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**Obstrucción cohomológica para extensión de deformaciones
de álgebras asociativas**

Tesis para optar el grado académico de Maestro en Matemáticas que presenta:

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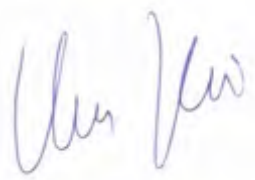
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Resumen

En el estudio de la teoría de deformaciones se observa que hay por lo menos tres tipos distintos, estos tipos aparecen en análisis, álgebra y geometría algebraica. La teoría de deformaciones es una idea que proviene desde Riemann con el estudio de las deformaciones de estructuras complejas de variedades Riemannianas. Por otro lado, las deformaciones en el área de la geometría algebraica datan casi desde la aparición de esta área, ya que los objetos algebro-geométricos pueden ser “deformados” con una variación de los coeficientes de sus ecuaciones de definición.

En el estudio de la teoría de deformaciones formales de álgebras aparecen algunas preguntas que aún se encuentran abiertas. Es en el caso particular de álgebras asociativas donde aparece un problema, no resuelto en general. Para explicar de que trata este problema debemos partir de la definición de deformación de un álgebra asociativa. Es a partir de la condición de asociatividad, donde se observa que el “infinitesimal” de una deformación es un cociclo de Hochschild. Se plantea entonces la pregunta “¿Dado un cociclo de Hochschild, resulta ser este cociclo el “infinitesimal” de una deformación?”.

Desglosaremos el problema en una construcción recursiva de deformaciones truncadas. La obstrucción a extender una deformación truncada de grado n a una de grado $n+1$ es un cociclo de Hochschild. Este resultado que es uno de los resultados principales en la teoría de deformaciones, se probará en la Proposición 10. 2. Para ello empleamos la teoría de álgebras graduadas y conceptos como anillos de Lie y pre-Lie graduados así como sistemas pre-Lie. En el desarrollo de este trabajo se mostrará, además del resultado, la manera de trabajar con distintos conceptos y como trabajar con operadores que aparecerán a lo largo del desarrollo.

Abstract

In the study of the theory of deformations it is observed that there are at least three different types, these types appear in analysis, algebra and algebraic geometry. The theory of deformations is an idea that comes from Riemann with the study of deformations of complex structures of Riemannian varieties. On the other hand, deformations in the area of algebraic geometry date almost from the appearance of this area since algebro-geometric objects can be "deformed" with a variation of the coefficients of their defining equations.

In the study of the theory of formal deformations of algebras, some questions remain open. In the particular case of associative algebras a problem appears that is not solved in general. To explain what this problem is about, we must start from the definition of deformation of an associative algebra. Considering the condition of associativity, where it is observed that the "infinitesimal" of a deformation is a Hochschild cocycle, the question arises "Given a Hochschild cocycle, does this cocycle happens to be the "infinitesimal" of a deformation?".

One can decompose the problem into a recursive construction of truncated deformations. The obstruction to extending a truncated deformation of degree n to a truncated deformation of degree $n+1$, is a Hochschild cocycle. This result, which is one of the most important results in deformation theory, is proven in Prop.10.2. For this purpose, the theory of graded algebras and concepts such as graded Lie and pre-Lie rings, pre-Lie systems will be used. In the development of this work we will also show, the ways of working with these different concepts and how to work with operators that will appear throughout the development.

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Introduction

Throughout this thesis K is going to be a commutative ring with 1. The objects will be K – modules and the tensor product will always be over K . Other objects that we will use are the K – algebras. A K – algebra A is a K – module together with a map $\mu_A : A \otimes_K A \longrightarrow A$ called multiplication, which is associative. This means that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes A} & A \otimes A \\ A \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

is commutative, where μ means μ_A and we write A for the map Id_A , in other words $\mu \otimes A$ means $\mu_A \otimes Id_A$. On elementary tensors we have

$$\mu(\mu \otimes A)(a \otimes b \otimes c) = \mu(A \otimes \mu)(a \otimes b \otimes c)$$

that is $(ab)c = a(bc)$.

A complex is a family of K – modules $C = \{C^n\}_{n \in \mathbb{Z}}$ and a family of K – linear maps $\rho^n : C^n \longrightarrow C^{n+1}$ such that $\rho^n \circ \rho^{n+1} = 0$.

$$C : 0 \longrightarrow C^0 \xrightarrow{\rho^0} C^1 \xrightarrow{\rho^1} C^2 \xrightarrow{\rho^2} C^3 \dots$$

The condition $\rho^2 = 0$ implies that $Im(\rho^{n-1}) \subseteq Ker(\rho^n)$. The quotient obtained from these modules,

$$H^n(C) \cong \frac{Ker \rho^n}{Im \rho^{n-1}}$$

is called the cohomology of the complex. The Hochschild co-complex of A (with coefficient in A) is defined in the following way.

$$C^0(A) = \{f : K \longrightarrow A, K\text{-linear}\} \approx A$$

$$C^1(A) = \{f : A \longrightarrow A, K\text{-linear}\}$$

$$C^2(A) = \{f : A \otimes A \longrightarrow A, K\text{-linear}\}$$

$$C^3(A) = \{f : A \otimes A \otimes A \longrightarrow A, K\text{-linear}\}$$

⋮

$$C^n(A) = \{f : A^{\otimes n} \longrightarrow A, K\text{-linear}\}$$

Now we are able to define the differential maps.

$$\begin{aligned} \rho^n : C^n(A) &\rightarrow C^{n+1}(A) \\ f^n &\mapsto \rho^n f^n(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) \end{aligned}$$

where

$$\begin{aligned} \rho^n f^n(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) &= a_1 f^n(a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f^n(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} f^n(a_1 \otimes a_2 \otimes \cdots \otimes a_n) a_{n+1} \end{aligned}$$

We define $Z^n(A, A) = \text{Ker} \rho^n$ and $B^n(A, A) = \text{Im} \rho^{n-1}$, for $n \geq 1$. and $B^0(A, A) = 0$. One verifies that $\rho^n \circ \rho^{n+1} = 0$, which means that $B^n(A, A) \subset Z^n(A, A)$. We define.

$$HH^n(A) = \frac{\text{Ker} \rho^n}{\text{Im} \rho^{n-1}} = \frac{Z^n(A, A)}{B^n(A, A)}$$

We will now briefly mention some concepts that we will develop in the following chapters. We are presenting them here in order to state the problem that we want to solve.

Consider the algebra of formal power series

$$K[[t]] = \{\lambda_0 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 + \dots, \lambda_i \in K\}$$

This algebra shares several properties with the algebra of polynomials contained in it. However, a formal power series does not generally determine a function because evaluating it at any value implies an infinite series of summands that does not necessarily converge.

Definition 0.1 A deformation of an algebra is a family of K -bilinear applications $\{F_n : A \times A \rightarrow A\}_{n=1,2,\dots}$ satisfying

$$\sum_{i+j=n} F_i(a, F_j(b, c)) = \sum_{i+j=n} F_i(F_j(a, b), c) \quad (0.1)$$

and $F_0(a, b) = ab$.

Let the K -module $A_t = A[[t]]$ of the formal power series in A with elements of the form

$$a_0 + a_1 t + a_2 t^2 + \dots$$

Given a deformation $\{F_n\}_{n \in \mathbb{N}}$ we define a $K[[t]]$ – bilinear application, $F : A_t \otimes A_t \rightarrow A_t$ by setting

$$F(a, b) = ab + F_1(a, b)t + F_2(a, b)t^2 + F_3(a, b)t^3 + \dots,$$

Then (A_t, F) is an associative algebra as we will see in Proposition (1.1).

We call F_1 the infinitesimal of the deformation. By the definition that we have just given, we have $F_1 \in \text{Ker}(\delta^1)$, which means that in the context of Hochschild cohomology, F_1 is a cocycle called the *infinitesimal* of the deformation. One of the main problems of the deformation theory, not solved in general, is the question of when a cocycle of Hochschild is the infinitesimal of a deformation. The following problem is more accessible and its solution answers partially the first question. If F_1 is a Hochschild cocycle, that satisfies the definition of deformation that we gave for $n = 1$, under wich circumstances there exists an F_2 such that the definition of deformation is satisfied for $n = 2$?

The following theorem responds a more general question.

Theorem 0.1 *Let $F_1, \dots, F_{k-1} : A \otimes A \rightarrow A$ be K – bilinear applications that satisfy the definition of deformation, for $n = 1, \dots, k - 1$. Defining $G : A \otimes A \otimes A \rightarrow A$ by*

$$G(a, b, c) = \sum_{\substack{i+j=k \\ i, j > 0}} F_i(a, F_j(b, c)) - F_i(F_j(a, b), c),$$

we obtain a cocycle of Hochschild, that is $G \in \text{Ker}(\delta^3)$. It defines a cohomology class $[G] \in HH^3(A)$. Furthermore $[G] = 0$ if and only if there exists F_k such that F_1, \dots, F_k satisfy the definition of deformation for $n = k$.

Proof. The proof of the first part of this theorem is the main content of the present thesis.

From the expression $F_\lambda \circ F_\mu = \sum_{i=0}^{\lambda} (-1)^i F_\lambda \circ_i F_\mu = F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c))$, we see that deformations of associative rings turn out to be pre-Lie rings associated with a pre-Lie system. To arrive at this conclusion, we used Theorems 4.1 and 8.1, along with examples 4 and 5 from Chapter 7. It's important to note that Theorems 4.1 and 8.1 allow us to transition from a graded pre-Lie ring to a graded Lie ring and from a pre-Lie system to a pre-Lie ring, respectively. Following Theorem 9.1, we can observe how differentiation behaves on a pre-Lie ring. This theorem, combined with the earlier equality, is used to prove Proposition 10.1. Essentially, what we want to prove is that given a sequence of elements F_1, \dots, F_{m-1} in $C^2(A, A)$ satisfying $\sum_{\lambda+\mu=\nu} F_\lambda \circ F_\mu = \delta F_\nu$ for $\nu = 1, \dots, m - 1$ the element G defined by $G = \sum_{\lambda+\mu=m} F_\lambda \circ F_\mu$ is a cocycle. For the second statement assume first that there exist F_k such that F_1, \dots, F_k satisfy the definition of deformation for $n = k$. Then

$$\begin{aligned}
0 &= \sum_{i+j=k} F_i(a, F_j(b, c)) - F_i(F_j(a, b), c) \\
&= \sum_{\substack{i+j=k \\ i, j > 0}} F_i(a, F_j(b, c)) - F_i(F_j(a, b), c) + F_0(a, F_k(b, c)) - F_0(F_k(a, b), c) \\
&\quad + F_k(a, F_0(b, c)) - F_k(F_0(a, b), c) \\
&= G + aF_k(b, c) - F_k(a, b)c + F_k(a, bc) - F_k(ab, c) \\
&= G + aF_k(b, c) - F_k(ab, c) + F_k(a, bc) - F_k(a, b)c \\
&= G + \delta^2(F_k)(a, b, c)
\end{aligned}$$

Then $G = -\delta^2(F_k) \in \text{Im}(\delta^2)$ and so $[G] = 0$.

On the other hand if $[G] = 0 \in HH^3(A)$, then $G \in \text{Im}(\delta^2)$, and we have in this case $G = \delta^2(g)$ for some $g : A \otimes A \rightarrow A$. We simply set $F_k = -g$, and we obtain the definition of *deformation* of an algebra, for $n = k$.

Corollary 0.2 *If $HH^3(A) = 0$, then every element $F_1 \in \text{Ker}(\delta^3)$ is the infinitesimal of a deformation.*

The interpretation of $HH^3(A)$ in this context is that some classes are obstructions to continue the inductive construction of a deformation.

The following results will be presented in the following chapters.

Proposition 0.3 *Let F and G be deformations given by (F_i) and (G_i) respectively. If $\phi : A_t \rightarrow A_t$ is an $K[[t]]$ -linear isomorphism (given by $\phi_j : A \rightarrow A$) which is an algebra isomorphism from (A_t, F) to (A_t, G) , then the infinitesimals of F and G are cohomologous. That is $[F_1] = [G_1] \in HH^2(A)$, moreover $F_1 = G_1 + \delta^1(\phi_1)$.*

The reciprocal result is not valid, since there are examples of deformations such that $[F_1] = [G_1]$, but $(A_t, F) \not\approx (A_t, G)$. If a deformation G is isomorphic to the trivial deformation F ($F_i = 0, i > 1$) then it is also called trivial.

In the theorem that we have presented, we proved the second part, that is, $[G] = 0$ if and only if there exists F_k such that F_1, \dots, F_k satisfy the definition of deformation for $n = k$. However the first part has not yet been proven. In this thesis we will prove this part.

Chapter 1

Infinitesimal Deformation of an Algebra

In the *Introduction* we have defined the formal power series $K[[t]]$, now let's see the K -module $A_t = A[[t]]$ of the formal power series in A with elements of the form

$$a_0 + a_1t + a_2t^2 + \dots$$

We can see the elements of $A \otimes K[[t]]$ naturally included in $A[[t]]$, via $a \otimes t^n \mapsto at^n$. Therefore $A[[t]]$ is the t -adic completion of the tensor product $A \otimes K[[t]]$ and we are going to write indistinctly

$$a_0 + a_1t + a_2t^2 + \dots = a_0 \otimes 1 + a_1 \otimes t + a_2 \otimes t^2 + \dots$$

Moreover $A[[t]]$ has a natural structure of a right $K[[t]]$ -module, given by $(a \otimes t^i)t^k = a \otimes t^{i+k}$. Our principal interest is the algebra structure over A_t , that is, K -linear maps

$$F : A_t \otimes A_t \rightarrow A_t$$

that are associative and also satisfy $F(a \otimes 1, 1 \otimes t^n) = a \otimes t^n$. The canonical structure given by

$$F(a \otimes t^i, b \otimes t^j) = ab \otimes t^{i+j}$$

in the elementary tensors and linearly extended to all $A_t \otimes A_t$, is evidently associative. Furthermore, the following properties are satisfied for the canonical structure.

A. The inclusion $K[[t]] \hookrightarrow A_t$ given by $t^j \mapsto 1 \otimes t^j$, is a morphism of algebras, this condition is equivalent to $(1 \otimes t^i)(1 \otimes t^j) = 1 \otimes t^{i+j}$.

B. The canonical projection $\pi_A : A_t \rightarrow A$ given by

$$\pi_A : (a_0 \otimes 1 + a_1 \otimes t + a_2 \otimes t^2 + \dots) \mapsto a_0$$

is an algebra morphism.

C. The map F is $K[[t]]$ – *bilinear*. This property means that

$$F((a \otimes t^i)t^k, (b \otimes t^j)t^l) = F(a \otimes t^i, b \otimes t^j)t^{k+l}$$

One can check that (C) implies (A).

D. The canonical inclusion $A \hookrightarrow A_t$, given by $a \mapsto a \otimes 1$, is a morphism of algebras.

We give another definition for deformations and we will see that it is equivalent to Definition (0.1).

Definition 1.1 *A deformation of A is a structure of associative algebra over A_t , such that the properties (A),(B) and (C) are satisfied.*

Note that by (C), if $F : A_t \otimes A_t \rightarrow A_t$ is a deformation of A , then

$$F(a_0 \otimes 1 + a_1 \otimes t + a_2 \otimes t^2 + \dots, b_0 \otimes 1 + b_1 \otimes t + b_2 \otimes t^2 + \dots) = \sum_{i,j} F(a_i t^i, b_j t^j) \quad (1.1)$$

$$= \sum_{i,j} F(a_i, b_j) t^{i+j} \quad (1.2)$$

and so it is enough to know the restriction $F|_{A \otimes A}$ in order to completely characterize F (we identify A with $A \otimes Id \subset A[[t]]$). Hence F is characterized by a family of K -bilinear maps $(F_n : A \times A \rightarrow A)_{n=0,1,2,\dots}$ such that

$$F(a, b) = F_0(a, b) + F_1(a, b)t + F_2(a, b)t^2 + F_3(a, b)t^3 + \dots \quad (1.3)$$

By the property (B), $F_0(a, b) = ab$.

