x_1, x_2, ..., y)(g - y).$$

If we replace y by $g(x_1, ..., x_n)$, we obtain

$$g^r = \sum \bar{a}_i(x_1, ..., x_n, g(x_1, ..., x_n))f_i(x_1, ..., x_n), \text{with} \quad \bar{a}_i \in k[x_1, x_2, ..., x_{n+1}].$$

Since $\bar{a}_i(x_1, ..., x_n, g(x_1, x_2, ..., x_n)) \in k[x_1, ..., x_n]$ we have, $g^r \in \langle f_1, f_2, ..., f_r \rangle = J$, for some $r > 0$. Hence, $g \in Rad(J)$, as desired. This proves $I(V(J)) \subset Rad(J)$, which concludes the proof.

$\square$

Chapter 3

Groebner bases

The application of Gröbner bases allows one to transform a given system of polynomial equations into an equivalent but much simpler form: the canonical Gröbner basis representation. This is done in such a way that the solution set is not changed but the equations are greatly simplified. This greatly aids in both solvability and more efficient calculations, as the structure of the equations may contain information about the number of solutions or other properties of interest. For example, this is useful in areas like algebraic geometry, computer-aided geometric design, and economics, where solving polynomial systems is a typical task.

3.1 Monomial Ordering

Before applying the division algorithm, it is crucial to classify all the monomials involved. Since we are working within a multivariate polynomial ring, we need to assign a weight to each monomial to determine their relative ordering. In this dissertation, we will not only work with monomials but also consider all types of polynomials.

Definition 3.1.1 (Monomial Ordering). $>$ on $k[x_1, x_2, x_3, ..., x_n]$ is a relation $>$ on $\mathbb{Z}^n_{\geq 0}$ or equivalently, a relation on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satysfying:

- $>$ is a total (or linear) ordering: this means that every pair of monomials must be classified in at least one of the following ways: $x^\alpha > x^\beta$, $x^\alpha < x^\beta$ or $x^\alpha = x^\beta$,
- If $\theta > \epsilon$ such that $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\theta + \gamma > \epsilon + \gamma$,
- $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. If $A \subset \mathbb{Z}^n_{\geq 0}$ is non-empty, then there is $\alpha \in \mathcal{A}$ such that $\beta > \alpha$, for all $\beta \neq \alpha$ in $\mathcal{A}$

3.2 Division Algorithm in $k[x_1, ..., x_n]$

Definition 3.2.1. Let $f \in k[x_1, ..., x_n]$, such that we can express $f = \sum_\alpha a_\alpha x^\alpha$

a) The multidegree of f:

$$multideg(f) = \max\{\alpha \in \mathbb{Z}_{\geq 0}, \text{such that} \quad a_\alpha \neq 0\}$$

b) The leading coefficient (LC):

$$LC(f) := \{a_\alpha, \quad \text{such that}, \alpha = multideg(f)\}$$

c) The leading monomial (LM):

$$LM(f) := \{x^\alpha, \quad \text{such that}, \alpha = multideg(f)\}$$

d) The leading term (LT):

$$LT(f) := LC(f).LM(f)$$

Proposition 3.2.2. *Let $F = (f_1, f_2, ..., f_s)$ be an s-tuple of polynomials such that $f_1, f_2, ..., f_s \in k[x_1, ..., x_n]$. Then any $f \in F$ can be expressed as follows,*

$$f = q_1 f_1 + \cdots + q_s f_s + r, \quad \forall i \quad q_i, r \in k[x_1, ..., x_n].$$

such that, $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of $LT(f_1), ..., LT(f_s)$. Moreover, for each i either $q_i f_i = 0$ or

$$multideg(q_i f_i) \leq multideg(f). \tag{3.1}$$

Proof. We have an algorithm that modifies q_i, r, p and such that the equality

$$f = q_1 f_1 + \cdots + q_s f_s + p + r \tag{3.2}$$

holds at each step. We begin with $r = 0, q_i = 0, p = f$ and we end with $p = 0$. Moreover, at each step either $q_i f_i = 0$ or

$$multideg(q_i f_i) \leq multideg(f).$$

The algorithm summarizes in the two principal steps:

(Division step) : If some $LT(f_i)$ divides $LT(p)$ then the algorithm proceeds as the division algorithm of one variable. i.e

$$q_i \to q_i + \frac{LT(p)}{LT(f_i)}$$

$$p \to p - \frac{LT(p)}{LT(f_i)} f_i$$

(Remainder Step) : If no $LT(f_i)$ divides $LT(p)$ then LT(p) is added to the remainder. i.e

$$r \to r + LT(p)$$

$$p \to p - LT(p)$$

To prove that the algorithm works, we will show that the equality (4.2) holds in every stage.

If the step is the division step at f_i, then

$$q_1 f_1 + \cdots + q_i f_i + q_s f_s + p + r \to q_1 f_1 + \cdots + \left(q_i + \frac{LT(p)}{LT(f_i)}\right) f_i + \cdots + q_s f_s + \left(p - \frac{LT(p)}{LT(f_i)} f_i\right) + r$$

$$q_1 f_1 + \cdots + q_i f_i + q_s f_s + p + r \to q_1 f_1 + q_2 f_2 + \cdots + q_s f_s + p + r$$

So (4.7) still holds after the division step.

At the remainder step, we have:

$$q_1 f_1 + \cdots + q_s f_s + p + r \to q_1 f_1 + \cdots + q_s f_s + q_s f_s + (p - LT(p)) + r + LT(p) = q_1 f_1 + ... + q_s f_s + p + r$$

$$\tilde{p} + r = (\tilde{p} - LT(p)) + (r + LT(p))$$

So (4.2) still hold after the remainder step.

Additionally, we need to show that the algorithm finishes. We need to prove that in each step of the algorithm p drops in multidegree or it becomes 0. Note that in

$$\tilde{p} = p - \frac{LT(p)}{LT(f_i)} f_i$$

The leading term of p cancels out, since

$$LT\left(\frac{LT(p)}{LT(f_i)} f_i\right) = \frac{LT(p)}{LT(f_i)} LT(f_i) = LT(p).$$

This implies that: $LT(\tilde{p}) < LT(p)$, which proves that the multidegree of $\tilde{p}$ drops in the division step.

On the other hand, if it's the remainder step then it's obvious that the inequality holds since $\tilde{p} = p - LT(p)$ which implies $LT(\tilde{p}) < LT(p)$.

The algorithm finishes, since otherwise we will get a decreasing sequence of multidegrees, $... < multideg(p''') < multideg(p'') < multideg(p')$ but this contradicts that every decreasing sequence of monomials should end. Therefore, the algorithm ends.

Finally, we need to prove that the inequality (4.6) holds,
Since $multideg(p)$ is decreasing and at the beginning we have $multideg(p) = multideg(f)$ we always have $multideg(p) \leq multideg(f)$. At the remainder step $q_i f_i$ is not changed, and at the beginning $q_i = 0$, so (4.6) hold after this step. After a division step, $LT(q_i f_i) \leq LT(f)$ or $q_i f_i = 0$ and $LT(q_i + \frac{LT(p)}{LT(f)}) f_i \leq \max\{LT(q_i f_i), LT(p)\}$

So either $LT(q_i f_i)$ remains the same, or $LT(q_i f_i) \leq LT(p) \leq LT(f)$ as desired.

□

3.3 Monomial Ideals and Dickson Lemma

Definition 3.3.1 (Monomial Ideal)**.** An ideal $J \subset k[x_1, ..., x_n]$ is a monomial ideal if it is generated by a set $\langle x^\alpha : \alpha \in A \rangle$, where $A \subset \mathbb{Z}^n_{\geq 0}$. Thus the elements of I are finite sums (even if A is infinite) of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, with $h_\alpha \in k[x_1, \ldots, x_n]$.

Theorem 3.3.2 (Dickson Lemma)**.** *Let $J = \langle x^\alpha | \alpha \in A \rangle \subset k[x_1, ..., x_n]$. Then J can be written in the form $J = \langle x^{\alpha_1}, ..., x^{\alpha_s} \rangle$ such that $\alpha_1, ..., \alpha_s \in A$. Therefore J is finitely generated.*

Proof. If $n = 1$,it is trivial,taking $\alpha_1 = \min A \subset \mathbb{Z}_{\geq 0}$ and so $J = \langle x_1^{\alpha_1} \rangle$ .We use induction when

$n > 1$. Let $I \triangleleft k[x_1, ..., x_{n-1}, y]$. We write $I = \langle x^\alpha y^\beta, \quad (\alpha, \beta) \in A \subset \mathbb{Z}^n_{\geq 0}\rangle$. Let $J \triangleleft k[x_1, ..., x_{n-1}]$ be such that, $J = \langle x^\alpha, \exists m \geq 0, x^\alpha y^m \in I\rangle$.

Because of the induction hypothesis J is finitely generated, and so $J = \langle x^{\alpha_1}, ..., x^{\alpha_s}\rangle$.Then $\exists m_i \geq 0, \quad x^{\alpha(i)} y^{m_i} \in I$, for all $1 \leq i \leq s$. Let $m = max(m_i)$. We define the ideals $J_l \in k[x_1, ..., x_{n-1}]$, such that $J_l = \langle x^\beta, x^\beta y^l \in I\rangle$ for all $0 < l < m - 1$. By inductive hypothesis we have $J_l = \langle x^{\alpha_l(1)}, ..., x^{\alpha_l(s_l)}\rangle$. We want to prove that $I = \langle Jy^m, J_0, yJ_1..., y^{m-1}J_{m-1}\rangle$. First, we will prove that, $I \subset \langle Jy^m, J_0, yJ_1..., y^{m-1}J_{m-1}\rangle$. Let $x^\alpha y^p \in I$. Then $x^\alpha y^m | x^\alpha y^p$ and assume that $p \geq m$ which implies that $x^\alpha y^p \in \langle x^\alpha y^m\rangle \subset Jy^m$.If $p \leq m - 1$, then $x^\alpha \in J_p$ and so $x^\alpha y^p \in y^p J_p$. This proves that $I \subset \langle Jy^m, J_0, yJ_1..., y^{m-1}J_{m-1}\rangle$.To prove the converse, let $x^\alpha y^m \in Jy^m$. Then $\exists i$, $x^{\alpha_i} | x^\alpha$. But then $x^{\alpha_i} y^{m_i} | x^\alpha y^m$ and so $x^\alpha y^m \in I$. If $x^\alpha y^i \in J_i y^i$, then by definition of J_i we have $x^\alpha y^i \in I$.Therefore $I = \langle Jy^m, J_0, yJ_1..., y^{m-1}J_{m-1}\rangle$. □

3.4 Hilbert Basis Theorem

Using Monomial ideals, we can give another proof of Theorem 1.1.4.

Theorem 3.4.1. *Every ideal $I \in k[x_1, ..., x_n]$ has a finite generating set. In other words, $I = \langle g_1, .., g_t\rangle$ for some $g_1, \cdots, g_t \in I$.*

Proof. If $I = \langle 0\rangle$, it is finitely generated. Else, by Lemma 3.3.2 $\exists g_1, \cdots, g_t \in I$,such that $LT(I) = \langle LT(g_1), LT(g_2), \cdots, LT(g_t)\rangle$. We claim that $I = \langle g_1, \cdots, g_t\rangle$. On one hand, clearly $\langle g_1, .., g_t\rangle \subset I$. On the other hand, let $f \in I$. By the division algorithm, $f = g_1 q_1 + \cdots + q_t g_t + r$. We claim that $r = 0$. Otherwise $r = f - q_1 f_1 ... - q_t f_t \in I$. Then, $LT(r) \in LT(I) = \langle LT(g_1), ..., LT(g_t)\rangle$. But this can't be possible since this would mean that $LT(g_i) | LT(r)$, which is impossible by Proposition 3.2.2. Thus we just proved that $f = q_1 g_1 + \cdots + q_t g_t \in \langle g_1, \cdots, g_t\rangle$. So, $I \subset \langle g_1, \cdots, g_t\rangle$. □

Definition 3.4.2. Fixing the monomial ordering in $k[x_1, ..., x_n]$ we call a finite susbet $G = \{g_1, ..., g_t\} \subset I$, $I \triangleleft k[x_1, ..., x_n]$ a Groebner basis, if

$$\langle LT(I)\rangle = \langle LT(g_1), ..., LT(g_t)\rangle.$$

3.5 Properties of Groebner Basis

Proposition 3.5.1. *Let $I \subset k[x_1, ..., x_n]$ and let $G = \{g_1, ..., g_t\}$ be a Groebner basis for I. Let $f \in k[x_1, ..., x_n]$, then there exists a unique $r \in k[x_1, ..., x_n]$, such that*
$i)$ No term of r is divisible by $LT(g_1), ..., LT(g_t)$,
$ii)$ Exists a $g \in I$, such that $f = g + r$

Proof. By Proposition 3.2.2, $f = q_1g_1 + q_2g_2 + ... + q_tg_t + r$, and no $LT(g_i)$ divides any term of r. Clearly $g = q_1g_1 + ... + q_tg_t$ is in I, so it remains to prove that the residual r is unique. For that, suppose that $f = g + r = g' + r'$ Since G is a Groebner basis, we have that $r - r' = g' - g \in I$. If $r \neq r'$ then $LT(r - r') \in \langle LT(I)\rangle$. Since G is a Groebner basis, $\langle LT(I)\rangle = \langle LT(g_1), ..., LT(g_t)\rangle$. Then, $LT(g_i)|LT(r - r')$ a contradiction since r, r' are residuals. Then, we have that $r = r'$. □

Definition 3.5.2. We define $\bar{f}^F$ as the remainder on the division of the polynomial $f \in k[x_1, ..., x_n]$ by the ordered s-tuple $F = (f_1, ..., f_s)$. If F is a Groebner basis for $\langle f_1, f_2, ..., f_s\rangle$, then we can regard F as a set (without any particular order).

Corollary 3.5.3. *Let $G = \{g_1, ..., g_t\}$ be a Groebner basis for an ideal $I \subset k[x_1, ..., x_n]$. Then $f \in I$ if and only if $\bar{f}^G = 0$.*

Definition 3.5.4 (Reduced Groebner basis). A reduced Groebner Basis for a polynomial ideal I is a Groebner basis G for I such that:

$i) LC(p) = 1$ for all $p \in G$.

$ii)$ For all $p \in G$, no monomial of p lies in $\langle LT(G) - \{p\}\rangle$.

Proposition 3.5.5. *Let $I \neq \{0\}$ be a polynomial ideal. Then, I has a unique reduced Groebner basis for a given monomial ordering.*

Proof. Let J be a minimal Groebner basis for I. We say that $g \in J$ is reduced for G, if , no monomial of g lies in $\langle LT(J) - \{g\}\rangle$. Then, we define $g' = \bar{g}^{J-\{g\}}$ and $J' = (J - \{g\}) \cup \{g'\}$. One checks that J' is a minimal Groebner basis, since $LT(g) = LT(g')$, which shows that $\langle LT(J)\rangle = \langle LT(J')\rangle$. Apply this process to all the elements of J until they are all reduced, and so we get $\tilde{J}$. Note that once a term is reduced, it remains reduced, since we are not changing the leading terms of them.

To prove uniqueness, let's suppose we have two reduced basis G, G'.We will prove that $G \subset G'$. So take $g \in G$. Since $LT(g)$ is in $LT(I) = \langle LT(G')\rangle$, there exists $g' \in G'$ such that $LT(g')$ divides $LT(g)$. Since $LT(g')$ is in $LT(I) = \langle LT(G)\rangle$, there exists g'' in G such that $LT(g'')$ divides $LT(g')$, since both have leading coefficient equal to 1.

Since G is a Groebner basis $\overline{g - g'}^{G} = 0$, and $LT(g) = LT(g')$ implies that this term cancel out. After that, all the remaining terms can't be divisible by the $LT(G) = LT(G')$, since G and G' are reduced. Then, $g - g' = \overline{g - g'}^{G} = 0$, which proves that $g = g'$. This proves $G \subset G'$. The same argument shows that $G' \subset G$, and so we obtain $G = G'$, which shows that the reduced Groebner basis is unique. □

3.6 Buchberger's Algorithm

Definition 3.6.1. Let $f, g \in k[x_1, ..., x_n]$.

If $multideg(f) = \alpha$ *and* $multideg(g) = \beta$*. Let* $\gamma = (\gamma_1, ..., \gamma_n)$. *such that* $\gamma_i = max(\alpha_i, \beta_i) \forall i$*. We call* x^γ *the least common multiple of* $LM(f)$ *and* $LM(g)$ *and write*

$$x^\gamma = LCM(LM(f), LM(g)).$$

An ***S-polynomial*** *of two polynomials* $f, g \in k[x_1, ..., x_n]$ *is the combination*

$$S(f, g) = \frac{x^\gamma}{LT(f)}.f - \frac{x^\gamma}{LT(g)}.g$$

To be able to introduce Buchberger's Criterion we first need to introduce the following lemma.

Lemma 3.6.2. *Suppose we have a sum* $\sum_{i=1}^{s} c_t f_t$ *where* $multideg(f_t) = \delta \in \mathbb{Z}^n_{\geq 0}$ *. If*

$$multideg\Big(\sum_{i=1}^{s}\Big) c_t f_t < \delta \quad \textit{then} \quad \sum_{i=1}^{s} c_t f_t$$

is a linear combination of $S(p_j, p_l), \quad 1 \leq j, l \leq s$*. Moreover,* $multideg(S(p_j, p_l)) < \delta$*.*

Proof. Let $d_t = LC(f_t)$ then we have that $c_t d_t = LC(c_t f_t)$. Given that the $multideg(c_t f_t) = \delta$ and $multideg(\sum_{t=1}^{s} c_t f_t) < \delta$. Then

$$\sum_{t=1}^{s} c_t d_t = 0. \tag{3.4}$$

We define $p_t = \frac{f_t}{d_t}$ with leading coefficient equal to 1. Consider the telescopic sum:

$$\sum_{i=1}^t c_i f_i = \sum_{i=1}^t c_i d_i p_i =$$
$$c_1 d_1(p_1 - p_2) + (c_1 d_1 + c_2 d_2)(p_2 - p_3) + ... + (c_1 d_1 + ... + c_{t-1} d_{t-1})(p_{t-1} - p_t) + (c_1 d_1 + ... + c_t d_t) p_t.$$

Since $LT(f_t) = d_t x^\delta$, for any $j, k \in \{1, 2, ..., t\}$, we get the S-polynomials of f_j and f_k:

$$S(f_j, f_k) = \frac{x^\delta}{LT(f_j)} f_j - \frac{x^\delta}{LT(f_k)} f_k = \frac{x^\delta}{d_j x^\delta} f_j - \frac{x^\delta}{d_k x^\delta} f_k = p_j - p_k.$$

Using the equation (4.8)

$$\sum_{i=1}^t c_i f_i = c_1 d_1 S(f_1, f_2) + (c_1 d_1 + c_2 d_2) S(f_2, f_3) + ... + (c_1 d_1 + c_2 d_2 + ... + c_t d_t) S(f_{t-1}, f_t),$$

which is a sum of the desired form. Since $multideg(p_j) = \delta$ with $LC(p_j) = 1$ and $multideg(p_k) = \delta$ with $LC(p_k) = 1$, then $multideg(p_j - p_k) < \delta$. Then $multideg(S(f_j, f_k)) < \delta \quad \forall j.k \in \{1, ..., t\}$ proving the result. □

Lemma 3.6.3. *Suppose that* g_i *and* g_j *are monic polynomials and that* $x^{\beta(i)} g_i$ *and* $x^{\beta_j} g_j$ *have multidegree* δ*, then*

$$S(x^{\beta(i)} g_i, x^{\beta(j)} g_j) = x^{\delta - \gamma_{ij}} S(g_i, g_j)$$

where $x^{\gamma_{ij}} = LCM(LM(g_i), LM(g_j))$*, and* $multideg(S(x^{\beta_i g_i}, x^{\beta_j} g_j)) < \delta$.