On the other hand, given a family of K -bilinear maps $\{F_n : A \times A \rightarrow A\}_{n=0,1,2,\dots}$, such that $F_0(a, b) = ab$, we can define a K -bilinear map $F : A_t \otimes A_t \rightarrow A_t$ using (1.2) and (1.3).

Moreover, F is associative if and only if

$$F(F(a, b), c) = F(a, F(b, c)) \quad (1.4)$$

for all $a, b, c \in A$.

From the associativity of F we obtain.

$$\begin{aligned} F(F(a, b), c) &= F(a, F(b, c)) \\ \sum_{\lambda=0}^{\infty} F_{\lambda} \left(\sum_{\mu=0}^{\infty} F_{\mu}(a, b) t^{\mu}, c \right) t^{\lambda} &= \sum_{\lambda=0}^{\infty} F_{\lambda} \left(a, \sum_{\mu=0}^{\infty} F_{\mu}(b, c) t^{\mu} \right) t^{\lambda} \\ \sum_{\lambda=0}^{\infty} F_{\lambda} \left(\sum_{\mu=0}^{\infty} F_{\mu}(a, b) t^{\mu}, c \right) t^{\lambda+\mu} &= \sum_{\lambda=0}^{\infty} F_{\lambda} \left(a, \sum_{\mu=0}^{\infty} F_{\mu}(b, c) t^{\mu} \right) t^{\lambda+\mu} \end{aligned}$$

$$\sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} F_{\lambda}(F_{\mu}(a, b), c) t^{\lambda+\mu} = \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} F_{\lambda}(a, F_{\mu}(b, c)) t^{\lambda+\mu}$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} [F_{\lambda}(F_{\mu}(a, b), c) - F_{\lambda}(a, F_{\mu}(b, c))] t^{\lambda+\mu} = 0$$

$$\Rightarrow \sum_{\nu=0}^{\infty} \sum_{\mu+\lambda=\nu}^{\infty} [F_{\lambda}(F_{\mu}(a, b), c) - F_{\lambda}(a, F_{\mu}(b, c))] t^{\lambda+\mu} = 0$$

Therefore the condition (1.2) is equivalent to having for all a, b, c in V and all $\nu = 0, 1, 2, \dots$

$$\sum_{\substack{\mu+\lambda=\nu \\ \lambda, \mu \geq 0}}^{\infty} [F_{\lambda}(F_{\mu}(a, b), c) - F_{\lambda}(a, F_{\mu}(b, c))] = 0. \quad (1.5)$$

Thus definition (1.1) of a deformation is equivalent to the first definition we gave.

Proposition 1.1 *The expression (1.5) is satisfied if and only if (A_t, F) is an associative algebra.*

Therefore Definition 0.1 is equivalent to Definition 1.1.

We will see what equality (0.1) means for small n .

For $n = 0$, this is the associativity of the original multiplication. In fact

$$\begin{aligned} 0 &= \sum_{i+j=0} (F_i(a, F_j(b, c)) - F_i(F_j(a, b), c)) \\ &= F_0(a, F_0(b, c)) - F_0(F_0(a, b), c) \\ &= a(bc) - (ab)c \end{aligned}$$

For $n = 1$, the condition (1.5) may be expressed in the form.

$$\begin{aligned}
0 &= \sum_{i+j=1} (F_i(a, F_j(b, c)) - F_i(F_j(a, b), c)) \\
&= F_0(a, F_1(b, c)) - F_0(F_1(a, b), c) + F_1(a, F_0(b, c)) - F_1(F_0(a, b), c) \\
&= aF_1(b, c) - F_1(a, b)c + F_1(a, bc) - F_1(ab, c) \\
&= \delta^2(F_1)(a, b, c)
\end{aligned}$$

i.e. in terms of the Hochschild theory, F_1 is an element of $Z^2(A, A)$ i.e., a two co-cycles, of A with coefficients in A . That means, that for all deformations $F_1 \in \text{Ker}(\delta^1)$ that is, F_1 is a co cycle of Hochschild called the *infinitesimal* of a deformation.

1.1 Lie Algebras

Suppose now that A is assumed to be a Lie algebra rather than an associative one, and f_t a one-parameter family of deformations of A , that is, for a fixed $t_0 \in K$ we consider the multiplication $f_{t_0} : A \otimes A \rightarrow A$ given by $f_{t_0} = ab + F_1(a, b)t_0 + F_2(a, b)t_0^2 + \dots$, the generic element of which is again a Lie algebra. Analogously to (1.4) we have

$$f_t(a.b) = -f_t(b, a) \quad (1.6)$$

$$f_t(f_t(a, b), c) + f_t(f_t(b, c), a) + f_t(f_t(c, a), b) = 0 \quad (1.7)$$

Analogously to (1.3).

$$F_{\nu}(a, b) = -F_{\nu}(b, a) \quad (1.8)$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda, \mu \geq 0}} F_{\lambda}(F_{\mu}(a, b), c) + F_{\lambda}(F_{\mu}(b, c), a) + F_{\lambda}(F_{\mu}(c, a), b) = 0, \quad (1.9)$$

For $\nu = 0$ this express that A is a Lie algebra, similar as in the associative case. *i.e.* $F_0[a, b] = [a, b]$.

For $\nu = 1$ we write

$$\begin{aligned} F_1(a, b) &= -F_1(b, a) \\ F_1([a, b], c) + F_1([b, c], a) - F_1([a, c], b) - [a, F_1(b, c)] + [b, F_1(a, c)] - [c, F_1(a, b)] &= 0 \end{aligned}$$

Considering F_1 as an element of $C^2(A, A)$, the group of 2-co chains of the Lie algebra A with coefficients in itself, the quantity on the left in the last equation is just $\delta F_1(a, b, c)$ and this means, when $\nu = 1$, F_1 is again a 2- co cycle.

1.2 Obstructions

We have seen that every '*differential*' or '*infinitesimal deformation*' of a one-parameter family of deformations of A is an element of the group $Z^2(A, A)$ of two-cocycles of A with coefficients in A . However, the inverse assertion ist not true in general. That is, an arbitrary element F_1 in $Z^2(A, A)$ need not be the differential of a one-parameter family of deformations of A . If it is such, then we shall say that F_1 is integrable. The integrability of F_1 implies an infinite

sequence of relations, which may be interpreted as the vanishing of the 'obstructions' to the integration of F_1 . In the associative case, obstructions are all deducible from equation (1.5) by rewriting that equation in the form

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu \geq 0}}^{\infty} F_{\lambda}(F_{\mu}(a,b),c) - F_{\lambda}(a,F_{\mu}(b,c)) = \delta F_{\nu}(a,b,c) \quad (1.10)$$

In the case when $\nu = 2$.

$$F_1(F_1(a,b),c) - F_1(a,F_1(b,c)) = \delta F_2(a,b,c)$$

The function of three variables on the left is the *associator* of F_1 , and is an element of $Z^2(A,A)$ whenever F_1 is in $Z^2(A,A)$. The cohomology class of this element is the first obstruction to the integration of F_1 , and if F_1 is integrable this must be the zero class.

The associator of F is quadratic in F . Linearizing, one finds that if F and G are in $Z^2(A,A)$, then the 3-co chain $[F,G]$ defined by

$$[F,G](a,b,c) = F(G(a,b),c) - F(a,G(b,c)) + G(F(a,b),c) - G(a,F(b,c))$$

is an element of $Z^3(A,A)$. This is a special case of the Lie product that we will introduce in the following chapters. It will be shown that if F_1, \dots, F_{n-1} satisfy (1.9) for $\nu = 1, \dots, n-1$, then the left side of (1.9) in fact defines an element of $Z^2(A,A)$. The $n-1$ "obstruction cocycle" which, is a function of the sequence F_1, \dots, F_{n-1} .

Assertions analogous to the foregoing hold in the Lie theory, for that we need to use (1.3).

Chapter 2

Trivial Deformation

Definition 2.1 *One parameter families f_t and g_t of deformations of an algebra A , which need not be associative, will be called equivalent if there exist a non-singular linear automorphism Φ_t of V_K of the form*

$$\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \dots \quad (2.1)$$

where all $\varphi_i : V_K \rightarrow V_K$ are linear maps defined over k , such that

$$g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b) \quad (2.2)$$

holds.

From the definition, we have that this determines an equivalence relation.

- Reflexivity: $f_t \approx f_t$

Is straightforward to see, $f_t(a, b) = I_t^{-1} f_t(I_t a, I_t b)$, with I_t the identity automorphism.

- Simetry: $g_t \approx f_t \Rightarrow f_t \approx g_t$

If $g_t \approx f_t$, then $\exists \Phi : A_K \Rightarrow A_K$ such that $g_t = \Phi^{-1} f_t(\Phi_t, \Phi_t)$.

$$g_t(a, b) = \Phi^{-1} f_t(\Phi_t a, \Phi_t b)$$

$$g_t(\Phi^{-1} \Phi_t a, \Phi^{-1} \Phi_t b) = \Phi^{-1} f_t(\Phi_t a, \Phi_t b)$$

setting $\Phi_t a = a'$ and $\Phi_t b = b'$.

$$g_t(\Phi^{-1} a', \Phi^{-1} b') = \Phi^{-1} f_t(a', b')$$

$$\Phi_t g_t(\Phi_t^{-1} a', \Phi_t^{-1} b') = f_t(a', b')$$

$$(\Phi_t^{-1})^{-1} g_t(\Phi_t^{-1} a', \Phi_t^{-1} b') = f_t(a', b')$$

then $f_t \approx g_t$.

- Transitivity: Let f_t, g_t, h_t be families of deformations such that $g_t \approx f_t$ and $f_t \approx h_t$, then there exist non singular automorphism Φ_t, Λ_t of the algebra A_t .

$$g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b) \text{ and } f_t(a, b) = \Lambda_t^{-1} h_t(\Lambda_t a, \Lambda_t b)$$

$$g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b)$$

$$g_t(a, b) = \Phi_t^{-1} \Lambda_t^{-1} h_t(\Lambda_t \Phi_t a, \Lambda_t \Phi_t b)$$

$$g_t(a, b) = (\Lambda_t \Phi_t)^{-1} h_t(\Lambda_t \Phi_t a, \Lambda_t \Phi_t b)$$

then $g_t \approx h_t$.

We say that a one-parameter family of deformations g_t of A , is trivial if it is equivalent to the identity deformation $f_t(a, b)$ defined by $f_t(a, b) = ab$. *i.e* there is a non-singular linear mapping Φ_t of V_K onto itself of the form

$$\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \dots$$

where all the $\varphi_i : V_K \rightarrow V_K$ are linear maps defined over k , such that

$$g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b) = \Phi_t^{-1}(\Phi_t a, \Phi_t b) \quad (2.3)$$

The algebra A_t so obtained is isomorphic to $A_K = A \otimes_k K$, the isomorphism being in fact, the linear mapping Φ_t considered as a mapping from A_t to A_K . This mapping is defined on the power series ring $R = k[[t]]$, therefore on $K = k((t))$. Setting

$$g_t(a, b) = ab + tG_1(a, b) + \dots \quad \text{and} \\ f_t(a, b) = ab + tF_1(a, b) + \dots$$

and if we have that $g_t \approx f_t$, *i.e.* $\exists \Phi_t : A_t \rightarrow A_t$ automorphism such that $g_t = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b)$, for all $a, b \in V_K$.

•

$$\begin{aligned} \Phi_t g_t(a, b) &= \Phi_t(ab + G_1(a, b)t + G_2(a, b)t^2 + \dots) \\ &= \Phi_t(ab) + \Phi_t(G_1(a, b))t + \Phi_t(G_2(a, b))t^2 + \dots \\ &= (ab + \varphi_t(ab) + \dots) + (G_1(a, b) + \varphi(G_1(a, b))t + \dots)t + \dots \\ &= ab + (\varphi_t(ab) + G_1(a, b))t + \dots \end{aligned}$$

•

$$\begin{aligned}
f_t(\Phi_t a, \Phi_t b) &= \Phi_t a \Phi_t b + F_1(\Phi_t a, \Phi_t b)t + F_2(\Phi_t a, \Phi_t b)t^2 + \dots \\
&= (a + \varphi_1(a)t + \dots)(b + \varphi_1(b)t + \dots) + F_1(a + \varphi_1(a)t + \dots, b + \varphi_1(b)t + \dots) + \dots \\
&= (ab + (a\varphi_1(b) + \varphi_1(a)b)t + \dots) + F_1(a, b)t + \dots \\
&= ab + (a\varphi_1(b) + \varphi_1(a)b + F_1(a, b))t + \dots
\end{aligned}$$

Since $\Phi_t g_t(a, b)$ and $f_t(\Phi_t a, \Phi_t b)$ are equal.

$$\begin{aligned}
\varphi_1(ab) + G_1(ab) &= a\varphi_1(b) + \varphi_1(a)b + F_1(a, b) \\
G_1(a, b) &= F_1(a, b) + a\varphi_1(b) - \varphi_1(ab) + \varphi_1(a)b
\end{aligned}$$

then we get

$$G_1(a, b) = F_1(a, b) + \delta\varphi_1(a, b) \quad (2.4)$$

That means, if two multiplications g_t and f_t (defined as before) that are equivalent, then $G_1(a, b) = F_1(a, b) + \delta\varphi_1(a, b)$.

It follows that if an element in a class is integrable, then the elements of that class may be integrated to give one-parameter families of algebras whose generic elements are isomorphic. The isomorphism is defined over $R = k[[t]]$. We may interpret the elements of $H^2(A, A)$ as being the infinitesimal deformations. In particular if a multiplication $g_t(a, b) = ab + tG_1(a, b) + t^2G_2(a, b) + \dots$ is trivial, then $G_1(a, b) = \delta\varphi_1(a, b)$.

Proposition 2.1 *Let f_t be a one-parameter family of deformations of an algebra A . Then f_t is equivalent to a family $g_t(a, b) = ab + t^n G_n(a, b) + t^{n+1} G_{n+1}(a, b) + \dots$, where the first non-vanishing cochain F_n is in $Z^2(A, A)$, and is not cohomologous to zero.*

Proof.

A one parameter family of an algebra A , f_t may have $F_1 = F_2 = \dots = F_{n-1} = 0$. Now from (1.10) we have.

$$0 = \delta F_n = \sum_{\substack{\lambda+\mu=n \\ \lambda, \mu \geq 0}} F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c))$$

Then $F_n \in Z^2(A, A)$. Further, if we have $F_n \in B^2(A, A)$, then $F_n = -\delta\varphi_n$ with $\varphi_n \in C^1(A, A)$. And from (2.4),

$$0 = F_n + \delta\varphi_n = G_n$$

where G_n is part of a family of deformations of the form $g_t(a, b) = ab + t^n G_n(a, b) + t^{n+1} G_{n+1}(a, b) + \dots$ where $f_t \approx g_t$ i.e. exist an automorphism of A_t , Φ_t , such that $g_t(a, b) = \Phi_t^{-1} f_t(\Phi_t a, \Phi_t b)$.

2.1 Obstructions to Derivations and the squaring operations

Let A be an algebra A which need not be associative, one may exhibit obstructions now lying in $Z^2(A, A)$ in a analogous way to that given in the previous analysis.

Suppose that $A_K = A \otimes_k k((t))$ posses an automorphism Φ_t of the form

$$\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \dots \quad (2.5)$$

the φ_i being linear functions from A_K to A_K defined over k , and φ_0 being interpreted as the identity mapping I . As in the case of deformations, we may consider Φ_t of (2.5) to be the generic element of a 'one-parameter family of automorphism' of A . Analogous to (1.5), the condition that Φ_t be an automorphism has a consequence.

$$\begin{aligned} \Phi_t(ab) &= \Phi_t(a) \cdot \Phi_t(b) \\ ab + t\varphi_1(ab) + t^2\varphi_2(ab) + \dots &= (a + t\varphi_1(a) + t^2\varphi_2(a) + \dots)(b + t\varphi_1(b) + t^2\varphi_2(b) + \dots) \\ ab + t\varphi_1(ab) + t^2\varphi_2(ab) + \dots &= ab + t(a\varphi_1(b) + \varphi_1(a)b) + t^2(a\varphi_2(b) + \varphi_1(a)\varphi_1(b) + \varphi_2(a)b) + \dots \\ \Rightarrow \varphi_1(ab) &= a\varphi_1(b) + \varphi_1(a)b = \varphi_0(a)\varphi_1(b) + \varphi_1(a)\varphi_0(b) \\ \Rightarrow \varphi_2(ab) &= a\varphi_2(b) + \varphi_1(a)\varphi_1(b) + \varphi_2(a)b = \varphi_0(a)\varphi_2(b) + \varphi_1(a)\varphi_1(b) + \varphi_2(a)\varphi_0(b) \end{aligned}$$

Then

$$\sum_{\substack{\lambda+\mu=v \\ \lambda, \mu \geq 0}} \varphi_\lambda(a)\varphi_\mu(b) = \varphi_v(ab) \quad (2.6)$$

And analogous to (1.10).

$$\sum_{\substack{\lambda+\mu=v \\ \lambda, \mu > 0}} \varphi_\lambda(a)\varphi_\mu(b) = -\delta\varphi_v(ab) \quad (2.7)$$

The condition is vacuous for $v = 0$. For $v = 1$ it asserts that $\delta_1(ab) = a\varphi_1(b) - \varphi_1(ab) + \varphi_1(a)b = 0$. *i.e* φ_1 is a derivation of A into itself, or an element of $Z^1(A, A)$. For $v = 2$, we have

$$\varphi_1(a)\varphi_1(b) = -\delta_2(a,b) \quad (2.8)$$

The left-hand side of Equation (2.8) represents the cup product $\varphi_1 \smile \varphi_1$ of φ_1 with itself, which is an element of $Z^2(A,A)$. This expression is known as the 'first obstruction' to the integration of φ_1 . From Equation (2.8), we can see that it represents the zero cohomology class if and only if φ_1 is the differential of a one-parameter family of automorphisms.

Now, if we assume that the characteristic of A is not 2, then in fact $\varphi_1 \smile \varphi_1 = \frac{-1}{2}\delta\varphi_1^2$, and the first obstruction vanishes. Linearizing, we find that for ϕ_1, ψ_1 in $Z^1(A,A)$, $\phi_1 \smile \psi_1 + \psi_1 \smile \phi_1$ is a 2-coboundary, suggesting that the cohomology ring of a ring with coefficients in itself is a graded commutative ring.

If the characteristic of A is p , then setting $\phi_i = \frac{\phi_1^i}{i!}$ for $i = 1, 2, \dots, p-1$, the elements of $C^1(A,A)$ so defined satisfy (2.7) for $v = 1, \dots, p-1$, and the expression on the left in (2.7) for $v = p$ is an element of $Z^2(A,A)$, the *primary obstruction* to the integration of ϕ_1 .

It can be shown that primary obstruction induce an additive mapping of $H^1(A,A)$ into $H^2(A,A)$. The obstruction is analogous in some respect to a p^{th} power and will be denoted $Sq_p\phi_1$. If k is the prime field F_p of p elements, then Sq_p is linear.

Being an element of $Z^1(A,A)$, Sq_p may be interpreted as an *infinitesimal* deformation. In fact, letting $K = K((t))$, there exists an associative multiplication f_t on V_k of the form

$$f_t = ab + tSq_p\phi_1(a,b) + t^2F_2(a,b) + t^3F_3(A,A) + \dots$$

that is, such an infinitesimal deformation is always integrable. If A_t denotes, as usual, the corresponding algebra, then $A_t \otimes_K K(t^{1/p})$ is isomorphic to $A \otimes_k K(t^{1/p})$, but A_t is generally not

isomorphic to $A_K = A \otimes_k K$.

One thing that is interesting to note is that while the cohomology groups of an algebra are essentially independent of the ground field, the deformation theory may distinguish between non-isomorphic algebras even if they are isomorphic over some larger field.

Since all obstructions to the integration of an element of $Z^1(A,A)$ will be elements of $Z^2(A,A)$, and only their cohomology classes are of interest, the vanishing of $H^2(A,A)$ implies not only the rigidity of A but also the integrability of any ϕ_1 in $Z^1(A,A)$ to a one-parameter family of automorphisms of A .



Chapter 3

Graded Rings

Definition 3.1 A graded ring A , is one which, as an S -module, is a direct sum of sub modules $A = \sum A_\lambda$, indexed by the elements of an additive group G and in which $A_\lambda A_\mu \subset A_{\lambda+\mu}$.

If in G , there is a subgroup G^+ of index two, the elements of which will be called even; the elements of the complementary coset G^- will be called odd. Commonly G will be the additive group of integers, and G^+ is, in that case, the even integers.

Any element of A_λ will be called homogeneous of degree λ . Setting $A^+ = \sum A_{\lambda'}, \lambda' \in G^+$, and $A^- = \sum A_{\lambda''}, \lambda'' \in G^-$, we have $A^+A^+, A^-A^- \subset A^+, A^+A^-, A^-A^+ \subset A^-$; in particular A^+ is a subring of A .

The group G , G^+ being fixed, we define $1^\lambda = 1$, for all $\lambda \in G$ and $(-1)^\lambda$ is 1 or -1 , if λ is even or odd, respectively. Also, we set $(-1)^{\lambda\mu} = ((-1)^\lambda)^\mu$ for all λ, μ in G . There is an involutive automorphism J of A defined by $Ja = (-1)^\lambda a$. If μ is even, $J^\mu a = a$, if μ is odd, $J^\mu = J$.

Definition 3.2 A graded ring $A = \sum A_\lambda$ will be call commutative if given elements a, b in A of degree λ, μ respectively, we have

$$ab = (-1)^{\lambda\mu} ba$$

and will be called skew if

$$ab = -(-1)^{\lambda\mu}ba$$

The anti isomorph or opposite A' of a ring A is that ring, which as an S -module, is identical with A , but in which the product of a and b is defined to be ba .

A trivial example is a commutative ungraded ring A , which is identical with A' , while if A is skew, the map $\varphi : A \rightarrow A'$, defined by $a \mapsto -a$ is an isomorphism. In fact, is straightforward to see that it is a bijection.

If A is skew *i.e.* $ab = -ba$, with $a, b \in A$.

$$\varphi(ab) = -ab = ba = (-b)(-a) = \varphi(b)\varphi(a) = \varphi(a)\varphi(b)$$

then φ is a morphism.

For graded commutative and skew rings, A and A' may fail to be isomorphic. However, if G^+ contains a subgroup G^{++} such that $\frac{G}{G^{++}}$ is the cyclic group of four elements, then A is isomorphic to A' . In this case, choosing a generator u of $\frac{G}{G^{++}}$ we may define an element λ of G to be congruent to $n \pmod{4}$, $n = 0, 1, 2$ or 3 if $\lambda \equiv u^n \pmod{G^{++}}$.

If is a graded commutative ring, an S -module automorphism σ of A , defined by setting for a^λ in A_λ .

$$\sigma(a^\lambda) = \begin{cases} a^\lambda & \text{if } \lambda \equiv 0, 1 \pmod{4} \\ -a^\lambda & \text{if } \lambda \equiv 2, 3 \pmod{4} \end{cases}$$

It is straightforward to see that is a bijection.

Injectivity: Let $a^\lambda, b^\mu \in A$, such that, $\sigma(a^\lambda) = \sigma(b^\mu)$, that is $\pm a^\lambda = \pm b^\mu$. We have the following cases.

A. $a^\lambda = b^\mu$ for $\lambda \equiv 0, 1 \pmod{4}$ and $\mu \equiv 0, 1 \pmod{4}$

B. $a^\lambda = -b^\mu$ for $\lambda \equiv 0, 1 \pmod{4}$ and $\mu \equiv 2, 3 \pmod{4}$

C. $-a^\lambda = b^\mu$ for $\lambda \equiv 2, 3 \pmod{4}$ and $\mu \equiv 0, 1 \pmod{4}$

D. $-a^\lambda = -b^\mu$ for $\lambda \equiv 2, 3 \pmod{4}$ and $\mu \equiv 2, 3 \pmod{4}$

For the cases A) and D) we have the injectivity. For the cases B) and C) we have $a^\lambda + b^\mu = 0$, $\Rightarrow a^\lambda = b^\mu = 0$. And we get the injectivity.

Surjectivity: Let $a^\lambda \in A$, then a^λ is homogeneous of degree λ , where $\lambda \equiv n \pmod{4}$.

If n is 0 or 1 then $\sigma(a^\lambda) = a^\lambda$.

If n is 2 or 3 then $\sigma(-a^\lambda) = -(-a^\lambda) = a^\lambda$.

Therefore, is surjective.

Lets see if it is a morphism. We have to verify that $\sigma(a^\lambda b^\mu) = \sigma(a^\lambda)\sigma(b^\mu)$.

We know that $a^\lambda b^\mu \in A_{\lambda+\mu}$. Then $\sigma(a^\lambda b^\mu) = a^\lambda b^\mu$ if $\lambda + \mu \equiv 0, 1 \pmod{4}$. On the other hand $\sigma(a^\lambda b^\mu) = -a^\lambda b^\mu$ if $\lambda + \mu \equiv 2, 3 \pmod{4}$.

A. For $\lambda + \mu \equiv 0 \pmod{4}$, this happen when

- $\lambda = \mu \equiv 0 \pmod{4}$ then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = b^\mu$.

$$\Rightarrow \sigma(a^\lambda b^\mu) = a^\lambda b^\mu = (-1)^0 a^\lambda (-1)^0 b^\mu = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

- $\lambda \equiv 1 \pmod{4}, \mu \equiv 3 \pmod{4}$. then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = -1$.

$$\Rightarrow \sigma(a^\lambda b^\mu) = a^\lambda b^\mu = (-a^\lambda)(-b^\mu) = -\sigma(a^\lambda) \sigma(b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

The case $\lambda \equiv 3 \pmod{4}, \mu \equiv 1 \pmod{4}$ is analogous.

- $\lambda = \mu \equiv 2 \pmod{4}$. then $\sigma(a^\lambda) = -a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = 1$

$$\Rightarrow \sigma(a^\lambda b^\mu) = a^\lambda b^\mu = (-a^\lambda)(-b^\mu) = \sigma(a^\lambda) \sigma(b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

B. For $\lambda + \mu \equiv 1 \pmod{4}$, this happen when

- $\lambda \equiv 0 \pmod{4}, \mu \equiv 1 \pmod{4}$ then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = b^\mu, (-1)^{\lambda\mu} = 1$

$$\Rightarrow \sigma(a^\lambda b^\mu) = a^\lambda b^\mu = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

The case $\lambda \equiv 1 \pmod{4}, \mu \equiv 0 \pmod{4}$ is analogous.

- $\lambda \equiv 2 \pmod{4}, \mu \equiv 3 \pmod{4}$ then $\sigma(a^\lambda) = -a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = 1$

$$\Rightarrow \sigma(a^\lambda b^\mu) = a^\lambda b^\mu = (-a^\lambda)(-b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

C. For $\lambda + \mu \equiv 2 \pmod{4}$, this happen when

- $\lambda \equiv 0 \pmod{4}, \mu \equiv 2 \pmod{4}$ then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = 1$
 $\Rightarrow \sigma(a^\lambda b^\mu) = -a^\lambda b^\mu = a^\lambda(-b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$

The case $\lambda \equiv 2 \pmod{4}, \mu \equiv 0 \pmod{4}$ is analogous.

- $\lambda = \mu \equiv 1 \pmod{4}$ then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = b^\mu, (-1)^{\lambda\mu} = -1$
 $\Rightarrow \sigma(a^\lambda b^\mu) = -a^\lambda b^\mu = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$
- $\lambda = \mu \equiv 3 \pmod{4}$ then $\sigma(a^\lambda) = -a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = -1$
 $\Rightarrow \sigma(a^\lambda b^\mu) = -a^\lambda b^\mu = -(-a^\lambda)(-b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$

D. For $\lambda + \mu \equiv 3 \pmod{4}$, this happen when

- $\lambda \equiv 0 \pmod{4}, \mu \equiv 3 \pmod{4}$. then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = 1$
 $\Rightarrow \sigma(a^\lambda b^\mu) = -a^\lambda b^\mu = a^\lambda(-b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$

The case $\lambda \equiv 3 \pmod{4}, \mu \equiv 0 \pmod{4}$ is analogous.

- $\lambda \equiv 1 \pmod{4}, \mu \equiv 2 \pmod{4}$ then $\sigma(a^\lambda) = a^\lambda, \sigma(b^\mu) = -b^\mu, (-1)^{\lambda\mu} = 1$
 $\Rightarrow \sigma(a^\lambda b^\mu) = -a^\lambda b^\mu = a^\lambda(-b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda) \sigma(b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$

The case $\lambda \equiv 2 \pmod{4}, \mu \equiv 1 \pmod{4}$, is analogous.

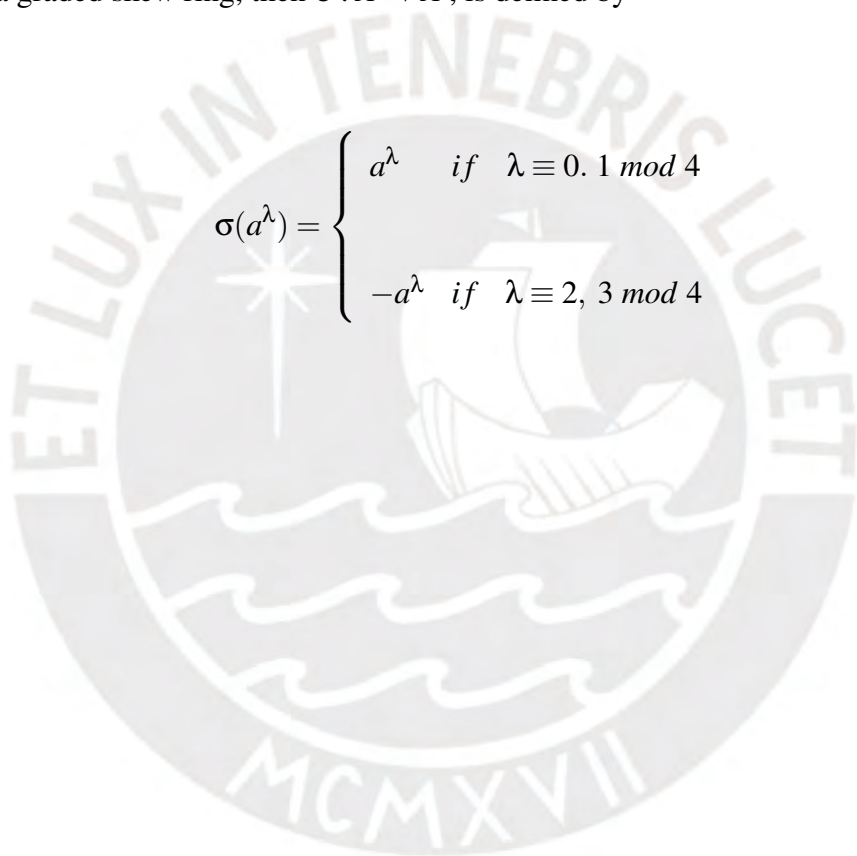
Therefore the automorphism satisfies

$$\sigma(a^\lambda b^\mu) = (-1)^{\lambda\mu} \sigma(a^\lambda b^\mu) = \sigma(b^\mu) \sigma(a^\lambda)$$

i.e. σ is an anti automorphism of A , or an isomorphism of A onto A' .

If A is a graded skew ring, then $\sigma : A \rightarrow A'$, is defined by

$$\sigma(a^\lambda) = \begin{cases} a^\lambda & \text{if } \lambda \equiv 0, 1 \pmod{4} \\ -a^\lambda & \text{if } \lambda \equiv 2, 3 \pmod{4} \end{cases}$$



Chapter 4

Graded Lie and pre-Lie rings

Definition 4.1 A ring A is a graded Lie ring if it is a graded ring with a skew multiplication (usually denoted $[,]$) satisfying the graded Jacoby identity. i.e. if given a^λ, b^μ, c^ν in A of degree λ, μ, ν , respectively we have.

$$[a^\lambda, b^\mu] = -(-1)^{\lambda\mu}[b^\mu, a^\lambda] \quad (4.1)$$

and

$$(-1)^{\lambda\nu}[[a^\lambda, b^\mu], c^\nu] + (-1)^{\mu\lambda}[[b^\mu, c^\nu], a^\lambda] + (-1)^{\nu\mu}[[c^\nu, a^\lambda], b^\mu] = 0 \quad (4.2)$$

If A is a graded Lie ring, from (4.1) and (4.2) one verifies that the opposite ring A' is also a graded ring. Let R_a be a module homomorphism of A defined by $bR_a = [b, a]$.