Proof. On one hand we have

$$S(x^{\beta_i} g_i, x^{\beta_j} g_j) = \frac{x^\delta}{LT(x^{\beta_i} g_i)} x^{\beta_i} g_i - \frac{x^\delta}{LT(x^{\beta_j} g_j)} x^{\beta_j} g_j = \frac{x^\delta}{LT(g_i)} g_i - \frac{x^\delta}{LT(g_j)} g_j.$$

On the other hand we have,

$$S(g_i, g_j) = \frac{x^{\gamma_{ij}}}{LT(g_i)} g_i - \frac{x^{\gamma_{ij}}}{LT(g_j)} g_j.$$

Then,

$$x^{\delta-\gamma_{ij}}S(g_i,g_j)=\frac{x^\delta}{LT(g_i)}g_i-\frac{x^\delta}{LT(g_j)}g_j=S(x^{\beta_i}g_i,x^{\beta_j}g_j)$$

and we get the result as desired. Since $x^{\beta_i}g_i$ and $x^{\beta_j}g_j$ have the same $multideg=\delta$, we obtain:

$$multideg(S(x^{\beta_i}g_i,x^{\beta_j}g_j))<\delta.$$

□

Theorem 3.6.4. ***Buchberger's Criterion*** *Let an ideal $I\lhd k[x_1,...x_n]$. Then a basis $G=(g_1,...,g_t)$ of I is a Groebner basis of I if and only if for all pairs $i\neq j$, the remainder of* $\overline{S(g_i,g_j)}^G=0$.

Proof. ($\Rightarrow$) If $G=(g_1,...,g_t)$ is a Groebner basis of, then $S(g_i,g_j)\in I$ and so by Corollary 3.5.3, $\overline{S(g_i,g_j)}^G=0$.

($\Leftarrow$) For the other implication, we have to prove that if $f\in I$, then $LT(f)\in\langle LT(g_1),LT(g_2),...,LT(g_t)\rangle$. Let $f\in I$, then, we can write

$$f=\sum_{i=1}^{t}h_ig_i,\qquad h_i\in k[x_1,...,x_n]. \tag{3.5}$$

We define $\delta=\max_i\{multideg(h_ig_i)\}$. For each representation as in (3.5) we have a certain δ associated with it. Since the monomial ordering is a well-ordering, we can choose among all representations as in (3.5) a representation such that δ is minimal.

We claim that for such a representation $multideg(f) = \delta$. If the claim is true, then there is an i and an h_i such that $multideg(f) = multideg(h_i g_i)$. But then $LT(g_i)$ divides $LT(f)$ and so $LT(f) \in \langle LT(g_1), LT(g_2), ..., LT(g_t)\rangle$, as desired.

In order to prove the claim, note that the $multideg(f) \leq \delta$; hence it suffices to prove that $multideg(f) < \delta$, is impossible. So assume that $multideg(f) < \delta$. Write f as follows:

$$f = \sum_{mutlideg(h_i g_i)<\delta}^{t} h_i g_i + \sum_{multideg(h_i g_i)=\delta}^{t} h_i g_i.$$

$$f = \sum_{mutlideg(h_i g_i)<\delta}^{t} h_i g_i + \sum_{multideg(h_i g_i)=\delta} LT(h_i) g_i + \sum_{multideg(h_i g_i)=\delta}^{t} (h_i - LT(h_i)) g_i. \quad (3.6)$$

It is straightforward that the first and third summand have $multideg < \delta$. Since we assume that $multideg(f) < \delta$, we know that the second summand also has $multideg < \delta$:

$$\sum_{multideg(h_i g_i)=\delta} LT(h_i) g_i < \delta.$$

Let $p_i = LT(h_i)g_i$, by Lemma 3.6.2 we have that, since $multideg\Big(\sum_i p_i\Big) < \delta$, we can express this sum as a linear combination of $S(p_i, p_j)$.

Let $LT(h_i) = c_i x^{\alpha(i)}$. Then the second sum, can be expressed as follows:

$$\sum_{multideg(h_i g_i)=\delta} LT(h_i) g_i = \sum_{multideg(h_i g_i)=\delta} c_i x^{\alpha(i)} g_i \quad (3.7)$$

By Lemma 3.6.3, we can express this as a combination of the S-polynomials $S(x^{\alpha j} g_j, x^{\alpha(k)} g_k)$. By Lemma 3.6.2, we have

$$S(x^{\alpha(j)}g_j, x^{\alpha(k)}g_k) = x^{\delta-\gamma_{j,k}}S(g_j, g_k),$$

where $x^{\gamma_{j,k}} = LCM(LT(g_i, LT(g_j)))$. Then we can express (4.12) as follows:

$$\sum_{multideg(h_i g_i)=\delta} LT(h_i)g_i = \sum_{i,j} c_{j,k}\Big(x^{\delta-\gamma_{j,k}}S(g_j, g_k)\Big). \tag{3.8}$$

By assumption each $\overline{S(g_j, g_k)}^G = 0$, and so we can write each S-polynomial as

$$S(g_j, g_k) = \sum_{i=1}^{t} a_{ijk}g_i,$$

where $a_{ijk} \in k[x_1, ..., x_n]$. By the division algorithm, we have that:

$$multideg(a_{ijk}g_i) \leq multideg(S(g_j, g_k)).$$

Then,

$$x^{\delta-\gamma_{j,k}}S(g_j, g_k) = \sum_{i=1}^{t} b_{ijk}g_i \tag{3.9}$$

where $b_{ijk} = x^{\delta-\gamma_{jk}}a_{ijk}$. Moreover, by Lemma 3.6.3 we have $multideg(x^{\delta-\gamma_{ij}}S(g_i, g_j)) = multideg(S(x^{\beta_i g_i}, x^{\beta_j}g_j)) < \delta$, and so

$$multideg(b_{ijk}g_i) \leq multideg(x^{\delta-\gamma_{jk}}S(g_j, g_k)) < \delta$$

If we replace (3.9) in (3.8): we obtain

$$\sum_{multideg(h_i g_i)=\delta} LT(h_i)g_i = \sum_{j,k} c_{j,k}x^{\delta-\gamma_{j,k}}S(g_j, g_k) = \sum_{j,k}\Big(\sum_i b_{ijk}g_i\Big) = \sum_i \tilde{h}_i g_i \tag{3.10}$$

with $multideg(\tilde{h}_i g_i) < \delta$.

If we replace (3.10) in (4.10), we get an expression for f with $multideg < \delta$, but this contradicts the minimality of δ, therefore is an absurd. Thus we have proved that $multideg(f) = \delta$, therefore there is an i and h_i, such that $multideg(f_i) = multideg(h_i g_i)$. Then, $LT(f) \in \langle LT(g_1), ..., LT(g_t) \rangle$, as desired.

□

Chapter 4

Equation related to the Jacobian conjecture

In this chapter, we will discuss a system of polynomial equations related to the Jacobian Conjecture. Following the paper of Guccione and Solorzano (2014) [VS14], we will compute the system of polynomial equations for $n = 3$ with $m = 3r + 1$ and $m = 3r + 2$. In the first section, we will explore the system of polynomial equations associated with these cases and develop a general form by providing a general formula. In the second section, we will delve deeper to reduce this system of equations to a simpler form. Additionally, we will compute a Groebner basis for a partial system of polynomials. Finally, in the third section, we will examine the system of solutions for these polynomials. We will catalog all possible cases and discuss whether or not a solution exists.

4.1 Previous results

Let K be a characteristic zero field and let $K[y]((x^{-1}))$ be the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. We will start from the following theorem, proved in Theorem 1.9 of Guccione and Valqui (2024) [GGV24]

Theorem 4.1.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x, y]$ *and* $C, F \in K[y]((x^{-1}))$,
- $n, m \in N$ *such that* $n \nmid m$ *and* $m \nmid n$,
- $\nu_i \in K, \quad i = 0, \dots, m + n - 2$ *with* $\nu_0 = 1$,

  such that

- C *has the following form:*

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots$$

  with each $C_{-i} \in K[y]$

- $gr(C) = 1$ *and* $gr(F) = 2 - n$, *where* gr *is the total degree,*
- $F_+ = x^{1-n}y$, *where* F_+ *is the term of maximal degree in* x *of* F,
- $C^n = P$ *and* $Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F$.

  Furthermore, under these conditions (P, Q) *is a counterexample to the Jacobian conjecture.*

In Guccione and Valqui (2024) [GGV24], the authors consider the following slightly more general situation. Let D be a K-algebra (for example, in Theorem 4.1.1 we have $D = K[y]$), n, m positive integers such that $n \nmid m$ and $n \nmid m$, $(\nu_i)_{1 \leq i \leq n+m-2}$ a family of elements in K with $\nu_0 = 1$ and $F_{1-n} \in D$ (in Theorem 4.1.1 we take $F_{1-n} = y$. A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \textit{with } C_{-i} \in D,$$

is a solution of the system $S(n, m, (\nu_i), F_{1-n})$, if there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \cdots, \quad \textit{with } F_{1-n}, F_{-n}, \ldots \textit{ in } D$$

$$P = C^n \qquad \textit{and} \qquad Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F.$$

For example, if $n = 3$, then

$$\begin{aligned} P(\mathbf{x}) = C^3 = \mathbf{x}^3 + 3C_{-1}\,\mathbf{x} + 3C_{-2} + (3C_{-1}^2 + 3C_{-3})\,\mathbf{x}^{-1} + (6C_{-1}C_{-2} + 4C_{-4})\,\mathbf{x}^{-2} \\ + (C_{-1}^3 + 3C_{-2}^2 + 6c_{-2}C_{-3} + 3C_{-5})\,\mathbf{x}^{-3} \\ + (3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6})\,\mathbf{x}^{-4} \\ + (3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 6C_{-7})\,\mathbf{x}^{-5} \\ + \ldots \end{aligned}$$

and the condition $C^3 \in D[x]$ translates into the following conditions on C_{-k}:

$$\begin{aligned} 0 &= (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}, \\ 0 &= (C^3)_{-2} = 6C_{-1}C_{-2} + 4C_{-4}, \\ 0 &= (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-2}C_{-3} + 3C_{-5}, \\ 0 &= (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6}, \\ 0 &= (C^3)_{-5} = 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 6C_{-7}, \\ &\vdots \end{aligned}$$

In general, the condition $P(x) = C^n \in D[x]$ yields equations $(C^n)_{-k} = 0$, whereas the condition $Q(x) = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \in D[x]$ gives us the equations $\left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F\right)_{-k} = 0$, where we note that $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. Remark 1.13 of Guccione and Valqui (2024) [GGV24]) that the first $m + n - 2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m + n - 2$. Moreover, the F_{-k} for $k > n - 1$ depend only on F_{1-n} and C. Consequently, having a solution C to the system $S(n, m, (\nu_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \dots, C_{-m-n+2})$ to the system

$$\begin{aligned}
E_k &:= (C^n)_{-k} = 0, && \text{for } k = 1, \dots, m-1, \\
E_{m-1+k} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{-k} = 0, && \text{for } k = 1, \dots, n-2, \\
E_{m+n-2} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{1-n} + F_{1-n} = 0,
\end{aligned} \tag{4.1}$$

with $m + n - 2$ equations E_k and $m + n - 2$ unknowns C_{-k}.

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1}, \dots, C_{m+n-2}]$. In this paper we compute such a Groebner basis of (4.1) in a very particular case: we assume $n = 3$, $m = 3r + 1$ or $m = 3r + 2$ for some integer $r > 0$, and $\nu_i = 0$ for $i > 0$. Moreover we consider $D = C[y]$ and $F_{1-n} = y$.

4.2 Computation of Groebner Basis for I_{m-1}

Assume $n = 3$, $3 \nmid m > 3$ and $\nu_i = 0$ for $i > 0$. Set also $D = C[y]$ and $F_{1-n} = y$. Then the system (4.1) reads

$$E_i = \begin{cases} (C^3)_{-i}, & i = 1, \ldots, m-1 \\ (C^m)_{-1}, & i = m, \\ (C^m)_{-2} + y, & i = m+1, \end{cases} \tag{4.1}$$

where $(C^2)_{-i}$ *denotes the coefficient of* x^{-i} *in the Laurent series* C^3*. Explicitly, the polynomials* E_i *are given by:*

$$\begin{aligned} E_1 &= 3C_{-1}^2 + 3C_{-3}, \\ E_2 &= 6C_{-1}C_{-2} + 4C_{-4}, \\ E_3 &= C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}, \\ E_4 &= 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6}, \\ E_5 &= 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 6C_{-7}, \\ &\vdots \\ E_{m-1} &= (C^3)_{1-m}, \\ E_m &= (C^m)_{-1}, \\ E_{m+1} &= (C^m)_{-2} + y, \end{aligned} \tag{4.2}$$

Each E_i *is a polynomial in the ring* $\mathbb{C}[C_{-1}, C_{-2}, \ldots, C_{m+1}, y]$*, and the* $m+1$ *polynomials yield the ideal*

$$I = \langle E_1, \ldots, E_m, E_{m+1} \rangle.$$

Our goal is to find a Groebner basis for the ideal I*, but we find it nearly explicit only for* $I_{m-1} := \langle E_1, E_2, \ldots, E_{m-2}, E_{m-1} \rangle$*. For this we note that the equations are homogeneous, for the weight obtained by setting*

$$w(C_{-i}) = i + 1, \quad \textit{and} \quad w(y) = m + n - 1 = m + 2.$$

We consider y as a variable, so the equations remain homogeneous. Then

$$w(E_k) = k+3, \text{ for } k = 1, \dots, m-1, \quad w(E_m) = m+1 \quad \text{and} \quad w(E_{m+1}) = m+2.$$

Note that for $k = 1\dots, m-1$ we have

$$E_k := 3\Big(\sum_{\substack{i=-1\\ 3i\neq k}}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)}\Big) + 6\Big(\sum_{\substack{0<i<j\\ i+j=k+1}} C_{-i}C_{-j}\Big) + 6\Big(\sum_{\substack{0<i<j<l\\ i+j+l=k}} C_{-i}C_{-j}C_{-l}\Big) + \varepsilon(C_{-\frac{k}{3}})^3, \tag{4.3}$$

where $\varepsilon = \begin{cases} 1, & 3|k \\ 0, & 3\nmid k \end{cases}$ *. Note that $C_1 = 1$ and $C_0 = 0$, and so*

$$3\sum_{i=-1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)} = 3C_{k+2} + 3\sum_{i=1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)} \tag{4.4}$$

In order to compute a Groebner basis we will consider the degree reverse lexicographic monomial order, but for the degree given by the above mentioned weight. This means that the monomial order is given by the matrix

$$wmat = \begin{pmatrix} m+2 & m+1 & m & \dots & 4 & 3 & 2 & m+2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

on the variables $C_{-(m+1)}, C_{-m}, C_{-(m-1)}, \dots, C_{-3}, C_{-2}, C_{-1}, y$. We first compute a Groebner basis $(\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{m-1})$ for the ideal $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1}\rangle$.

Proposition 4.2.1. *The set $\{E_1, \ldots, E_{m-1}\}$ is a Groebner basis of I_{m-1}. The reduced Groebner basis of I_{m-1} is given by polynomials $\tilde{E}_k$ for $k = 1, \ldots, m-1$, each of the form*

$$\tilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

where $R_k(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ is an homogeneous polynomial in the variables C_{-1} and C_{-2} of weight $w(\tilde{E}_k) = w(E_k) = k + 3$.

Proof. By (4.3) and (4.4) we know that E_k is of the form

$$E_k = 3C_{-k-2} + T(C_{-1}, \ldots, C_{-k}), \quad \text{for } k = 1, \ldots, m-1,$$

where T is a polynomial in the variables $C_{-1}, \ldots, C_{-k}$. Then by Proposition 2.9.4 of [CLO13], since

$$LCM(LT(E_i)/3, LT(E_j)/3) = LCM(C_{-i-2}, C_{-j-2}) = C_{-i-2}C_{-j-2} = (LT(E_i)/3)(LT(E_j)/3),$$

we have $S(E_i, E_j) \longrightarrow_G 0$, and so, by Theorem 2.9.3 of [CLO13], the set $G = \{E_1/3, \ldots, E_{m-1}/3\}$ is a Groebner basis of I_{m-1}. One verifies directly that it is a minimal Groebner basis, according to Definition 2.7.4 of [CLO13]. If we apply the process described in the proof Proposition 2.7.6 of [CLO13] to the Groebner basis $G = \{E_1/3, \ldots, E_{m-1}/3\}$ we obtain that

$\tilde{E}_1 = \overline{E_1/3}^{G \setminus E_1/3} = E_1/3$ and $\tilde{E}_1 = \overline{E_2/3}^{G \setminus E_2/3} = E_2/3$. Moreover, for $k = 3, \ldots, m-1$, we define $G_k = \{\tilde{E}_1, \ldots, \widetilde{E}_{k+1}, E_k, \ldots, E_{m-1}\}$ and then $\widetilde{E}_k = \overline{E_k}^{G_k \setminus E_k}$,

Clearly the remainder can have only the variables C_{-1} and C_{-2}, hence $\widetilde{E}_k$ is of the form $\widetilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2})$, as desired. □

Although we have no explicit formula for $R_k(C_{-1}, C_{-2})$, we can compute it for small k.

$$
\begin{aligned}
\widetilde{E}_1 &= C_{-3} + C_{-1}^2, \\
\widetilde{E}_2 &= C_{-4} + 2C_{-1}C_{-2}, \\
\widetilde{E}_3 &= C_{-5} + C_{-2}^2 - \tfrac{5}{3}C_{-1}^3, \\
\widetilde{E}_4 &= C_{-6} - 5C_{-1}^2C_{-2}, \\
\widetilde{E}_5 &= C_{-7} + \tfrac{10}{3}C_{-1}^4 - 5C_{-1}C_{-2}^2, \\
&\vdots
\end{aligned}
\tag{4.5}
$$

Dividing the polynomials E_m and E_{m+1} by the polynomials $\{\widetilde{E}_{m-1}, ..., \widetilde{E}_2, \widetilde{E}_1\}$ with respect to the given order, we obtain

$$
\overline{\frac{E_m}{3}}^{G_m \backslash \{\frac{E_m}{3}\}} = \tilde{E}_m = R_m(C_{-1}, C_{-2}) \text{ and } \overline{\frac{E_{m+1}}{3}}^{G_m \backslash \{\frac{E_{m+1}}{3}\}} = \widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2})
$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials such that $w(\widetilde{E_m}) = w(E_m) = m + 1$ and $w(\widetilde{E}_{m+1} = w(E_{m+1}) = m + 2$.