In (4.2) we have that.

•

$$(-1)^{\lambda\nu}[[a^\lambda, b^\mu], c^\nu] = (-1)^{\lambda\nu}(-(-1)^{(\lambda+\mu)\nu}[c^\nu, [a^\lambda, b^\mu]])$$

$$\begin{aligned}
&= -(-1)^{\lambda\nu}(-1)^{\lambda\nu}(-1)^{\mu\nu}[c^\nu, [a^\lambda, b^\mu]] \\
&= -(-1)^{\mu\nu}[c^\nu, [a^\lambda, b^\mu]] \\
&= -(-1)^{\mu\nu}c^\nu R_{[a^\lambda, b^\mu]}
\end{aligned}$$

•

$$\begin{aligned}
(-1)^{\mu\lambda}[[b^\mu, c^\nu], a^\lambda] &= (-1)^{\mu\lambda}[-(-1)^{\mu\nu}[c^\nu, b^\mu], a^\lambda] \\
&= -(-1)^{\mu\lambda}(-1)^{\mu\nu}[[c^\nu, b^\mu], a^\lambda] \\
&= -(-1)^{\mu\lambda}(-1)^{\mu\nu}c^\nu R_{b^\mu} R_{a^\lambda}
\end{aligned}$$

•

$$(-1)^{\nu\mu}[[c^\nu, a^\lambda], b^\mu] = (-1)^{\nu\mu}c^\nu R_{a^\lambda} R_{b^\mu}$$

Consequently,

$$\begin{aligned}
-(-1)^{\mu\nu}c^\nu R_{[a^\lambda, b^\mu]} - (-1)^{\mu\lambda}(-1)^{\mu\nu}c^\nu R_{b^\mu} R_{a^\lambda} + (-1)^{\nu\mu}c^\nu R_{a^\lambda} R_{b^\mu} &= 0 \\
\Rightarrow (-1)^{\nu\mu}R_{a^\lambda} R_{b^\mu} - (-1)^{\mu\lambda}(-1)^{\mu\nu}R_{b^\mu} R_{a^\lambda} &= (-1)^{\mu\nu}R_{[a^\lambda, b^\mu]} \\
(-1)^{\mu\nu}(-1)^{\nu\mu}R_{a^\lambda} R_{b^\mu} - (-1)^{\mu\nu}(-1)^{\mu\lambda}(-1)^{\mu\nu}R_{b^\mu} R_{a^\lambda} &= (-1)^{\mu\nu}(-1)^{\mu\nu}R_{[a^\lambda, b^\mu]}
\end{aligned}$$

$$R_{a^\lambda} R_{b^\mu} - (-1)^{\lambda\mu} R_{b^\mu} R_{a^\lambda} = R_{[a^\lambda, b^\mu]} \quad (4.3)$$

Therefore (4.1) and (4.2) are equivalent to (4.1) and (4.3).

Definition 4.2 A ring A will be called a graded right pre-Lie ring if for elements a^λ, b^μ, c^ν of A of degrees λ, μ, ν respectively, we have denoting the product of a and b in A by $a \circ b$.

$$(c \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu} (c \circ b^\mu) \circ a^\lambda = c \circ (a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \quad (4.4)$$

Where $a \circ b$ denotes the product in A .

(A graded left pre-Lie ring will be one whose anti isomorph is a graded pre-Lie ring).

Denoting by R_a again the right multiplications by a , (4.4) is equivalent to

$$R_{a^\lambda} R_{b^\mu} - (-1)^{\lambda\mu} R_{b^\mu} R_{a^\lambda} = R_{a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda} \quad (4.5)$$

It follows from the defining identity (4.4), that a graded associative ring is also a graded pre-Lie ring. However, a graded Lie ring in general is not a graded pre-Lie ring.

Theorem 4.1 Let A be a graded pre-Lie ring, and define for elements in A a new multiplication by setting for a^λ, b^μ of degree λ, μ respectively

$$[a^\lambda, b^\mu] = a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda$$

Then with the bracket product A is a graded Lie ring.

Proof. First we show that the bracket product is skew.

$$[a^\lambda, b^\mu] = a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda = -(-1)^{\lambda\mu} (b^\mu \circ a^\lambda - (-1)^{\lambda\mu} a^\lambda \circ b^\mu)$$

$$= -(-1)^{\lambda\mu}[b^\mu, a^\lambda]$$

we prove that it satisfies the graded Jacobi identity.

The first term in (4.2) is

$$\begin{aligned} (-1)^{\lambda\nu}[[a^\lambda, b^\mu], c^\nu] &= (-1)^{\lambda\nu}[(a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \circ c^\nu - \\ &\quad (-1)^{(\lambda+\mu)\nu} c^\nu \circ (a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda)] \end{aligned}$$

which by hypothesis (4.4) is equal to

$$(-1)^{\lambda\nu}[(a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \circ c^\nu - (-1)^{(\lambda+\mu)\nu} ((c^\nu \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu} (c^\nu \circ b^\mu) \circ a^\lambda)]$$

Applying similar transformations to the other terms in (4.2).

$$\begin{aligned} &(-1)^{\lambda\nu}[(a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \circ c^\nu - (-1)^{(\lambda+\mu)\nu} ((c^\nu \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu} (c^\nu \circ b^\mu) \circ a^\lambda)] \\ &+ (-1)^{\mu\lambda}[(b^\mu \circ c^\nu - (-1)^{\mu\nu} c^\nu \circ b^\mu) \circ a^\lambda - (-1)^{(\mu+\nu)\lambda} ((a^\lambda \circ b^\mu) \circ c^\nu - (-1)^{\mu\nu} (a^\lambda \circ c^\nu) \circ b^\mu)] \\ &+ (-1)^{\nu\mu}[(c^\nu \circ a^\lambda - (-1)^{\nu\lambda} a^\lambda \circ c^\nu) \circ b^\mu - (-1)^{(\nu+\lambda)\mu} ((b^\mu \circ c^\nu) \circ a^\lambda - (-1)^{\nu\lambda} (b^\mu \circ a^\lambda) \circ c^\nu)] \\ &= (-1)^{\lambda\nu} (a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \circ c^\nu - (-1)^{\mu\nu} ((c^\nu \circ a^\lambda) \circ b^\mu - (-1)^{\lambda\mu} (c^\nu \circ b^\mu) \circ a^\lambda) \\ &\quad + (-1)^{\mu\lambda} (b^\mu \circ c^\nu - (-1)^{\mu\nu} c^\nu \circ b^\mu) \circ a^\lambda - (-1)^{\nu\lambda} ((a^\lambda \circ b^\mu) \circ c^\nu - (-1)^{\mu\nu} (a^\lambda \circ c^\nu) \circ b^\mu) \\ &\quad + (-1)^{\nu\mu} (c^\nu \circ a^\lambda - (-1)^{\nu\lambda} a^\lambda \circ c^\nu) \circ b^\mu - (-1)^{\lambda\mu} ((b^\mu \circ c^\nu) \circ a^\lambda - (-1)^{\nu\lambda} (b^\mu \circ a^\lambda) \circ c^\nu) \\ &= 0 \end{aligned}$$

Therefore (4.2) hold.

It follows from Theorem 4.1, that the bracket product may be introduced in a graded associative ring, yielding a graded Lie ring.



Chapter 5

Modules over associative Lie and pre-Lie rings

Let $A = \sum_{\lambda} A_{\lambda}$, be a graded ring and $P = \sum_{\lambda} P_{\lambda}$ be a module which is a direct sum of modules indexed by the same group as A . Furthermore, let $\rho : P \otimes A \rightarrow P$ be a module homomorphism such that $\rho(P_{\lambda} \otimes A_{\mu}) \subseteq P_{\lambda+\mu}$. The graded ring A will be an associative, pre-Lie, or a Lie ring, and we shall say that P is a right A -module, provided the following conditions, depending on the type of ring of A , hold.

If A is associative then $\rho : P \otimes A \rightarrow P$, is defined by $\rho : z \otimes a \rightarrow za$, for $z \in P, a \in A$, and for $z \in P, a, b \in A$ we have

$$(za)b = z(ab)$$

In §4, we stated that a graded associative ring is also a pre-Lie ring. In Sections 5.1 and 5.2, the definitions of modules over graded Lie rings and pre-Lie rings will be provided.

If $P = \sum P_{\lambda}$ has both a right and left module structure over a graded ring A , with the same

gradation for both structures, then the direct sum module $A + P$ is graded by setting $(A + P)_\lambda = A_\lambda + P_\lambda$. This module may be made into a ring by defining the product of two elements of P to be zero. This ring contains A as a subring and P as an ideal with $P^2 = 0$.

If A is an associative or Lie ring and $A + P$ has a similar type of structure, then P is a two-sided module. That is, if we impose the same conditions on A and $A + P$, then we can conclude that P is a two-sided module.

Now, if we impose certain conditions on A with P being only a right module, then P and $A + P$ are subject to certain conditions.

- If A is a graded commutative ring and P a right A -module, then setting for $a \in A_\lambda$, $z \in P_\mu$.

$$a^\lambda z^\mu = (-1)^{\lambda\mu} z^\mu a^\lambda$$

P becomes a two-sided A module and $A + P$ is commutative.

- If A is a graded Lie ring and P a right A -module, then setting for $a \in A_\lambda$, $z \in P_\mu$.

$$[a^\lambda, z^\mu] = -(-1)^{\lambda\mu} [z^\mu, a^\lambda]$$

P becomes a two-sided A -module and $A + P$ is a graded Lie ring.

It is possible to define a new right A -module structure on P , if P is a right module over a graded associative, pre-Lie or Lie ring A , given by a homomorphism $\rho : P \otimes A \rightarrow P$. Let ϕ be a gradation-preserving ring endomorphism of A . Then, we can define a new right A -module structure on P by defining a new homomorphism $\rho^\phi : P \otimes A \rightarrow P$, given by:

$$\rho^\phi(z \otimes a) = \rho(z \otimes \phi(a))$$

for $z \in P, a \in A$.

In particular, we may take $\phi = J^v$.(i.e. $\phi = J$ if v odd, $\phi = 1$ if v even).

5.1 Modules over graded Lie rings

Let $A = \sum_{\lambda} A_{\lambda}$, be a graded Lie ring and $P = \sum_{\lambda} P_{\lambda}$ be a module which is a direct sum of modules indexed by the same group as A . Let $\rho : P \otimes A \rightarrow P$ be a module homomorphism such that $\rho(P_{\lambda} \otimes A_{\mu}) \subseteq P_{\lambda+\mu}$.

The module homomorphism $\rho : P \otimes A \rightarrow P$, is defined by $\rho : z \otimes a \mapsto [z, a]$ for $z \in P, a \in A$, and for $z \in P, a^{\lambda}, b^{\mu} \in A$, of degrees λ, μ , respectively.

$$[[z, a^{\lambda}], b^{\mu}] - (-1)^{\lambda\mu} [[z, b^{\mu}], a^{\lambda}] = [z, [a^{\lambda}, b^{\mu}]] \quad (5.1)$$

Let R_a the endomorphism of P defined by $zR_a = [z, a]$, (5.1) is formally equivalent to (4.3). Therefore, a graded Lie ring, is a right module over itself.

In the context of tensorial products, let P and P' be right modules over a graded Lie A . Then, $P \otimes P'$ may be given the structure of a right module over A by setting

$$(P \otimes P')_v = \sum_{\lambda+\mu=v} P_{\lambda} \otimes P_{\mu}$$

and for $y^{\mu} \in P_{\mu}, x \in P_{\lambda}, a^{\lambda} \in A_{\lambda}$, by setting.

$$[x \otimes y^\mu, a^\lambda] = (-1)^{\lambda\mu} [x, a^\lambda] \otimes y^\mu + x \otimes [y^\mu, a^\lambda]$$

In this subsection, we will refrain from discussing the tensorial product and solely focus on examining the definition specifically for Lie rings.

5.2 Modules over graded pre-Lie rings

Let $A = \sum_{\lambda} A_{\lambda}$, be a graded pre-Lie ring and $P = \sum_{\lambda} P_{\lambda}$ be a module which is a direct sum of modules indexed by the same group as A . Let $\rho : P \otimes A \rightarrow P$ be a module homomorphism such that $\rho(P_{\lambda} \otimes A_{\mu}) \subseteq P_{\lambda+\mu}$.

The module homomorphism $\rho : P \otimes A \rightarrow P$, is defined by $\rho : z \otimes a \mapsto z \circ a$ for $z \in P, a \in A$, and for $z \in P, a^\lambda, b^\mu \in A$, of degree λ, μ respectively.

$$((z \circ a^\lambda) \circ b^\mu) - (-1)^{\lambda\mu} ((z \circ b^\mu) \circ a^\lambda) = z \circ (a^\lambda \circ b^\mu - (-1)^{\lambda\mu} b^\mu \circ a^\lambda) \quad (5.2)$$

Let R_a , the endomorphism of P defined by $zR_a = z \circ a$, Then (5.2) is formally equivalent to (4.5). Therefore a graded pre-Lie ring, is a right module over itself. Once we have defined the concepts of modules over associative, Lie and pre-Lie rings, we can explore the potential connections between these two concepts.

If P is a right module over a pre-Lie ring A , then it is also a right module over the associated Lie ring obtained by introducing the bracket product.

$$[z, a] = z \circ a \quad (5.3)$$

for $z \in P, a \in A$

But with respect to this definition, we must have certain caution. If A is a graded pre-Lie ring, then it is a right module over itself and using (5.3) it is also a right module over its associative graded Lie ring (Remember that the associated graded Lie ring is defined in Theorem 4.1). Therefore, if P is a right module over an associative ring A , then it is also a right module over A considered as a pre-Lie ring. For this we just define $z \circ a = za$ for $z \in P$ and $a \in A$.

For example, suppose A is associative and ungraded (i.e. $A_\lambda = 0$ for $\lambda \neq 0$). Let a, b , and z be elements of A , where A is a pre-Lie ring. We have $a \circ b = ab$, and the associated Lie structure is given by $[a, b] = ab - ba$. By Equation (5.3), considering the pre-Lie ring A as a right A -module, we have $[z, a] = z \circ a = za$. Then a module P over a Lie ring A must satisfy Equation (5.1), that is, $[[z, a], b] - [[z, b], a] = [z, [a, b]]$. The left side being $zab - zba$ and the right side being $z(ab - ba)$.

However, considering A as a Lie ring in the multiplication $[a, b] = ab - ba$, we have $[z, a] = za - az$, and again $[[z, a], b] - [[z, b], a] = [z, [a, b]]$, where the left side is.

$$\begin{aligned}
[z a - a z, b] - [z b - b z, a] &= ((z a - a z) b - b(z a - a z)) - ((z b - b z) a - a(z b - b z)) \\
&= ((z a b - a z b) - (b z a - b a z)) - ((z b a - b z a) - (a z b - a b z)) \\
&= (z a b - a z b - b z a + b a z) - (z b a - b z a - a z b + a b z) \\
&= z a b - a z b - b z a + b a z - z b a + b z a + a z b - a b z \\
&= (z a b - z b a) + (b a z - a b z) \\
&= z(a b - b a) + (b a - a b) z \\
&= z[a, b] - [a, b] z
\end{aligned}$$

$$= [z, [a, b]]$$

Therefore taking in A a graded pre-Lie ring structure then it is a right module over itself and therefore, using (5.3), it is also a right module over its "associated" graded Lie ring. However, taking in A the graded Lie ring structure it is also a right module over itself, but this is not the same module structure.

A left module P over a graded associative, pre-Lie or Lie ring A , is a right module over its anti-isomorphism A' . The notations are similar to those for right modules, the operation of the elements of A on P being written on the left.