Although we don't give an explicit description of the Groebner Basis of the whole system, in the next section we show how to determine the solution of the set of polynomial system, using that

$$
I = \langle E_1, E_2, ..., E_m, E_{m+1}\rangle = \langle \widetilde{E}_1, \widetilde{E}_2, ..., \widetilde{E}_m, \widetilde{E}_{m+1}\rangle
$$

4.3 The solution of the system of polynomial equations

In this section we analyze the solutions of the system of equations. Note that the partial system I_{m-1} shows that the values of C_{-1} and C_{-2} determine univocally the *values of* C_{-k} *for* $k > 2$. *Moreover,* C_{-1} *and* C_{-2} *can be computed using the following two equations:*

$$\widetilde{E}_m = R_m(C_{-1}, C_{-2}). \tag{4.6}$$

and

$$\widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}). \tag{4.7}$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ *are homogeneous polynomials with respect to the weight considered before, i.e* $w(C_{-1}) = 2, w(C_{-2}) = 3$.*Moreover* $w(\widetilde{E}_m) = m + n - 2 = m + 1$ *and* $w(\widetilde{E}_{m+1}) = m + n - 1 = m + 2$.. *Then* (4.6) *and* (4.7) *read*

$$\widetilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j \tag{4.8}$$

and

$$\widetilde{E}_{m+1} = y + \sum_{2i+3j=m+2} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j \tag{4.9}$$

for some constants $\lambda_m^{ij}, \lambda_{m+1}^{ij} \in K$. *By* (4.9) *the two variables cannot be zero at the same time. We compute the solutions in the cases where one of the variables is zero.*

1. ***FIRST CASE****: For* $C_{-1} = 0$, *and* $C_{-2} \neq 0$.

   In this case the only term surviving in (4.8) *is*

$$0 = \widetilde{E}_m = \lambda_m^{0j} C_{-2}^j$$

   with $3j = m + 1$. *So necessarily*

$$\lambda^{0,\frac{(m+1)}{3}} = 0 \quad \textit{if} \quad 3|m+1 \tag{4.10}$$

Similarly, the only term surviving in the sum has $i = 0$, *and so we obtain*

$$0 = \tilde{E}_{m+1} = y + \lambda^{0j}_{m+1} C^{j}_{-2} \quad \textit{with} \quad 3j = m+2$$

Since $y \neq 0$, *necessarily* $\lambda^{0j}_{m+1} \neq 0$ *for* $3j = m+2$, *and so* $3|m+2$, *i.e.* $m \equiv 1(mod3)$. *This shows that the condition 4.10 is trivially satisfied.*

Lemma 4.3.1. *If* $3|m+2$ *and* $C_{-1} = 0$, *then* $\lambda^{0j}_{m+1} \neq 0$ *for* $3j = m+2$

Proof. It is easy to check that $P = x^3 + 3C_{-2}$, and then, by Newton binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty} \binom{m/3}{k} (3C_{-2})^k (x^3)^{\frac{m}{3}-k} \tag{4.11}$$

Thus $\lambda^{0j}_{m+1} C^{j}_{-2} = (C^m)_{-2}$ is the coefficient of $x^{-2} = (x^3)^{\frac{m}{3}-j}$, since $m = 3j-2$. Then

$$\lambda^{0j}_{m+1} = \binom{m/3}{j} 3^j \neq 0$$

as desired. □

Thus we have proved the following proposition.

Proposition 4.3.2. *If* $(C_{-1}, C_{-2}, ..., C_{-(m+1)})$ *is a solution of the system 4.1, with* $C_{-1} = 0$ *and* $C_{-2} \neq 0$, *then*

- $m \equiv 1 mod 3$
- $\lambda^{0j}_{m+1} \neq 0$ *for* $j = \frac{m+2}{3}$

– *There are j solutions of the system 4.1 in $K[y^{1/j}]$ given by*
$C_{-1}=0, \quad C_{-2}=\left(\frac{-y}{\lambda^{0j}_{m+1}}\right)^{\frac{1}{j}}$ *and* $C_{-(k)}=-R_{k-2}(C_{-1},C_{-2})$ *for* $3\leq k\leq m+1$

2. ***SECOND CASE:*** *For $C_{-1}\neq 0$ and $C_{-2}=0$.*

In this case the only term surviving in 4.8 is

$$0=\tilde{E}_m=\lambda^{i0}_m C^i_{-1}$$

with $2i=m+1$. So necessarily

$$\lambda^{(m+1)/2,0}_m=0 \quad \textit{if} \quad 2|m+1 \tag{4.12}$$

Similarly, the only term surviving in the sum 4.8 has $j=0$, and so we obtain

$$0=\tilde{E}_{m+1}=y+\lambda^{i0}_{m+1}C^i_{-1} \quad \textit{with} \quad 2i=m+2$$

Since $y\neq 0$, necessarily $\lambda^{i0}_{m+1}\neq 0$ for $2i=m+2$, and so $2|m+2$, i.e. m is even. This shows that the condition 4.12 is trivially satisfied.

Lemma 4.3.3. *If $2|m$ and $C_{-2}=0$, then $\lambda^{i0}_{m+1}\neq 0$ for $2i=m+2$*

Proof. It is easy to check that $P=x^3+3xC_{-1}$, and then, by Newtons binomial theorem we have

$$C^m=P^{m/3}=\sum_{k=0}^{\infty}\binom{m/3}{k}(3xC_{-1})^k(x^3)^{\frac{m}{3}-k}$$

Thus $\lambda^{i0}_{m+1}C^i_{-1}=(C^m)_{-2}$ is the coefficient of $x^{-2}=(x)^i(x^3)^{\frac{m}{3}-i}$, since $m=2i-2$. Then

$$\lambda^{i0}_{m+1} = \binom{m/3}{i} 3^i \neq 0$$

as desired □

Thus we have proved the following proposition.

Proposition 4.3.4. *If $(C_{-1}, C_{-2}, ..., C_{-(m+1)})$ is a solution of the system , with $C_{-1} \neq 0$ and $C_{-2} = 0$, then*

- *$m \equiv 1 mod 3$*
- *$\lambda^{i0}_{m+1} \neq 0$ for $i = \frac{m+2}{2}$*
- *There are i solutions of the system in $K[y^{1/i}]$, given by*

$C_{-1} = \left(\frac{-y}{\lambda^{i0}_{m+1}}\right)^{\frac{1}{i}}, C_{-2} = 0$ *and* $C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad for \quad 3 \leq k \leq m+1$

3. ***THIRD CASE***: *For $C_{-1} \neq 0$ and $C_{-2} \neq 0$ and m even.*

In this case we introduce a new auxiliary variable t satisfying $C^2_{-2} = tC^3_{-1}$. The equality 4.8 now reads

$$\tilde{E}_m = \sum_{2i+3j=m+1} \lambda^{ij}_m C^i_{-1} C^j_{-2} = \sum_{2i+6r+3=m+1} \lambda^{i,2r+1}_m C^i_{-1} C^{2r+1}_{-2} = \sum_{2i+6r+2=m} \lambda^{i,2r+1}_m C^{i+3r}_{-1} C_{-2} t^r$$

since m even implies that the weight $2i + 3j = m + 1$ is odd, so j is odd and can be written as $2r + 1$. Moreover, for the terms in the sum we have $i + 3r = \frac{m-2}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m-2}{2}} C_{-2} \sum_{\substack{2i+6r=m-2 \\ j=2r+1}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m-2}{6} \rfloor} a_r t^r, \quad \textit{where } a_r = \lambda_m^{\frac{m-2-6r}{2}, 2r+1}. \tag{4.13}$$

Let $\{t_1, \ldots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality 4.9 the power j has to be even, since m is even and $2i + 3j = m + 2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in 4.9, we obtain

$$\widetilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2 \\ j=2r}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r=m+2} \lambda_{m+1}^{i,2r} C_{-1}^{i+3r} t_l^r.$$

Note that for each of the terms in the last sum we have $i + 3r = \frac{m+2}{2}$, and so

$$0 = y + C_{-1}^{\frac{m+2}{2}} g(t_l), \quad \textit{where } g(t) = \sum_{r=0}^{\lfloor \frac{m+2}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m+2-6r}{2}, 2r}$. It follows that

$$C_{-1} = \left(\frac{-y}{g(t_l)} \right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.3.5. *If $(C_{-1}, C_{-2}, \ldots, C_{-(m+1)})$ is a solution of the system 4.1, with $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m even, then the system has at most $s \cdot (m+2)$ solutions, where s is the number of roots of $f(t)$ defined in 4.13. Moreover, for every choice of a root t_l of f, the solutions are given by*

$$C_{-1} = \left(\frac{-y}{g(t_l)} \right)^{\frac{2}{m+2}}, \qquad \frac{m+2}{2} \quad \textit{choices},$$

$$C_{-2} = \left(t_l C_{-1}^3 \right)^{\frac{1}{2}}, \qquad 2 \quad \textit{choices},$$

$$C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \textit{for} \quad 3 \leq k \leq m+1.$$

4. ***FOURTH CASE:*** *For $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m odd*

In this case we introduce a new auxiliary variable t satisfying $C_{-2}^2 = tC_{-1}^3$. The equality 4.8 now reads

$$\tilde{E}_m = \sum_{2i+3j=m+1} \lambda^{ij} C_{-1}^i C_{-2}^j = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^i C_{-2}^{2r} = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^{i+3r} t^r$$

since m odd implies that the weight $2i+3j = m+1$ is even, so j is even and can be written as $2r$. Moreover, for the terms in the sum we have $i + 3r = \frac{m+1}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m+1}{2}} \sum_{\substack{2i+6r=m+1 \\ j=2r}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m+1}{6} \rfloor} a_r t^r, \quad \textit{where } a_r = \lambda_m^{\frac{m+1-6r}{2},2r}. \tag{4.14}$$

Let $\{t_1, \ldots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality 4.9 the power j has to be odd, since m is odd and $2i+3j = m+2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in 4.9, we obtain

$$\widetilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2 \\ j=2r+1}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r+3=m+2} \lambda_{m+1}^{i,2r+1} C_{-1}^{i+3r} C_{-2} t_l^r.$$

Note that for each of the terms in the last sum we have $i + 3r = \frac{m-1}{2}$, and so

$$0 = y + C_{-1}^{\frac{m-1}{2}} C_{-2} g(t_l), \quad \textit{where } g(t) = \sum_{r=0}^{\lfloor \frac{m-1}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m-1-6r}{2},2r+1}$. *We also replace* C_{-2} *by* $\left(t_l C_{-1}^3\right)^{\frac{1}{2}}$. *It follows that*

$$0 = y + C_{-1}^{\frac{m+2}{2}}(t_l)^{\frac{1}{2}} g(t_l),$$

and so

$$C_{-1} = \left(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.3.6. *If* $(C_{-1}, C_{-2}, ..., C_{-(m+1)})$ *is a solution of a system 4.1, with* $C_{-1} \neq 0, C_{-2} \neq 0$ *and* m *odd, then the system has at most* $2s.(m+2)$ *solutions, where* s *is the number of roots of* $f(t)$ *defined in . Moreover, for every choice of a root* t_l *of f, the solutions are given by*

$$\begin{aligned} C_{-1} &= \left(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\right)^{\frac{2}{m+2}}, && \frac{m+2}{2} \quad \text{choices,} \\ C_{-2} &= \left(t_l C_{-1}^3\right)^{\frac{1}{2}}, && 2 \quad \text{choices,} \\ C_{-k} &= -R_{k-2}(C_{-1}, C_{-2}) \quad \text{for} \quad 3 \leq k \leq m+1. && \end{aligned}$$

Chapter 5

Application in Mathematica

The following is a Mathematica code used to determine the complete system of equations $E_1, E_2, ..., E_{m-1}, E_m, E_{m+1}$, the Groebner basis for the $m-1$ equations, $\{E_1, E_2, ..., E_{m-1}\}$ and the system of solutions of them.

Unfortunately, since the exercise of computing the Groebner basis is quite demanding, we will only be able to calculate results up to the threshold of $M = 12$. It is left for future work to generate a program that can calculate above that threshold. The following is the code that generates the results mentioned before.

```
GroebnerProgram[k_] :=
  Module[{E, EM, EM1, cVars, equations, G, S},(*For k>12,
   skip the solution  calculation *)
   If [k > 13, Return["<p>k-is-too- large , - calculation -skipped.</p>"]];
   M = k;
   CC = x;
   For[i = 1, i <= M + 1, i++,
    CC = CC + Symbol["c" <> ToString[i]]*x^(-i)];
   E = Table[0, {i, 1, M - 1}];
```

```
For[i = 1, i <= M - 1, i++, E[[i]] = Coefficient[CC^3, x, -i]];
EM = Coefficient[CC^M, x, -1];
EM1 = Coefficient[CC^M + F, x, -2];
cVars = Table[Symbol["c" <> ToString[i]], {i, M + 1, 1, -1}];
G = GroebnerBasis[E, cVars];
cVars = Table[Symbol["c" <> ToString[i]], {i, 3, M + 1}];
(*Calculate the solutions of the (E_{1},E_ {2},...,
E_{m-1}) equations*)S = Solve[E == 0, cVars];
(*Format results as HTML*)
StringJoin["<html><head><style>",
 "body { background-color: #e6f2ff; font-family: Arial, \
sans-serif; }", "h1 { color: #004080;}", "p { font-size: 16px; }",
 ".result-name { font-weight: bold; font-size: 18px; color: \
#004080; }", "</style></head><body>",
 "<h1>Program of Jacobian Conjecture for n=3</h1>",
 "<p class='result-name'>System of Equations for M-1 = ",
 ToString[k - 1], ":</p>",
 StringJoin[
  Table["<p class='result-name'>E_" <> ToString[i] <> ":</p><p>" <>
     ToString[E[[i]], InputForm] <> "</p>", {i, 1, M - 1}]],
 "<p class='result-name'>E_" <> ToString[k] <> ":</p><p>",
 ToString[EM, InputForm], "</p>",
 "<p class='result-name'>E_" <> ToString[k + 1] <> ":</p><p>",
 ToString[EM1 + F, InputForm], "</p>",
 "<p class='result-name'>Groebner Basis of E_{1}, E_{2}, ..., \
E_{m-1} for M-1 = ", ToString[k - 1], ":</p><p>",
  StringRiffle[ToString[#, InputForm] & /@ G, "<br>"], "</p>",
 "<p class='result-name'>Solutions of the system of M-1 = ",
 ToString[k - 1], ":</p><p>",
  StringRiffle[ToString[#, InputForm] & /@ S, "<br>"], "</p>",
 "</body></html>"]];
```

```

form = FormFunction[{"k" -> "Integer"}, GroebnerProgram[#k] &,
   "HTMLFragment"];
CloudDeploy[form, "GroebnerProgramWebApp"]
```

Finally, we are going to give the explicit results that are obtained from this program.

1. *For* $m = 4$*, we get the following results:*

   Firstly, we get the whole system of equations for $m-1=3$*:* $\{E_1, E_2, E_3\}$

$$E_1 = (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}$$

   and the two remaining equations $\{E_m, E_{m+1}\}$

$$E_4 = (C^4)_{-1} = 12C_{-1}C_{-2} + 4C_{-4}$$

$$E_5 = (C^4)_{-2} + y = 4C_{-1}^3 + 6C_{-2}^2 + 12C_{-1}C_{-3} + 4C_{-5} + y$$

   Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$$G = \langle C_{-1}^2 + C_{-3}, 2C_{-1}C_{-2} + C_{-4}, -5C_{-1}^3 + 3C_{-2}^2 + 3C_{-5}\rangle$$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C_{-1}^2,$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \Big(\frac{5C_{-1}^3 - 3C_{-2}^2}{3}\Big)$$

2. *For* $m = 5$*,*

   Firstly, we get the whole system of equations for $m - 1 = 4$*:* $\{E_1, E_2, E_3, E_4\}$

$$E_1 = (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}$$

$$E_4 = (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}$$

   and the two remaining equations $\{E_m, E_{m+1}\}$

$$E_5 = (C^5)_{-1} = 10C_{-1}^3 + 10C_{-2}^2 + 20C_{-1}C_{-3} + 5C_{-5}$$

$$E_6 = (C^5)_{-2} + y = 30C_{-1}^2C_{-2} + 20C_{-2}C_{-3} + 20C_{-1}C_{-4} + 5C_{-6} + y$$

Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$$G = \langle C_{-1}^2 + C_{-3}, 2C_{-1}C_{-2} + C_{-4}, -5C_{-1}^3 + 3C_{-2}^2 + 3C_{-5}, -5C_{-1}^2C_{-2} + C_{-6}\rangle$$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}, C_{-6}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C_{-1}^2,$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \left(\frac{5C_{-1}^3 - 3C_{-2}^2}{3}\right)$$

$$C_{-6} = 5C_{-1}^2C_{-2}$$

3. *For* $m = 7$*,*

   Firstly, we get the whole system of equations for $m - 1 = 6$*:* $\{E_1, E_2, E_3, E_4, E_5, E_6\}$

$$E_1 = (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}$$

$$E_4 = (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}$$

$$E_5 = (C^3)_{-5} = 3C_{-1}C^2_{-2} + 3C^2_{-1}C_{-3} + 3C^2_{-3} + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}$$

$$E_6 = (C^3)_{-6} = C^3_{-2} + 6C_{-1}C_{-2}C_{-3} + 3C^2_{-1}C_{-4} + 6C_{-3}C_{-4} + 6C_{-2}C_{-5} + 6C_{-1}C_{-6} + 3C_{-8}$$

and the two remaining equations $\{E_m, E_{m+1}\}$

$$E_7 = (C^7)_{-1} = 35C^4_{-1} + 105C_{-1}C^2_{-2} + 105C^2_{-1}C_{-3} + 21C^2_{-3} + 42C_{-2}C_{-4} + 42C_{-1}C_{-5} + 7C_{-7}$$