Just like there is a tensorial product over Lie rings, there is also a tensorial product defined for pre-Lie rings. Let P and P' be right modules over pre-Lie ring A . Then, $P \otimes P'$ may be given the structure of a right module over A by setting

$$(P \otimes P')_{\nu} = \sum_{\lambda+\mu=\nu} P_{\lambda} \otimes P_{\mu}$$

and for $y^{\mu} \in P_{\mu}$, $x \in P_{\lambda}$, $a^{\lambda} \in A_{\lambda}$, by setting.

$$(x \otimes y^{\mu}) \circ a^{\lambda} = (-1)^{\lambda\mu} (x \circ a^{\lambda}) \otimes y^{\mu} + x \otimes (y^{\mu} \circ a^{\lambda})$$

As previously mentioned, we will refrain from discussing the tensorial product and solely focus on examining the definition specifically for pre-Lie rings.

Chapter 6

Derivations

Definition 6.1 If $P = \sum P_\lambda$, $P' = \sum P'_\lambda$, are direct sum of modules indexed by the same group G then a module homomorphism $\phi : P \rightarrow P'$ will be said to be homogeneous of degree $\nu \in G$, if $\phi(P_\lambda) \subset P'_{\lambda+\nu}$ for all λ in G .

With this definition, is possible to define a derivation of a graded ring.

Definition 6.2 If $A = \sum A_\lambda$ is a graded ring, then a module endomorphism D of A will be called a "left derivations of degree ν " of A if D is a homomorphism of degree ν of A into itself and if given a^λ , b^μ of degree λ and μ in A , we have

$$D(a^\lambda b^\mu) = (Da)b + (-1)^{\nu\lambda} a(Db) \quad (6.1)$$

(Analogously, a module endomorphism D of A will be called a "right derivations of degree ν " of A if D is a homomorphism of degree ν of A into itself and if given a^λ , b^μ of degree λ and μ in A , we have $D(a^\lambda b^\mu) = (-1)^{\nu\mu} (Da)b + a(Db)$).

A left derivation of A is a right derivation of the opposite ring A' of A .

If D and D' are left derivation of A of degrees ν , ν' , respectively, then

$$[D, D'] = DD' - (-1)^{\nu\nu'} D'D$$

is a left derivation of degree $\nu + \nu'$. In fact, we compute.

$$\begin{aligned}
[D, D'](a^\lambda b^\mu) &= (DD')a^\lambda b^\mu - (-1)^{\nu\nu'} (D'D)a^\lambda b^\mu \\
&= D[(D'a^\lambda)b^\mu + (-1)^{\nu'\lambda} a^\lambda D'b^\mu] - (-1)^{\nu\nu'} D'[(Da^\lambda)b^\mu + (-1)^{\nu\lambda} a^\lambda Db^\mu] \\
&= D((D'a^\lambda)b^\mu) + (-1)^{\nu'\lambda} D(a^\lambda D'b^\mu) - (-1)^{\nu\nu'} [D'((Da^\lambda)b^\mu) + (-1)^{\nu\lambda} D'(a^\lambda Db^\mu)] \\
&= D(D'a^\lambda)b^\mu + (-1)^{(\lambda+\nu')\nu} D'a^\lambda Db^\mu + (-1)^{\nu'\lambda} [Da^\lambda D'b^\mu + (-1)^{\lambda\nu} a^\lambda DD'b^\mu] \\
&\quad - (-1)^{\nu\nu'} [(D'(Da^\lambda)b^\mu + (-1)^{\nu'(\lambda+\nu)} Da^\lambda D'b^\mu) + (-1)^{\nu\lambda} (D'a^\lambda Db^\mu + (-1)^{\nu'\lambda} a^\lambda D'Db^\mu)] \\
&= (DD'a^\lambda)b^\mu + (-1)^{(\lambda+\nu')\nu} D'a^\lambda Db^\mu + (-1)^{\nu'\lambda} Da^\lambda D'b^\mu + (-1)^{\lambda\nu} (-1)^{\nu'\lambda} a^\lambda DD'b^\mu \\
&\quad - (-1)^{\nu\nu'} (D'Da^\lambda)b^\mu - (-1)^{\nu\nu'} (-1)^{\nu'\lambda} (-1)^{\nu'\nu} Da^\lambda D'b^\mu - (-1)^{\nu\nu'} (-1)^{\nu\lambda} D'a^\lambda Db^\mu \\
&\quad \quad \quad - (-1)^{\nu\nu'} (-1)^{\nu\lambda} (-1)^{\nu'\lambda} a^\lambda D'Db^\mu \\
&= ((DD'a^\lambda)b^\mu - (-1)^{\nu\nu'} (D'Da^\lambda)b^\mu) + ((-1)^{(\lambda+\nu')\nu} D'a^\lambda Db^\mu - (-1)^{\nu\nu'} (-1)^{\nu\lambda} D'a^\lambda Db^\mu) \\
&\quad + ((-1)^{\nu'\lambda} Da^\lambda D'b^\mu - (-1)^{\nu\nu'} (-1)^{\nu'\lambda} (-1)^{\nu'\nu} Da^\lambda D'b^\mu) \\
&\quad + ((-1)^{\lambda\nu} (-1)^{\nu'\lambda} a^\lambda DD'b^\mu - (-1)^{\nu\nu'} (-1)^{\nu\lambda} (-1)^{\nu'\lambda} a^\lambda D'Db^\mu) \\
&= ((DD'a^\lambda) - (-1)^{\nu\nu'} (D'Da^\lambda))b^\mu + ((-1)^{(\lambda+\nu')\nu} D'a^\lambda Db^\mu - (-1)^{\nu(\nu'+\lambda)} D'a^\lambda Db^\mu) \\
&\quad + ((-1)^{\nu'\lambda} Da^\lambda D'b^\mu - (-1)^{\nu'\lambda} Da^\lambda D'b^\mu) + (-1)^{\lambda\nu+\nu'\lambda} (a^\lambda DD'b^\mu - (-1)^{\nu\nu'} a^\lambda D'Db^\mu) \\
&= ([D, D']a^\lambda)b^\mu + (-1)^{\lambda(\nu+\nu')} a^\lambda ([D, D']b^\mu)
\end{aligned}$$

This shows that $[D, D']$ is a left derivation of degree $\nu + \nu'$.

Letting $\mathcal{D}_v = \mathcal{D}_v(A)$ denote the module of all left derivations of A of degree v , $D = \sum \mathcal{D}_v$ is therefore in a natural way a graded Lie algebra, graded by the same group as A , and the same is true for right derivations. These algebras are anti-isomorphic to each other.

Now we define, a new class of derivation, called inner derivations.

Definition 6.3 *If A is a graded associative ring and a^λ an element of degree λ , then the module endomorphism D_a , defined by setting for b^μ in A of degree μ .*

$$D_{a^\lambda} b^\mu = a^\lambda b^\mu - (-1)^{\lambda\mu} b^\mu a^\lambda \quad (6.2)$$

is the left derivation. (Analogously, a right derivation exists in a graded associative ring A , where a^λ is an element of degree λ . This right derivation, denoted by D'_a , is a module endomorphism defined as follows. For b^μ in A of degree μ , the following expression holds:

$$D'_{a^\lambda} b^\mu = b^\mu a^\lambda (-1)^{\lambda\mu} a^\lambda b^\mu.$$

Likewise, if A is a graded Lie ring of degree λ , then

$$D_{a^\lambda} b^\mu = [a^\lambda, b^\mu] \quad (6.3)$$

is a left derivation. (Analogously, a right derivation of degree λ is defined such that it satisfies the following property: $D'_{a^\lambda} b^\mu = [b^\mu, a^\lambda]$).

It is possible to extend the definitions of derivations of a ring A to a module P .

Definition 6.4 *Let P be a two-sided module over a graded ring A . Then a module homomorphism $D : A \rightarrow P$ is a left derivation respectively, right derivation, of degree v of A into P if it is*

homogeneous of degree ν and if $a^\lambda \in A_\lambda$, $b^\mu \in P_\mu$. The definitions of left and right derivations that we have previously provided hold, respectively, where now the multiplication is the module operation of A on P . Inner derivations can be defined as before, based on the definitions that we have previously provided, where b^μ is taken in P_μ .

To show that the homomorphism $D : A \rightarrow P$ becomes a right derivation of degree ν with respect to the new module structure on P , we need to show that for any $a, b \in A$ and $z \in P$, we have

$$D(ab)z = (D(a)J^\nu b + J^\nu aD(b))z$$

which proves that D is a right derivation of degree ν with respect to the new module structure on P .

If D is a left derivation of degree ν of A into P , and if we define a new two-sided A -module structure on P by defining the new product. For $a \in A$ and $z \in P$, the new product is the old product of $J^\nu a$ with z , i.e. $a \in A, z \in P$ then $az = (J^\nu a)z$, then the homomorphism $D : A \rightarrow P$, relative to the new module structure, becomes a right derivation of degree ν . In fact, Lets see

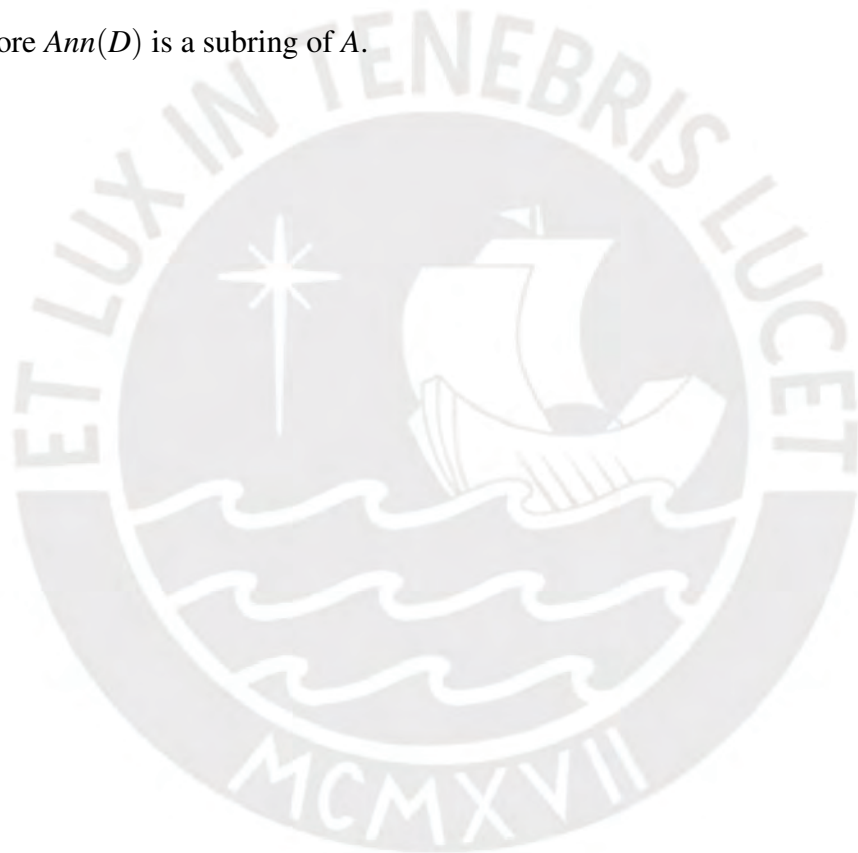
$$\begin{aligned} D(a^\lambda b^\mu) &= (Da^\lambda)b^\mu + (-1)^{\lambda\nu} a^\lambda (Db^\mu) \\ &= (Da^\lambda)J^\nu b^\mu + (-1)^{\lambda\nu} J^\nu a^\lambda (Db^\mu) \\ &= (Da^\lambda)((-1)^{\mu\nu} b^\mu) + (-1)^{\lambda\nu}((-1)^{\lambda\nu} a^\lambda)(Db^\mu) \\ &= (-1)^{\mu\nu} (Da^\lambda)b^\mu + a^\lambda (Db^\mu) \end{aligned}$$

And the last expression shows that D is right derivation of degree ν of A into P .

Let D be a derivation of A either into itself or into a two-sided module over A . Let the subset $\mathcal{K}(D) = \{a \in A / D(a) = 0\}$, the elements annihilated by D . We notice that

- For $a^\lambda, b^\mu \in \text{Ann}(D)$, we have $D(a - b) = Da - Db = 0$, then $a - b \in \text{Ann}(D)$.
- For $a^\lambda, b^\mu \in \text{Ann}(D)$, we have $D(ab) = (Da^\lambda)b^\mu - (-1)^{\lambda\nu}a^\lambda Db^\mu = 0$, then $ab \in \text{Ann}(D)$.

Therefore $\text{Ann}(D)$ is a subring of A .



Chapter 7

Pre-Lie System

Definition 7.1 Let $\dots V_{-1}, V_0, V_1, V_2, \dots$ be a sequence of S -modules and an assignment for every triple of integers $m, n, i \geq 0$ with $i \leq m$ of a homomorphism $\circ_i = \circ_i(m, n) : V_m \otimes V_n \rightarrow V_{m+n}$. If $f^m \in V_m, g^n \in V_n$, we shall denote $\circ_i(f \otimes g)$ by $f \circ_i g$. We define a right pre-Lie system by a pair $\{V_m, \circ_i\}$ with the following properties

$$(f^m \circ_i g^n) \circ_j h^p = \begin{cases} (f^m \circ_j h^p) \circ_{i+p} g^n & \text{if } 0 \leq j \leq i-1 \\ f^m \circ_i (g^n \circ_{j-i} h^p) & \text{if } i \leq j \leq n+i \end{cases} \quad (7.1)$$

f, g and h belongs to V_m, V_n and V_p respectively. From the first case of (7.1), we may deduce further that.

$$(f^m \circ_j h^p) \circ_{i+p} g^n = (f^m \circ_{i+p} g^n) \circ_{j+n} h^p \quad \text{if } 0 \leq i+p \leq j$$

$$(f^m \circ_i g^n) \circ_j h^p = (f^m \circ_{j-n} h^p) \circ_i g^n \quad \text{if } n+i+1 \leq j \leq m+n \quad (7.2)$$

We present some examples of right pre-Lie system.

- 1 Given any sequence V_{-m} , $m = 1, 2, \dots$, indexed by the negative integers, define $V_m = 0$, for $m \geq 0$ and define all the \circ_i to be zero.
- 2 Let A be an associative ring and set $V_0 = A$, $V_m = 0$ for $m \neq 0$. Then $\circ_i(m, n)$ is only be defined for $i = m = n = 0$, and we take to be the multiplication in A .
- 3 Let V be an S -module, and let V^m , be the tensor product of V , with itself m times, $m = 1, 2, \dots$, and let $V_m = V^m$, $m = 1, 2, \dots$. Set $V_0 = S$, $V_{-1} = V_{-2} = \dots = 0$. If $f \in V_m$, $g \in V_n$ are of the form $f = \alpha_1 \otimes \dots \otimes \alpha_m$, $g = \beta_1 \otimes \dots \otimes \beta_n$, we define $\circ_i = \circ_i(m, n)$ by

$$f^m \circ_i g^n = \alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m$$

Extending $\circ_i(m, n)$ linearly. Let $f^m = \alpha_1 \otimes \dots \otimes \alpha_m$, $g^n = \beta_1 \otimes \dots \otimes \beta_n$, $h^p = \rho_1 \otimes \dots \otimes \rho_p$. elements of V^m , V^n , V^p respectively.

- if $0 \leq j \leq i-1$

$$\begin{aligned} (f^m \circ_i g^n) \circ_j h^p &= ((\alpha_1 \otimes \dots \otimes \alpha_m) \circ_i (\beta_1 \otimes \dots \otimes \beta_n)) \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ &= (\alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m) \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \end{aligned}$$

$$= \alpha_1 \otimes \dots \otimes \alpha_j \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \alpha_{j+1} \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \\ \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m$$

And the other expression

$$(f^m \circ_j h^p) \circ_{i+p} g^n = ((\alpha_1 \otimes \dots \otimes \alpha_m) \circ_j (\rho_1 \otimes \dots \otimes \rho_p)) \circ_{i+p} (\beta_1 \otimes \dots \otimes \beta_n) \\ = (\alpha_1 \otimes \dots \otimes \alpha_j \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \alpha_{j+1} \otimes \dots \otimes \alpha_m) \circ_{i+p} (\beta_1 \otimes \dots \otimes \beta_n) \\ = (\alpha_1 \otimes \dots \otimes \alpha_j \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \tilde{\alpha}_{j+p+1} \otimes \dots \otimes \tilde{\alpha}_{m+n+p}) \\ \circ_{i+p} (\beta_1 \otimes \dots \otimes \beta_n) \\ = \alpha_1 \otimes \dots \otimes \alpha_j \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \tilde{\alpha}_{j+p+1} \otimes \dots \otimes \tilde{\alpha}_{i+p} \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \\ \tilde{\alpha}_{i+p+1} \otimes \dots \otimes \tilde{\alpha}_{m+n+p} \\ = \alpha_1 \otimes \dots \otimes \alpha_j \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \alpha_{j+1} \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \\ \alpha_{i+1} \otimes \dots \otimes \alpha_{m+n}$$

thus $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_j h^p) \circ_{i+p} g^n$, are equal when $0 \leq j \leq i-1$.