$E_8 = (C^7)_{-2} + y = 140C^3_{-1}C_{-2} + 35C^3_{-2} + 210C_{-1}C_{-2}C_{-3} + 105C^2_{-1}C_{-4} + 42C_{-3}C_{-4} + 42C_{-2}C_{-5} + 42C_{-1}C_{-6} + 7C_{-8} + y$

Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$G = \langle C^2_{-1}+C_{-3}, 2C_{-1}C_{-2}+C_{-4}, -5C^3_{-1}+3C^2_{-2}+3C_{-5}, -5C^2_{-1}C_{-2}+C_{-6}, 10C^4_{-1}-15C_{-1}C^2_{-2}+3C_{-7}, 40C^3_{-1}C_{-2} - 5C^3_{-2} + 3C_{-8}\rangle$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}, C_{-6}, C_{-7}, C_{-8}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C^2_{-1},$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \Big(\frac{5C^3_{-1} - 3C^2_{-2}}{3}\Big)$$

$$C_{-6} = 5C^2_{-1}C_{-2}$$

$$C_{-7} = -5\Big(\frac{2C^4_{-1} - 3C_{-1}C^2_{-2}}{3}\Big)$$

$$C_{-8} = -5\Big(\frac{8C^3_{-1}C_{-2} - C^3_{-2}}{3}\Big)$$

4. *For* $m = 8$*,*

Firstly, we get the whole system of equations for $m-1 = 7$*:* $\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$

$$E_1 = (C^3)_{-1} = 3C^2_{-1} + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C^3_{-1} + 3C^2_{-2} + 6C_{-1}C_{-3} + 3C_{-5}$$

$$E_4 = (C^3)_{-4} = 3C^2_{-1}C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}$$

$$E_5 = (C^3)_{-5} = 3C_{-1}C^2_{-2} + 3C^2_{-1}C_{-3} + 3C^2_{-3} + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}$$

$$E_6 = (C^3)_{-6} = C^3_{-2} + 6C_{-1}C_{-2}C_{-3} + 3C^2_{-1}C_{-4} + 6C_{-3}C_{-4} + 6C_{-2}C_{-5} + 6C_{-1}C_{-6} + 3C_{-8}$$

$$E_7 = (C^3)_{-7} = 3C^2_{-2}C_{-3} + 3C_{-1}C^2_{-3} + 6C_{-1}C_{-2}C_{-4} + 3C^2_{-4} + 3C^2_{-1}C_{-5} + 6C_{-3}C_{-5} + 6C_{-2}C_{-6} + 6C_{-1}C_{-7} + 3C_{-9}$$

and the two remaining equations $\{E_m, E_{m+1}\}$

$E_8 = (C^8)_{-1} = 280C^3_{-1}C_{-2} + 56C^3_{-2} + 336C_{-1}C_{-2}C_{-3} + 168C^2_{-1}C_{-4} + 56C_{-3}C_{-4} + 56C_{-2}C_{-5} + 56C_{-1}C_{-6} + 8C_{-8}$

$E_9 = (C^8)_{-2} + y = 56C^5_{-1} + 420C^2_{-1}C^2_{-2} + 280C^3_{-1}C_{-3} + 168C^2_{-2}C_{-3} + 168C_{-1}C^2_{-3} + 336C_{-1}C_{-2}C_{-4} + 28C^2_{-4} + 168C^2_{-1}C_{-5} + 56C_{-3}C_{-5} + 56C_{-2}C_{-6} + 56C_{-1}C_{-7} + 8C_{-9} + y$

Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$G = \langle C^2_{-1} + C_{-3}, 2C_{-1}C_{-2} + C_{-4}, -5C^3_{-1} + 3C^2_{-2} + 3C_{-5}, -5C^2_{-1}C_{-2} + C_{-6}, 10C^4_{-1} - 15C_{-1}C^2_{-2} + 3C_{-7}, 40C^3_{-1}C_{-2} - 5C^3_{-2} + 3C_{-8}, -22C^5_{-1} + 60C^2_{-1}C^2_{-2} + 3C_{-9}\rangle$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}, C_{-6}, C_{-7}, C_{-8}, C_{-9}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C^2_{-1},$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \Big(\frac{5C^3_{-1} - 3C^2_{-2}}{3}\Big)$$

$$C_{-6} = 5C^2_{-1}C_{-2}$$

$$C_{-7} = -5\Big(\frac{2C^4_{-1} - 3C_{-1}C^2_{-2}}{3}\Big)$$

$$C_{-8} = -5\Big(\frac{8C^3_{-1}C_{-2} - C^3_{-2}}{3}\Big)$$

$$C_{-9} = 2\Big(\frac{11C^5_{-1} - 30C^2_{-1}C^2_{-2}}{3}\Big)$$

5. *For* $m = 10$*,*

Firstly, we get the whole system of equations for $m - 1 = 9$*:*

$\{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9\}$

$$E_1 = (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}$$

$$E_4 = (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}$$

$$E_5 = (C^3)_{-5} = 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}$$

$E_6 = (C^3)_{-6} = C_{-2}^3 + 6C_{-1}C_{-2}C_{-3} + 3C_{-1}^2C_{-4} + 6C_{-3}C_{-4} + 6C_{-2}C_{-5} + 6C_{-1}C_{-6} + 3C_{-8}$

$E_7 = (C^3)_{-7} = 3C_{-2}^2C_{-3} + 3C_{-1}C_{-3}^2 + 6C_{-1}C_{-2}C_{-4} + 3C_{-4}^2 + 3C_{-1}^2C_{-5} + 6C_{-3}C_{-5} + 6C_{-2}C_{-6} + 6C_{-1}C_{-7} + 3C_{-9}$

$E_8 = (C^3)_{-8} = 3C_{-10} + 3C_{-2}C_{-3}^2 + 3C_{-2}^2C_{-4} + 6C_{-1}C_{-3}C_{-4} + 6C_{-1}C_{-2}C_{-5} + 6C_{-4}C_{-5} + 3C_{-1}^2C_{-6} + 6C_{-3}C_{-6} + 6C_{-2}C_{-7} + 6C_{-1}C_{-8}$

$E_9 = (C^3)_{-9} = 3C_{-11} + C_{-3}^3 + 6C_{-2}C_{-3}C_{-4} + 3C_{-1}C_{-4}^2 + 3C_{-2}^2C_{-5} + 6C_{-1}C_{-3}C_{-5} + 3C_{-5}^2 + 6C_{-1}C_{-2}C_{-6} + 6C_{-4}C_{-6} + 3C_{-1}^2C_{-7} + 6C_{-3}C_{-7} + 6C_{-2}C_{-8} + 6C_{-1}C_{-9}$

and the two remaining equations $\{E_m, E_{m+1}\}$

$E_{10} = (C^{10})_{-1} = 10C_{-10} + 1260C^4_{-1}C_{-2} + 840C_{-1}C^3_{-2} + 2520C^2_{-1}C_{-2}C_{-3} + 360C_{-2}C^2_{-3} + 840C^3_{-1}C_{-4} + 360C^2_{-2}C_{-4} + 720C_{-1}C_{-3}C_{-4} + 720C_{-1}C_{-2}C_{-5} + 90C_{-4}C_{-5} + 360C^2_{-1}C_{-6} + 90C_{-3}C_{-6} + 90C_{-2}C_{-7} + 90C_{-1}C_{-8}$

$E_{11} = (C^{10})_{-2}+y = 210C^6_{-1}+10C_{-11}+2520C^3_{-1}C^2_{-2}+210C^4_{-2}+1260C^4_{-1}C_{-3}+2520C_{-1}C^2_{-2}C_{-3}+ 1260C^2_{-1}C^2_{-3} + 120C^3_{-3} + 2520C^2_{-1}C_{-2}C_{-4} + 720C_{-2}C_{-3}C_{-4} + 360C_{-1}C^2_{-4} + 840C^3_{-1}C_{-5} + 360C^2_{-2}C_{-5}+720C_{-1}C_{-3}C_{-5}+45C^2_{-5}+720C_{-1}C_{-2}C_{-6}+90C_{-4}C_{-6}+360C^2_{-1}C_{-7}+90C_{-3}C_{-7}+ 90C_{-2}C_{-8} + 90C_{-1}C_{-9} + y$

Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$$G = \langle C^2_{-1} + C_{-3}, 2C_{-1}C_{-2} + C_{-4}, -5C^3_{-1} + 3C^2_{-2} + 3C_{-5}, -5C^2_{-1}C_{-2} + C_{-6}, 10C^4_{-1} - 15C_{-1}C^2_{-2} + 3C_{-7}, 40C^3_{-1}C_{-2} - 5C^3_{-2} + 3C_{-8}, -22C^5_{-1} + 60C^2_{-1}C^2_{-2} + 3C_{-9}, 3C_{-10} - 110C^4_{-1}C_{-2} + 40C_{-1}C^3_{-2}, 154C^6_{-1} + 9C_{-11} - 660C^3_{-1}C^2_{-2} + 30C^4_{-2}\rangle$$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}, C_{-6}, C_{-7}, C_{-8}, C_{-9}, C_{-10}, C_{-11}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C^2_{-1},$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \left(\frac{5C^3_{-1} - 3C^2_{-2}}{3}\right)$$

$$C_{-6} = 5C^2_{-1}C_{-2}$$

$$C_{-7} = -5\Big(\frac{2C_{-1}^4 - 3C_{-1}C_{-2}^2}{3}\Big)$$

$$C_{-8} = -5\Big(\frac{8C_{-1}^3 C_{-2} - C_{-2}^3}{3}\Big)$$

$$C_{-9} = 2\Big(\frac{11C_{-1}^5 - 30C_{-1}^2 C_{-2}^2}{3}\Big)$$

$$C_{-10} = 10\Big(\frac{11C_{-1}^4 C_{-2} - 4C_{-1}C_{-2}^3}{3}\Big),$$

$$C_{-11} = -2\Big(\frac{77C_{-1}^6 - 330C_{-1}^3 C_{-2}^2 + 15C_{-2}^4}{9}\Big)$$

6. *For* $m = 11$*,*

Firstly, we get the whole system of equations for $m - 1 = 10$*:*

$\{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}\}$

$$E_1 = (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3}$$

$$E_2 = (C^3)_{-2} = 6C_{-1}C_{-2} + 3C_{-4}$$

$$E_3 = (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}$$

$$E_4 = (C^3)_{-4} = 3C_{-1}^2 C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}$$

$$E_5 = (C^3)_{-5} = 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}$$

$E_6 = (C^3)_{-6} = C_{-2}^3 + 6C_{-1}C_{-2}C_{-3} + 3C_{-1}^2C_{-4} + 6C_{-3}C_{-4} + 6C_{-2}C_{-5} + 6C_{-1}C_{-6} + 3C_{-8}$

$E_7 = (C^3)_{-7} = 3C_{-2}^2C_{-3} + 3C_{-1}C_{-3}^2 + 6C_{-1}C_{-2}C_{-4} + 3C_{-4}^2 + 3C_{-1}^2C_{-5} + 6C_{-3}C_{-5} + 6C_{-2}C_{-6} + 6C_{-1}C_{-7} + 3C_{-9}$

$E_8 = (C^3)_{-8} = 3C_{-10} + 3C_{-2}C_{-3}^2 + 3C_{-2}^2C_{-4} + 6C_{-1}C_{-3}C_{-4} + 6C_{-1}C_{-2}C_{-5} + 6C_{-4}C_{-5} + 3C_{-1}^2C_{-6} + 6C_{-3}C_{-6} + 6C_{-2}C_{-7} + 6C_{-1}C_{-8}$

$E_9 = (C^3)_{-9} = 3C_{-11} + C_{-3}^3 + 6C_{-2}C_{-3}C_{-4} + 3C_{-1}C_{-4}^2 + 3C_{-2}^2C_{-5} + 6C_{-1}C_{-3}C_{-5} + 3C_{-5}^2 + 6C_{-1}C_{-2}C_{-6} + 6C_{-4}C_{-6} + 3C_{-1}^2C_{-7} + 6C_{-3}C_{-7} + 6C_{-2}C_{-8} + 6C_{-1}C_{-9}$

$E_{10} = (C^3)_{-10} = 6C_{-1}C_{-10} + 3C_{-12} + 3C_{-3}^2C_{-4} + 3C_{-2}C_{-4}^2 + 6C_{-2}C_{-3}C_{-5} + 6C_{-1}C_{-4}C_{-5} + 3C_{-2}^2C_{-6} + 6C_{-1}C_{-3}C_{-6} + 6C_{-5}C_{-6} + 6C_{-1}C_{-2}C_{-7} + 6C_{-4}C_{-7} + 3C_{-1}^2C_{-8} + 6C_{-3}C_{-8} + 6C_{-2}C_{-9}$

and the two remaining equations $\{E_m, E_{m+1}\}$

$$E_{11} = (C^{11})_{-1} = 462C_{-1}^6 + 11C_{-11} + 4620C_{-1}^3C_{-2}^2 + 330C_{-2}^4 + 2310C_{-1}^4C_{-3} + 3960C_{-1}C_{-2}^2C_{-3} + 1980C_{-1}^2C_{-3}^2 + 165C_{-3}^3 + 3960C_{-1}^2C_{-2}C_{-4} + 990C_{-2}C_{-3}C_{-4} + 495C_{-1}C_{-4}^2 + 1320C_{-1}^3C_{-5} + 495C_{-2}^2C_{-5} + 990C_{-1}C_{-3}C_{-5} + 55C_{-5}^2 + 990C_{-1}C_{-2}C_{-6} + 110C_{-4}C_{-6} + 495C_{-1}^2C_{-7} + 110C_{-3}C_{-7} + 110C_{-2}C_{-8} + 110C_{-1}C_{-9}$$

$$E_{12} = (C^{11})_{-2} + y = 110C_{-1}C_{-10} + 11C_{-12} + 2772C_{-1}^5C_{-2} + 4620C_{-1}^2C_{-2}^3 + 9240C_{-1}^3C_{-2}C_{-3} + 1320C_{-2}^3C_{-3} + 3960C_{-1}C_{-2}C_{-3}^2 + 2310C_{-1}^4C_{-4} + 3960C_{-1}C_{-2}^2C_{-4} + 3960C_{-1}^2C_{-3}C_{-4} + 495C_{-3}^2C_{-4} + 495C_{-2}C_{-4}^2 + 3960C_{-1}^2C_{-2}C_{-5} + 990C_{-2}C_{-3}C_{-5} + 990C_{-1}C_{-4}C_{-5} + 1320C_{-1}^3C_{-6} + 495C_{-2}^2C_{-6} + 990C_{-1}C_{-3}C_{-6} + 110C_{-5}C_{-6} + 990C_{-1}C_{-2}C_{-7} + 110C_{-4}C_{-7} + 495C_{-1}^2C_{-8} + 110C_{-3}C_{-8} + 110C_{-2}C_{-9} + y$$

Next, we compute the Groebner basis, using the lexicographic order for the monomial ordering:

$$G =$$
$$\langle C_{-1}^{2} + C_{-3}, 2C_{-1}C_{-2} + C_{-4}, -5C_{-1}^{3} + 3C_{-2}^{2} + 3C_{-5}, -5C_{-1}^{2}C_{-2} + C_{-6}, 10C_{-1}^{4} - 15C_{-1}C_{-2}^{2} + 3C_{-7}, 40C_{-1}^{3}C_{-2} - 5C_{-2}^{3} + 3C_{-8}, -22C_{-1}^{5} + 60C_{-1}^{2}C_{-2}^{2} + 3C_{-9}, 3C_{-10} - 110C_{-1}^{4}C_{-2} + 40C_{-1}C_{-2}^{3}, 154C_{-1}^{6} + 9C_{-11} - 660C_{-1}^{3}C_{-2}^{2} + 30C_{-2}^{4}, 3C_{-12} + 308C_{-1}^{5}C_{-2} - 220C_{-1}^{2}C_{-2}^{3}\rangle$$

And the results obtained for $\{C_{-3}, C_{-4}, C_{-5}, C_{-6}, C_{-7}, C_{-8}, C_{-9}, C_{-10}, C_{-11}, C_{-12}\}$ *as variables of* C_{-1}, C_{-2}*:*

$$C_{-3} = -C_{-1}^{2},$$

$$C_{-4} = -2C_{-1}C_{-2},$$

$$C_{-5} = \Big(\frac{5C_{-1}^{3} - 3C_{-2}^{2}}{3}\Big)$$

$$C_{-6} = 5C_{-1}^{2}C_{-2}$$

$$C_{-7} = -5\Big(\frac{2C_{-1}^{4} - 3C_{-1}C_{-2}^{2}}{3}\Big)$$

$$C_{-8} = -5\Big(\frac{8C_{-1}^{3}C_{-2} - C_{-2}^{3}}{3}\Big)$$

$$C_{-9} = 2\Big(\frac{11C_{-1}^{5} - 30C_{-1}^{2}C_{-2}^{2}}{3}\Big)$$

$$C_{-10} = 10\Big(\frac{11C^4_{-1}C_{-2} - 4C_{-1}C^3_{-2}}{3}\Big),$$

$$C_{-11} = -2\Big(\frac{77C^6_{-1} - 330C^3_{-1}C^2_{-2} + 15C^4_{-2}}{9}\Big)$$

$$C_{-12} = -44\Big(\frac{7C^5_{-1}C_{-2} - 5C^2_{-1}C^3_{-2}}{3}\Big)$$

Conclusions

In this dissertation, we calculate the system of equations related to the Jacobian Conjecture, the corresponding Groebner Basis, and the corresponding system of solutions. In the first four chapters, we discuss the main theoretical concepts related to computational algebra. For the practical application, our main source was [GGV24] and [VS14].

The first one, explains with more technicality the origin of the system of polynomials related to the Jacobian Conjecture, which are not explained in this dissertation. The second one, explains the same calculations being made in this dissertation for the case $n = 2$*. In summary, in the present dissertation, we were able to calculate the system of equations and also the reduced form of the system of equations. Additionally, we implicitly calculated the Groebner basis for the* $m - 1$ *equations and proved it was one. This was enough to find the system of solutions related to them.*

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