- if $i \leq j \leq n+i$

$$(f^m \circ_i g^n) \circ_j h^p = ((\alpha_1 \otimes \dots \otimes \alpha_m) \circ_i (\beta_1 \otimes \dots \otimes \beta_n)) \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ = (\alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m) \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ = (\alpha_1 \otimes \dots \otimes \alpha_i \otimes \tilde{\alpha}_{i+1} \otimes \dots \otimes \tilde{\alpha}_{i+n} \otimes \tilde{\alpha}_{i+n+1} \otimes \dots \otimes \tilde{\alpha}_{m+n}) \\ \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ = \alpha_1 \otimes \dots \otimes \alpha_i \otimes \tilde{\alpha}_{i+1} \otimes \dots \otimes \tilde{\alpha}_j \otimes \rho_1 \otimes \dots \otimes \rho_n \otimes \tilde{\alpha}_{j+1} \otimes \dots \otimes \tilde{\alpha}_{i+n} \\ \otimes \tilde{\alpha}_{i+n+1} \otimes \dots \otimes \tilde{\alpha}_{m+n} \\ = \alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_{j-i} \otimes \rho_1 \otimes \dots \otimes \rho_n \otimes \beta_{j-i+1} \otimes \dots \otimes \beta_n$$

$$\otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m$$

And the other expression

$$\begin{aligned} f^m \circ_i (g^n \circ_{j-i} h^p) &= (\alpha_1 \otimes \dots \otimes \alpha_m) \circ_i ((\beta_1 \otimes \dots \otimes \beta_n) \circ_{j-i} (\rho_1 \otimes \dots \otimes \rho_p)) \\ &= (\alpha_1 \otimes \dots \otimes \alpha_m) \circ_i (\beta_1 \otimes \dots \otimes \beta_{j-i} \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \beta_{j-i+1} \otimes \dots \otimes \beta_n) \\ &= \alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_{j-i} \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \beta_{j-i+1} \otimes \dots \otimes \beta_n \\ &\quad \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m \end{aligned}$$

Thus $(f^m \circ_i g^n) \circ_j h^p$ and $f^m \circ_i (g^n \circ_{j-i} h^p)$, are equal when $i \leq j \leq n+i$.

- If $n+i+1 \leq j \leq m+n$

$$\begin{aligned} (f^m \circ_i g^n) \circ_j h^p &= ((\alpha_1 \otimes \dots \otimes \alpha_m) \circ_i (\beta_1 \otimes \dots \otimes \beta_n)) \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ &= (\alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_m) \\ &\quad \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ &= (\alpha_1 \otimes \dots \otimes \alpha_i \otimes \tilde{\alpha}_{i+1} \otimes \dots \otimes \tilde{\alpha}_{n+i} \otimes \tilde{\alpha}_{n+i+1} \otimes \dots \otimes \tilde{\alpha}_{m+n}) \\ &\quad \circ_j (\rho_1 \otimes \dots \otimes \rho_p) \\ &= \alpha_1 \otimes \dots \otimes \alpha_i \otimes \tilde{\alpha}_{i+1} \otimes \dots \otimes \tilde{\alpha}_{n+i} \otimes \tilde{\alpha}_{n+i+1} \otimes \dots \otimes \tilde{\alpha}_j \\ &\quad \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \tilde{\alpha}_{j+1} \otimes \dots \otimes \tilde{\alpha}_{m+n} \\ &= \alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_{j-n} \otimes \rho_1 \otimes \dots \otimes \rho_p \\ &\quad \otimes \alpha_{j-n+1} \otimes \dots \otimes \alpha_m \end{aligned}$$

And the other expression

$$\begin{aligned}
(f^m \circ_{j-n} h^p) \circ_i g^n &= ((\alpha_1 \otimes \dots \otimes \alpha_m) \circ_{j-n} (\rho_1 \otimes \dots \otimes \rho_p)) \circ_i (\beta_1 \otimes \dots \otimes \beta_n) \\
&= (\alpha_1 \otimes \dots \otimes \alpha_{j-n} \otimes \rho_1 \otimes \dots \otimes \rho_p \otimes \alpha_{j-n+1} \otimes \dots \otimes \alpha_m) \\
&\quad \circ_i (\beta_1 \otimes \dots \otimes \beta_n) \\
&= \alpha_1 \otimes \dots \otimes \alpha_i \otimes \beta_1 \otimes \dots \otimes \beta_n \otimes \alpha_{i+1} \otimes \dots \otimes \alpha_{j-n} \otimes \rho_1 \otimes \dots \otimes \rho_p \\
&\quad \otimes \alpha_{j-n+1} \otimes \dots \otimes \alpha_m
\end{aligned}$$

Thus $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_{j-n} h^p) \circ_i g^n$ are equal when $n + i + 1 \leq j \leq m + n$.

Therefore it is a pre-Lie system. In this example \circ_i is an isomorphism, and in fact $\circ_m(m, n)$ is the identity. If $m = 0$, in which case f is an element s of S , then $s \otimes g$ is identify with $sg = gs$, \circ_i is then defined for $i = 0$ and $s \circ_0 g = sg$. On the other hand if $n = 0$, then $f \otimes s = sf = f \circ_i s$, for all $i = 0, 1, \dots$

4 Let V, V^m be elements as in example (3), and define $V_m = Hom_S(S, S)$, $m = 1, 2, \dots$

Set $V_{-1} = V_{-2} = \dots = 0$ and $V_0 = Hom_S(S, S)$ which we identify with S . If $f \in V_m$, $g \in V_n$, we may define an element of V_{m+n} setting $fg \in V_{m+n}$ by $fg(a \otimes b) = f(a)g(b)$ where $a \in V^m$, $b \in V^n$ and then extending linearly. Then we may define homomorphism $\circ_i : V_m \otimes V_n \rightarrow V_{m+n}$ by $f \circ_i g(a \otimes b) = fg(a \circ_{i'} b)$, extended linearly, where $\circ_{i'} = \circ_{i'}(m, n)$ is the inverse of the application defined in (3). We are going to see that the application is a pre-Lie system.

- If $0 \leq j \leq i - 1$

$$\begin{aligned}
& (f^m \circ_i g^n) \circ_j h^p (a_1 \otimes \dots \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_i \otimes \dots \otimes a_{m+n+p}) = \\
& (f^m \circ_i g^n) \circ_j h^p (a_1 \otimes \dots \otimes a_j \otimes c_1 \otimes \dots \otimes c_p \otimes a_{j+p+1} \otimes \dots \otimes a_i \otimes \dots \otimes a_{m+n+p}) = \\
& (f^m \circ_i g^n) h^p (a_1 \otimes \dots \otimes a_j \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p} \otimes c_1 \otimes \dots \otimes c_p) = \\
& (f^m \circ_i g^n) h^p (a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n} \otimes c_1 \otimes \dots \otimes c_p) = \\
& (f^m \circ_i g^n) (a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes a_i \otimes \dots \otimes a_{i+n} \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p (c_1 \otimes \dots \otimes c_p) = \\
& (f^m \circ_i g^n) (a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_n \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p (c_1 \otimes \dots \otimes c_p) = \\
& f^m g^n (a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n} \otimes b_1 \otimes \dots \otimes b_n) h^p (c_1 \otimes \dots \otimes c_p) = \\
& f^m g^n (a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m \otimes b_1 \otimes \dots \otimes b_n) h^p (c_1 \otimes \dots \otimes c_p) = \\
& f^m (a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) g^n (b_1 \otimes \dots \otimes b_n) h^p (c_1 \otimes \dots \otimes c_p)
\end{aligned}$$

And the other expression

$$\begin{aligned}
& (f^m \circ_j h^p) \circ_{i+p} g^n (a_1 \otimes \dots \otimes a_{i+p-1} \otimes a_{i+p} \otimes a_{i+p+1} \otimes \dots \otimes a_{i+p+n} \otimes \\
& a_{i+p+n+1} \otimes \dots \otimes a_{m+p+n}) = \\
& (f^m \circ_j h^p) \circ_{i+p} g^n (a_1 \otimes \dots \otimes a_{i+p-1} \otimes a_{i+p} \otimes b_1 \otimes \dots \otimes b_n \otimes a_{i+p+n+1} \otimes \dots \otimes a_{m+p+n}) = \\
& (f^m \circ_j h^p) g^n (a_1 \otimes \dots \otimes a_{i+p-1} \otimes a_{i+p} \otimes a_{i+p+n+1} \otimes \dots \otimes a_{m+p+n} \otimes b_1 \otimes \dots \otimes b_n) = \\
& (f^m \circ_j h^p) (a_1 \otimes \dots \otimes a_{i+p-1} \otimes a_{i+p} \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) g^n (b_1 \otimes \dots \otimes b_n) = \\
& (f^m \circ_j h^p) (a_1 \otimes \dots \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{i+p} \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) \\
& g^n (b_1 \otimes \dots \otimes b_n) = \\
& f^m \circ_j h^p (a_1 \otimes \dots \otimes a_j \otimes c_1 \otimes \dots \otimes c_p \otimes a_{j+p+1} \otimes \dots \otimes a_{i+p} \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) g^n (b_1 \otimes \dots \otimes b_n) = \\
& f^m h^p (a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m \otimes c_1 \otimes \dots \otimes c_p) g^n (b_1 \otimes \dots \otimes b_n) =
\end{aligned}$$

$$f^m(a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) h^p(c_1 \otimes \dots \otimes c_p) g^n(b_1 \otimes \dots \otimes b_n)$$

Thus $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_j h^p) \circ_{i+p} g^n$ are equal when $0 \leq j \leq i - 1$.

- If $i \leq j \leq n + i$

$$\begin{aligned} (f^m \circ_i g^n) \circ_j h^p(a_1 \otimes \dots \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_i g^n) \circ_j h^p(a_1 \otimes \dots \otimes a_j \otimes c_1 \otimes \dots \otimes c_p \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_i g^n) h^p(a_1 \otimes \dots \otimes a_j \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p} \otimes c_1 \otimes \dots \otimes c_p) &= \\ (f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p(c_1 \otimes \dots \otimes c_p) &= \\ (f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes a_{i+n} \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p(c_1 \otimes \dots \otimes c_p) &= \\ (f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_n \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p(c_1 \otimes \dots \otimes c_p) &= \\ f^n g^n(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n} \otimes b_1 \otimes \dots \otimes b_n) h^p(c_1 \otimes \dots \otimes c_p) &= \\ f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) g^n(b_1 \otimes \dots \otimes b_n) h^p(c_1 \otimes \dots \otimes c_p) &= \end{aligned}$$

And the other expression

$$\begin{aligned} f^m \circ_i (g^n \circ_{j-i} h^p)(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+n+p} \otimes a_{i+n+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ f^m \circ_i (g^n \circ_{j-i} h^p)(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_{n+p} \otimes a_{i+n+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ f^m (g^n \circ_{j-i} h^p)(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m \otimes b_1 \otimes \dots \otimes b_{n+p}) &= \\ f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \end{aligned}$$

$$\begin{aligned}
(g^n \circ_{j-i} h^p)(b_1 \otimes \dots \otimes b_{j-i} \otimes b_{j-i+1} \otimes \dots \otimes b_{j-i+p} \otimes b_{j-i+p+1} \otimes \dots \otimes b_{n+p}) &= \\
f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) & \\
(g^n \circ_{j-i} h^p)(b_1 \otimes \dots \otimes b_{j-i} \otimes c_1 \otimes \dots \otimes c_p \otimes b_{j-i+p+1} \otimes \dots \otimes b_{n+p}) &= \\
f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) & \\
g^n h^p(b_1 \otimes \dots \otimes b_{j-i} \otimes \tilde{b}_{j-i+1} \otimes \dots \otimes \tilde{b}_n \otimes c_1 \otimes \dots \otimes c_p) &= \\
f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) g^n(b_1 \otimes \dots \otimes b_{j-i} \otimes \tilde{b}_{j-i+1} \otimes \dots \otimes \tilde{b}_n) & \\
h^p(c_1 \otimes \dots \otimes c_p) &
\end{aligned}$$

Thus $(f^m \circ_i g^n) \circ_j h^p$ and $f^m \circ_i (g^n \circ_{j-i} h^p)$ are equal when $0 \leq j \leq i-1$.

- If $n+i+1 \leq j \leq m+n$

$$\begin{aligned}
(f^m \circ_i g^n) \circ_j h^p(a_1 \otimes \dots \otimes a_j \otimes a_{j+1} \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\
(f^m \circ_i g^n) \circ_j h^p(a_1 \otimes \dots \otimes a_j \otimes c_1 \otimes \dots \otimes c_p \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\
(f^m \circ_i g^n) h^p(a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n} \otimes c_1 \otimes \dots \otimes c_p) &= \\
(f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) h^p(c_1 \otimes \dots \otimes c_p) &= \\
(f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+n} \otimes a_{i+n+1} \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) & \\
h^p(c_1 \otimes \dots \otimes c_p) &= \\
(f^m \circ_i g^n)(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_n \otimes a_{i+n+1} \otimes \dots \otimes a_j \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) & \\
h^p(c_1 \otimes \dots \otimes c_p) &= \\
(f^m g^n)(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m \otimes b_1 \otimes \dots \otimes b_n) & \\
h^p(c_1 \otimes \dots \otimes c_p) &=
\end{aligned}$$

$$f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m) g^n(b_1 \otimes \dots \otimes b_n) \\ h^p(c_1 \otimes \dots \otimes c_p)$$

And the other expression

$$\begin{aligned} (f^m \circ_{j-n} h^p) \circ_i g^n(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+n} \otimes a_{i+n+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_{j-n} h^p) \circ_i g^n(a_1 \otimes \dots \otimes a_i \otimes b_1 \otimes \dots \otimes b_n \otimes a_{i+n+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_{j-n} h^p) g^n(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{m+p} \otimes b_1 \otimes \dots \otimes b_n) &= \\ (f^m \circ_{j-n} h^p)(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{m+p}) g^n(b_1 \otimes \dots \otimes b_n) &= \\ (f^m \circ_{j-n} h^p)(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_{j-n+p} \otimes \tilde{a}_{j-n+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) &= \\ g^n(b_1 \otimes \dots \otimes b_n) &= \\ (f^m \circ_{j-n} h^p)(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes c_1 \otimes \dots \otimes c_p \otimes \tilde{a}_{j-n+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) &= \\ g^n(b_1 \otimes \dots \otimes b_n) &= \\ f^m h^p(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m \otimes c_1 \otimes \dots \otimes c_p) &= \\ g^n(b_1 \otimes \dots \otimes b_n) &= \\ f^m(a_1 \otimes \dots \otimes a_i \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n} \otimes \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m) h^p(c_1 \otimes \dots \otimes c_p) &= \\ g^n(b_1 \otimes \dots \otimes b_n) &= \end{aligned}$$

Thus $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_{j-n} h^p) \circ_i g^n$ are equal when $n + i + 1 \leq j \leq m + n$.

Therefore $\{V_m, \circ_i\}$ is a pre-Lie system.

The next example is the most relevant, and it will help us to understand that deformations form a form a pre-Lie system.

5 Let V, V^m as in example (3). W be an S -module, different from the firsts. Let $\phi : W \rightarrow V$ be an S -module homomorphism. Set $V_m = Hom_S(V^{m+1}, W)$, $m = 0, 1, \dots$, $V_{-1} =$

$Hom_S(S, W)$, which we may identify with W , and $V_{-2} = V_{-3} = \dots = 0$.

If $f \in V_m, g \in V_n, m, n \geq 0$, then $f \circ_i g \in V_{m+n} = Hom_S(V^{m+n+1}, W)$ is defined by

$$f \circ_i g(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_n \otimes a_{i+1} \otimes \dots \otimes a_m) = f(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g(b_0 \otimes \dots \otimes b_n) \otimes a_{i+1} \otimes \dots \otimes a_m)$$

We are going to see that the pre-Lie system conditions hold.

Let f^m, g^n, h^p belongs to V_m, V_n, V_p , respectively.

- If $0 \leq j \leq i-1$

$$\begin{aligned} (f^m \circ_i g^n) \circ_j h^p(a_0 \otimes \dots \otimes a_{j-1} \otimes a_j \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_i g^n) \circ_j h^p(a_0 \otimes \dots \otimes a_{j-1} \otimes c_0 \otimes \dots \otimes c_p \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) &= \\ (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) &= \\ (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes \tilde{a}_i \otimes \dots \otimes \tilde{a}_{i+n} &= \\ \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) &= \\ (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes b_0 \otimes \dots \otimes b_n &= \\ \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m(a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) &= \\ \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \end{aligned}$$

And the other expression

$$(f^m \circ_j h^p) \circ_{i+p} g^n(a_0 \otimes \dots \otimes a_{i+p-1} \otimes a_{i+p} \otimes \dots \otimes a_{i+p+n} \otimes a_{i+p+n+1} \otimes \dots \otimes a_{m+n+p}) =$$

$$\begin{aligned}
& (f^m \circ_j h^p) \circ_{i+p} g^n (a_0 \otimes \dots \otimes a_{i+p-1} \otimes b_0 \otimes \dots \otimes b_n \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_j h^p) (a_0 \otimes \dots \otimes a_{i+p-1} \otimes \phi g^n (b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_j h^p) (a_0 \otimes \dots \otimes a_{j-1} \otimes a_j \otimes \dots \otimes a_{j+p} \otimes \dots \otimes a_{i+p-1} \otimes \phi g^n (b_0 \otimes \dots \otimes b_n) \\
& \quad \otimes \tilde{a}_{i+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_j h^p) (a_0 \otimes \dots \otimes a_{j-1} \otimes c_0 \otimes \dots \otimes c_p \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes (b_0 \otimes \dots \otimes b_n) \\
& \quad \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) = \\
& f^m (a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p (c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes \phi g^n (b_0 \otimes \dots \otimes b_n) \\
& \quad \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m)
\end{aligned}$$

And we see that $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_j h^p) \circ_{i+p} g^n$ are equal when $0 \leq j \leq i-1$.

- If $i \leq j \leq n+i$

$$\begin{aligned}
& (f^m \circ_i g^n) \circ_j h^p (a_0 \otimes \dots \otimes a_{j-1} \otimes a_j \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) = \\
& (f^m \circ_i g^n) \circ_j h^p (a_0 \otimes \dots \otimes a_{j-1} \otimes c_0 \otimes \dots \otimes c_p \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n+p}) = \\
& (f^m \circ_i g^n) (a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p (c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) = \\
& (f^m \circ_i g^n) (a_0 \otimes \dots \otimes a_{i-1} \otimes a_i \otimes \dots \otimes a_{j-1} \otimes \phi h^p (c_0 \otimes \dots \otimes c_p) \otimes \\
& \quad \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i+n} \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) = \\
& (f^m \circ_i g^n) (a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_{j-i-1} \otimes \phi h^p (c_0 \otimes \dots \otimes c_p) \otimes \\
& \quad b_{j-i+1} \otimes \dots \otimes b_n \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) = \\
& f^m (a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n (b_0 \otimes \dots \otimes b_{j-i-1} \otimes \phi h^p (c_0 \otimes \dots \otimes c_p) \otimes
\end{aligned}$$

$$b_{j-i+1} \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m)$$

And the other expression

$$\begin{aligned} f^m \circ_i (g^n \circ_{j-i} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes a_i \otimes \dots \otimes a_{n+p+i} \otimes a_{n+p+i+1} \otimes \dots \otimes a_{m+n+p}) &= \\ f^m \circ_i (g^n \circ_{j-i} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_{n+p} \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi(g^n \circ_{j-i} h^p)(b_0 \otimes \dots \otimes b_{n+p}) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m \circ_i (g^n \circ_{j-i} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_{n+p} \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi(g^n \circ_{j-i} h^p)(b_0 \otimes \dots \otimes b_{j-i-1} \otimes b_{j-i} \otimes \dots \otimes b_{j-i+p} \otimes & \\ b_{j-i+p+1} \otimes \dots \otimes b_{n+p}) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi(g^n \circ_{j-i} h^p)(b_0 \otimes \dots \otimes b_{j-i-1} \otimes c_0 \otimes \dots \otimes c_p \otimes & \\ \tilde{b}_{j-i+1} \otimes \dots \otimes \tilde{b}_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) &= \\ f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_{j-i-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes & \\ \tilde{b}_{j-i+1} \otimes \dots \otimes \tilde{b}_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_m) & \end{aligned}$$

And we see that $(f^m \circ_i g^n) \circ_j h^p$ and $f^m \circ_i (g^n \circ_{j-i} h^p)$ are equal when $i \leq j \leq n+i$.

- If $n+i+1 \leq j \leq m+n$

$$\begin{aligned} (f^m \circ_i g^n) \circ_j h^p(a_0 \otimes \dots \otimes a_{j-1} \otimes a_j \otimes \dots \otimes a_{j+p} \otimes a_{j+p+1} \otimes \dots \otimes a_{m+n+p}) &= \\ (f^m \circ_i g^n) \circ_j h^p(a_0 \otimes \dots \otimes a_{j-1} \otimes c_0 \otimes \dots \otimes c_p \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) &= \\ (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{j-1} \otimes \phi h^p(c_0 \otimes \dots \otimes c_p) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) &= \end{aligned}$$

$$\begin{aligned}
& (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{i-1} \otimes a_i \otimes \dots \otimes a_{i+n} \otimes a_{i+n+1} \otimes \dots \otimes a_{j-1} \otimes \overline{\phi h^p}(c_0 \otimes \dots \otimes c_p) \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) = \\
& (f^m \circ_i g^n)(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_n \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n-1} \otimes \overline{\phi h^p}(c_0 \otimes \dots \otimes c_p) \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n}) = \\
& f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n-1} \otimes \overline{\phi h^p}(c_0 \otimes \dots \otimes c_p) \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{m+n})
\end{aligned}$$

And the other expression

$$\begin{aligned}
& (f^m \circ_{j-n} h^p) \circ_i g^n(a_0 \otimes \dots \otimes a_{i-1} \otimes a_i \otimes \dots \otimes a_{i+n} \otimes a_{i+n+1} \otimes \dots \otimes a_{m+n+p}) = \\
& (f^m \circ_{j-n} h^p) \circ_i g^n(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_n \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_{j-n} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_{j-n} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n-1} \otimes \tilde{a}_{j-n} \otimes \dots \otimes \tilde{a}_{j-n+p} \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j-n+p+1} \otimes \dots \otimes \tilde{a}_{m+p}) = \\
& (f^m \circ_{j-n} h^p)(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n-1} \otimes c_0 \otimes \dots \otimes c_p \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m) = \\
& f^m(a_0 \otimes \dots \otimes a_{i-1} \otimes \phi g^n(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+1} \otimes \dots \otimes \tilde{a}_{j-n-1} \otimes \overline{\phi h^p}(c_0 \otimes \dots \otimes c_p) \otimes \\
& \qquad \qquad \qquad \tilde{a}_{j-n+1} \otimes \dots \otimes \tilde{a}_m)
\end{aligned}$$

And we see that $(f^m \circ_i g^n) \circ_j h^p$ and $(f^m \circ_{j-n} h^p) \circ_i g^n$ are equal. when $n + i + 1 \leq j \leq m + n$.

We have showed that $\{V_m = Hom_S(V^{m+1}, W), \circ_i\}$ is a pre-Lie system.

Chapter 8

The graded Lie rings of a pre-Lie system

Given any pre-Lie system $\{V_m, \circ_i\}$, we can define a new homomorphism \circ of $V_m \otimes V_n$ into V_{m+n} , for every m and n , by setting for $f^m \in V_m, g^n \in V_n$, with $m \geq 0$.

$$f^m \circ g^n = \sum_{i=0}^m (-1)^{ni} f^m \circ_i g^n \quad (8.1)$$

In other words.

$$f^m \circ g^n = \begin{cases} f^m \circ_0 g^n + f^m \circ_1 g^n + f^m \circ_2 g^n + \dots + f^m \circ_m g^n & \text{if } n \text{ is even} \\ f^m \circ_0 g^n - f^m \circ_1 g^n + f^m \circ_2 g^n + \dots + (-1)^m f^m \circ_m g^n & \text{if } n \text{ is odd} \end{cases}$$

and for $m \leq 0$.

$$f^m \circ g^n = 0$$

For the following theorem, we are going to consider λ, μ, ν as integers.

Theorem 8.1 *Let $\{V_m, \circ_i\}$ be a pre-Lie system and f^m, g^n, h^p be elements of V_m, V_n, V_p respectively. Then*

$$i) (f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum' (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p$$

where the sum \sum' is extended over those i and j with either $0 \leq j \leq i-1$ or

$$n+i+1 \leq j \leq m+n.$$

$$i) (f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np} [(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$$

Proof.

- First we observe the term $(f^m \circ g^n) \circ h^p$ and extend this term according to (8.1). We see that the term occurs with coefficients $(-1)^{ni+pj}$. If we transform this term according to (7.1) and (7.2), we can transform this term to a term of the form $\pm f^m \circ_\lambda (g^n \circ_\mu h^p)$, which occurs if and only if one can find $i \geq 0$ and j with $i \leq j \leq n+i$ such that $\lambda = i$ and $\mu = i-1$. This says, that every term of the form $\pm f^m \circ_\lambda (g^n \circ_\mu h^p)$ occurs. It occurs with coefficient $(-1)^{\lambda n + (\lambda + \mu)p} = (-1)^{\mu p + \lambda(n+p)}$. And these coefficients appears when transform $f^m \circ (g^n \circ h^p)$ by (8.1). Therefore the terms that vanish from $(f^m \circ g^n) \circ h^p$ are those terms $(f^m \circ_i g^n) \circ_j h^p$ with $i \leq j \leq n+i$. And that complete the first part.

We can give a demonstration explicitly.

– By (8.1) we get

$$(f^m \circ g^n) \circ h^p = \sum_{j=0}^{m+n} (-1)^{pj} (f^m \circ g^n) \circ_j h^p, \quad \text{and} \quad f^m \circ g^n = \sum_{i=0}^m (-1)^{ni} f^m \circ_i g^n$$

$$\begin{aligned} (f^m \circ g^n) \circ h^p &= \sum_{j=0}^{m+n} (-1)^{pj} \left(\sum_{i=0}^m (-1)^{ni} f^m \circ_i g^n \right) \circ_j h^p \\ &= \sum_{j=0}^{m+n} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p \\ &= \sum_{j=0}^{i-1} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p + \\ &\quad \sum_{j=i}^{n+i} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p + \\ &\quad \sum_{j=n+i+1}^{m+n} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p \\ &= \sum_{j=0}^{i-1} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p + \\ &\quad \sum_{j=i}^{n+i} \sum_{i=0}^m (-1)^{ni+pj} f^m \circ_i (g^n \circ_{j-i} h^p) + \\ &\quad \sum_{j=n+i+1}^{m+n} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p \end{aligned}$$

– By (8.1) we get

$$f^m \circ (g^n \circ h^p) = \sum_{i=0}^m (-1)^{ni+pi} f^m \circ_i (g^n \circ h^p), \quad \text{and} \quad g^n \circ h^p = \sum_{j=0}^n (-1)^{pj} g^n \circ_j h^p$$

$$\begin{aligned} f^m \circ (g^n \circ h^p) &= \sum_{i=0}^m (-1)^{ni+pi} f^m \circ_i \left(\sum_{j=0}^n (-1)^{pj} g^n \circ_j h^p \right) \\ &= \sum_{i=0}^m \sum_{j=0}^n (-1)^{ni+p(i+j)} f^m \circ_i (g^n \circ_j h^p) \end{aligned}$$

$$= \sum_{i=0}^m \sum_{\hat{j}=i}^{n+i} (-1)^{ni+p\hat{j}} f^m \circ_i (g^n \circ_{\hat{j}-i} h^p)$$

where $\hat{j} = j + i$

– Finally

$$\begin{aligned} (f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) &= \sum_{j=0}^{i-1} \sum_{i=0}^m (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p + \\ &\sum_{i=0}^m \sum_{j=i}^{n+i} (-1)^{ni+pj} f^m \circ_i (g^n \circ_{j-i} h^p) + \\ &\sum_{i=0}^m \sum_{j=n+i+1}^{m+n} (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p - \\ &\sum_{i=0}^m \sum_{\hat{j}=i}^{n+i} (-1)^{ni+p\hat{j}} f^m \circ_i (g^n \circ_{\hat{j}-i} h^p) \\ &= \sum_{i=0}^m \sum_{j=0}^{i-1} (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p + \\ &\sum_{i=0}^m \sum_{j=n+i+1}^{m+n} (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p \end{aligned}$$

- From *i*) we have $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum' (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p$ with $0 \leq j \leq i-1$ or $n+i+1 \leq j \leq m+n$. This may be expressed as a sum of terms of the form $\pm (f^m \circ_j h^p) \circ_{i+p} g^n$ when $0 \leq j \leq i-1$ and also as a sum of terms of the form $\pm (f^m \circ_{j-n} h^p) \circ_i g^n$ when $n+i+1 \leq j \leq m+n$ that is $(f^m \circ_\lambda h^p) \circ_\mu g^n$ occurs if and only if either $\lambda = j$ and $\mu = i+p$ with $0 \leq j \leq i-1$ or $\lambda = j-n$ and $\mu = i$ with $n+i+1 \leq j \leq m+n$. From the first inequality we get $0 \leq \lambda \leq \mu - p - 1 \leq \mu - p$ and from the second inequality $i \leq i+1 \leq j-n$ or when $\mu \leq i+1 \leq \lambda$. In other words, the sum can be express as a sum of term in these to forms, when we do not have $\lambda \leq \mu \leq \lambda + p$.

And from (7.1) we have that the terms $(f^m \circ_\lambda h^p) \circ_\mu g^n$ can not be modified into $f^m \circ_\lambda (h^p \circ_\mu g^n)$. In other words, the terms that only occur are those occurring in $(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)$, As for the sign $(f^m \circ_j h^p) \circ_{i+p} g^n$ occurs with sign $(-1)^{pj+n(i+p)} = (-1)^{ni+pj}(-1)^{np}$. In the same way $(f^m \circ_{j-n} h^p) \circ_i g^n$ occurs with the sign $(-1)^{p(j-n)+ni} = (-1)^{ni+pj}(-1)^{np}$. Which is in both cases $(-1)^{np}$ times the factor with which it appears in $(f^m \circ g^n) \circ h^p$. Therefore

$$(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np}[(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$$

This completes the proof.

The next Corollary is of great importance because says how from a pre-Lie system $\{V_m, \circ_i\}$ we get a pre-Lie ring.

Corollary 8.2 *Let $\{V_m, \circ_i\}$ be a pre-Lie system and let $A = \sum V_m$ be the direct sum of the modules V_m made into a ring by extending \circ to be an S -module homomorphism of $A \otimes A$ into A . Then with this multiplication, A becomes a right pre-Lie ring graded by the integers.*

Proof. Form the theorem part *ii*), we have that

$$(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np}[(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$$

and this can also be expressed

$$\begin{aligned} (f^m \circ g^n) \circ h^p - (-1)^{np}(f^m \circ h^p) \circ g^n &= f^m \circ (g^n \circ h^p) - (-1)^{np}f^m \circ (h^p \circ g^n) \\ &= f^m \circ [(g^n \circ h^p) - (-1)^{np}(h^p \circ g^n)] \end{aligned}$$

and this is the definition of a pre-Lie ring.

If A is defined as in the previous corollary, we can say that A is the pre-Lie ring graded by integers that is associated with the pre-Lie system $\{V_m, \circ_i\}$. Similarly, the graded Lie ring derived from A by Theorem 4.1 in §4 will be referred to as the graded Lie ring associated with the pre-Lie system $\{V_m, \circ_i\}$.

If $\{V_m, \circ_i\}$ is a pre-Lie system and $W = \sum W_m$ is a direct sum of modules indexed by the integers, then W has the structure of a right module over the pre-Lie system $\{V_m, \circ_i\}$ if there exists a module homomorphism (which we shall denote by $\circ_i(m, n)$), $\circ_i : W_m \otimes V_n \rightarrow W_{m+n}$, such that if $f = f^m \in W_m$, $g = g^n \in V_n$, and $h_p \in V_p$, then equations (7.1) and (7.2) hold.

Once we have defined a module W over a pre-Lie system $\{V_m, \circ_i\}$, we can also define on W the structure of a right module over the pre-Lie ring associated with $\{V_m, \circ_i\}$ using equation (8.1), i.e., $f^m \circ g^n = \sum_{i=0}^m (-1)^{ni} f^m \circ_i g^n$, where $f^m \in V_m$ and $g^n \in V_n$. Therefore, it follows from equation (5.3) in §5 that W also has the structure of a module over the graded Lie ring associated with $\{V_n, \circ_i\}$. As a brief comment, note that a pre-Lie system, in particular those in the examples, naturally give rise to a graded Lie ring using Theorems 4.1 and 8.1.

Chapter 9

Cohomology of a ring and commutative of the cup product

Let A be an associative ring and P be a two-sided A -module. We have defined an m -cochain f^m of A with coefficients in P to be an S -module homomorphism

$$f^m : A^{(m)} : \underbrace{A \otimes A \otimes \dots \otimes A}_{m\text{-times}} \implies P$$

The module $\text{Hom}_S(A^{(m)}, P)$ of all such f^m will be denoted by $C^m(A, P)$, S being understood

$$\begin{aligned} C^0(A, P) &\approx P \\ C^1(A, P) &= \{f : A \longrightarrow P : f \text{ homomorphism}\} \\ C^2(A, P) &= \{f : A \otimes A \longrightarrow P : f \text{ homomorphism}\} \\ &\vdots \\ C^m(A, P) &= \{f^m : \underbrace{A \otimes A \otimes \dots \otimes A}_{m\text{-times}} \longrightarrow P : f \text{ homomorphism}\} \\ &\vdots \end{aligned}$$

There is defined a homomorphism $\delta_m : C^m(A, P) \longrightarrow C^{m+1}(A, P)$ defined by setting

$$\begin{aligned} \delta_m f(a_1 \otimes \dots \otimes a_{m+1}) = & a_1 f(a_2 \otimes \dots \otimes a_{m+1}) + \sum_{i=1}^m (-1)^i f(a_1 \otimes \dots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_{m+1}) \\ & + (-1)^{m+1} f(a_1 \otimes \dots \otimes a_m) a_{m+1} \end{aligned}$$

and extending δ_m linearly.

We have defined $Z^m(A, P) = \ker \delta_m$ and $B^m(A, P) = \text{Im } \delta_{m+1}$, for $m \geq 1$ and is zero for $m = 0$. We also have $\delta_{m+1} \delta_m = 0$ which implies $\text{Im } \delta_m \subseteq \ker \delta_{m+1}$ or $B^m(A, P) \subseteq Z^m(A, P)$ and $H^m(A, P)$ is defined by $\frac{Z^m(A, P)}{B^m(A, P)}$ for $m \geq 0, 1$. We will denote $\sum C^m(A, P)$, $\sum Z^m(A, P)$, $\sum H^m(A, P)$ by $C^*(A, P)$, $Z^*(A, P)$, $H^*(A, P)$ respectively.

Now we give P an associative structure, and the multiplications in P induces a homomorphism called the cup product.

$$\smile : C^m(A, P) \otimes C^n(A, P) \longrightarrow C^{m+n}(A, P)$$

for $m = 0, 1, 2, \dots$. The cup product is defined by setting

$$f^m \smile g^n(a_1 \otimes \dots \otimes a_m \otimes b_1 \otimes \dots \otimes b_n) = f^m(a_1 \otimes \dots \otimes a_m) g^n(b_1 \otimes \dots \otimes b_n)$$

under this multiplication $C^*(A, P)$ becomes an associative ring graded by integers. We assert that

$$\delta(f^m \smile g^n) = \delta f^m \smile g^n + (-1)^m f^m \smile \delta g^n \quad (9.1)$$

$\delta(f^m \smile g^n) \in C^{m+n+1}(A, P)$, then

$\delta(f^m \smile g^n)(c_1 \otimes c_2 \otimes \dots \otimes c_m \otimes c_{m+1} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1})$ is equal to

$$\begin{aligned}
&= c_1(f^m \smile g^n)(c_2 \otimes \dots \otimes c_m \otimes c_{m+1} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad \sum_{i=1}^{m+n} (-1)^i (f^m \smile g^n)(c_1 \otimes c_2 \otimes \dots \otimes c_i c_{i+1} \otimes c_{i+2} \otimes \dots \otimes c_{m+n+1}) + \\
&\quad (-1)^{m+n+1} (f^m \smile g^n)(c_1 \otimes \dots \otimes c_{m+n}) c_{m+n+1} \\
&= c_1(f^m \smile g^n)(c_2 \otimes \dots \otimes c_m \otimes c_{m+1} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad \sum_{i=1}^m (-1)^i (f^m \smile g^n)(c_1 \otimes \dots \otimes c_{i-1} \otimes c_i c_{i+1} \otimes c_{i+2} \otimes \dots \otimes c_m \otimes c_{m+1} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad \sum_{i=m+1}^{m+n} (-1)^i (f^m \smile g^n)(c_1 \otimes \dots \otimes c_m \otimes c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + \\
&\quad (-1)^{m+n+1} (f^m \smile g^n)(c_1 \otimes \dots \otimes c_{m+n}) c_{m+n+1} \\
&= c_1 f^m(c_2 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad \sum_{i=1}^m (-1)^i f^m(c_1 \otimes \dots \otimes c_{i-1} \otimes c_i c_{i+1} \otimes c_{i+2} \otimes \dots \otimes c_m \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad \sum_{i=m+1}^{m+n} (-1)^i f^m(c_1 \otimes \dots \otimes c_m) g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + \\
&\quad (-1)^{m+n+1} f^m(c_1 \otimes \dots \otimes c_m) g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1} \\
&= c_1 f^m(c_2 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n} \otimes c_{m+n+1}) + \\
&\quad g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) \sum_{i=1}^m (-1)^i f^m(c_1 \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+1}) + \\
&\quad f^m(c_1 \otimes \dots \otimes c_m) \sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + \\
&\quad (-1)^{m+n+1} f^m(c_1 \otimes \dots \otimes c_m) g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1} \\
&= g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) [c_1 f^m(c_2 \otimes \dots \otimes c_{m+1}) + \sum_{i=1}^m (-1)^i f^m(c_1 \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n})] + \\
&\quad f^m(c_1 \otimes \dots \otimes c_m) [\sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + \\
&\quad (-1)^{m+n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}] \\
&= g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) [c_1 f^m(c_2 \otimes \dots \otimes c_{m+1}) + \sum_{i=1}^m (-1)^i f^m(c_1 \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n})] +
\end{aligned}$$

$$\begin{aligned}
& (-1)^{m+1} f^m(c_1 \otimes \dots \otimes c_m) c_{m+1} - (-1)^{m+1} f^m(c_1 \otimes \dots \otimes c_m) c_{m+1}] + \\
& f^m(c_1 \otimes \dots \otimes c_m) \left[\sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + \right. \\
& \left. (-1)^{m+n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1} \right] \\
= & g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) [\delta f^m(c_1 \otimes \dots \otimes c_{m+1}) - (-1)^{m+1} f^m(c_1 \otimes \dots \otimes c_m) c_{m+1}] \\
& + f^m(c_1 \otimes \dots \otimes c_m) [-(-1)^m c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + (-1)^m c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + \\
& \sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^{m+n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}] \\
= & g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) [\delta f^m(c_1 \otimes \dots \otimes c_{m+1}) - (-1)^{m+1} f^m(c_1 \otimes \dots \otimes c_m) c_{m+1}] \\
& + f^m(c_1 \otimes \dots \otimes c_m) [-(-1)^m c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + ((-1)^m c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + \\
& \sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^{m+n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}] \\
= & g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) [\delta f^m(c_1 \otimes \dots \otimes c_{m+1}) + (-1)^m f^m(c_1 \otimes \dots \otimes c_m) c_{m+1}] \\
& - (-1)^m f^m(c_1 \otimes \dots \otimes c_m) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) c_{m+1} + \\
& f^m(c_1 \otimes \dots \otimes c_m) (-1)^m (c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + \\
& (-1)^m \sum_{i=m+1}^{m+n} (-1)^i g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^m (-1)^{m+n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}) \\
= & \delta f^m(c_1 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + f^m(c_1 \otimes \dots \otimes c_m) (-1)^m (c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) \\
& + \sum_{i=m+1}^{m+n} (-1)^{i+m} g^n(c_{m+1} \otimes \dots \otimes c_i c_{i+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^{n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}) \\
= & \delta f^m(c_1 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + f^m(c_1 \otimes \dots \otimes c_m) (-1)^m (c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) \\
& + \sum_{i'=1}^{m+n} (-1)^{i'+2m} g^n(c_{m+1} \otimes \dots \otimes c_{i'+m} c_{i'+m+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^{n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}) \\
= & \delta f^m(c_1 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + f^m(c_1 \otimes \dots \otimes c_m) (-1)^m (c_{m+1} g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) \\
& + \sum_{i'=1}^{m+n} (-1)^{i'} g^n(c_{m+1} \otimes \dots \otimes c_{i'+m} c_{i'+m+1} \otimes \dots \otimes c_{m+n+1}) + (-1)^{n+1} g^n(c_{m+1} \otimes \dots \otimes c_{m+n}) c_{m+n+1}) \\
= & \delta f^m(c_1 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + f^m(c_1 \otimes \dots \otimes c_m) (-1)^m \delta g^n(\tilde{c}_1 \otimes \dots \otimes \tilde{c}_{n+1}) \\
= & \delta f^m(c_1 \otimes \dots \otimes c_{m+1}) g^n(c_{m+2} \otimes \dots \otimes c_{m+n+1}) + (-1)^m f^m(c_1 \otimes \dots \otimes c_m) \delta g^n(\tilde{c}_1 \otimes \dots \otimes \tilde{c}_{n+1})
\end{aligned}$$

$$\therefore \delta(f^m \smile g^n) = \delta f^m \smile g^n + (-1)^m f^m \smile \delta g^n$$

then, δ is a left derivation of degree one of the ring $C^*(A, P)$.

From the formula $\delta(f^m \smile g^n) = \delta f^m \smile g^n + (-1)^m f^m \smile \delta g^n$, if we have $f^m \in Z^m(A, P)$ and $g^n \in Z^n(A, P)$. then $\delta(f^m \smile g^n) = 0$, then

$$\delta(f^m \smile g^n) = \underbrace{\delta f^m}_{0} \smile g^n + (-1)^m f^m \smile \underbrace{\delta g^n}_{0} = 0$$

then $f^m \smile g^n \in Z^{m+n}(A, P)$.

Further more if $f^m \in B^m(A, P)$ and $g^n \in C^n(A, P)$.

$$\begin{aligned} \delta(f^m \smile g^n) &= \delta f^m \smile g^n + (-1)^m f^m \smile \delta g^n \\ &= \underbrace{\delta \delta f^{m-1}}_0 \smile g^n + (-1)^m \underbrace{f^m \smile \delta g^n}_{\in C^{m+n+1}(A, P)} \in \text{Im } C^{m+n}(A, P) = B^{m+n}(A, P) \end{aligned}$$

and similarly if $f^m \in C^m(A, P)$ and $g^n \in B^n(A, P)$. Then $B^*(A, P)$ is an ideal of $Z^*(A, P)$.

We may therefore define the cup product of elements of $H^m(A, P)$ and $H^n(A, P)$ by choosing representatives for them in $Z^m(A, P)$ and $Z^n(A, P)$, respectively. The induced multiplications will be again denoted by \smile and makes $H^*(A, P) = \frac{Z^*(A, P)}{B^*(A, P)}$ into a associative ring, called the cohomology ring of A with coefficients in P .

Let P be a two-sided A module. Then $C^*(A, P)$ is endowed with the structure of a two sided $C^*(A, A)$ module. The operations will again be denoted by \smile and we define $f \smile g \in C^{m+n}(A, P)$ for $f \in C^m(A, A)$, $g \in C^n(A, P)$ by

$$f \smile g(a_1 \otimes \dots \otimes a_m \otimes b_1 \otimes \dots \otimes b_n) = f(a_1 \otimes \dots \otimes a_m)g(b_1 \otimes \dots \otimes b_n)$$

and similarly $g \smile f$.

Since $C^m(A, A) = \text{Hom}_S(A^{(m)}, A)$, setting $A = V$ and considering only its S -module structure, and setting $A^{(m)} = V^m$ and $V_m = C^{m+1}(A, A) = \text{Hom}_S(V^{m+1}, V)$, we obtain a pre-Lie system, as we saw before. We may say that $C^m(A, A)$ has dimension m , but degree $m - 1$. With this remark, we can say that $C^0(A, A)$, which is identified with A , is to be understood as having degree -1 . If we set $f^m \in C^m(A, A)$ and $g^n \in C^n(A, A)$.

$$f^m \circ_i g^n(a_0 \otimes \dots \otimes a_{i-1} \otimes b_0 \otimes \dots \otimes b_{n-1} \otimes a_{i+1} \otimes \dots \otimes a_{m+1}) = f(a_0 \otimes \dots \otimes a_{i-1} \otimes g(b_0 \otimes \dots \otimes b_{n-1}) \otimes a_{i+1} \otimes \dots \otimes a_{m+1})$$

for $i = 0, 1, \dots, m - 1$.

Note that $f^m \circ_i g^n \in C^{m+n-1}(A, A)$. Then we can define the structure of a right pre-Lie system on the set of modules of $C^m(A, A)$. If we take $f^m \in C^m(A, P)$ and $g^n \in C^n(A, A)$, then the formula we have just set defines an element $f^m \circ_i g^n$ of $C^{m+n-1}(A, P)$, and we see immediately that $C^*(A, P)$ has the structure of a right module over the right pre-Lie system $\{C^m(A, A), \circ_i\}$.

With this, we affirm that deformations in $V_1 = C^2(A, A)$, form a pre-Lie system and we can define.

$$F_\lambda \circ F_\mu = \sum_{i=0}^{\lambda} (-1)^i F_\lambda \circ_i F_\mu$$

$$\begin{aligned}
F_\lambda \circ F_\mu(a, b, c) &= \sum_{i=0}^1 (-1)^i F_\lambda \circ_i F_\mu(a, b, c) \\
&= F_\lambda \circ_0 F_\mu(a, b, c) - F_\lambda \circ_1 F_\mu(a, b, c) \\
&= F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c))
\end{aligned}$$

Once we have the structure of pre-Lie system in $C^*(A, A)$, by §8 it can be endowed with the structure of a right pre-Lie ring and also with the structure of a graded Lie ring.

In order to specify which product $C(A, A)$ is being considered as a ring, we will write it as $(C^*(A, A), \smile)$, $(C^*(A, A), \circ)$, or $(C^*(A, A), [,])$. From §8, we see that $C^*(A, P)$ has the structure of a right module over $(C^*(A, A), \circ)$. Therefore, it also has the structure of a two-sided module over $(C^*(A, A), [,])$.

Returning to $C(A, P)$, we have just seen that it is a right module over $(C^*(A, A), \circ_i)$, and thus, by §8, it has the structure of a module over $(C^*(A, A), \circ)$, a two-sided module over $(C^*(A, A), [,])$, and a two-sided module over $(C^*(A, A), \smile)$.

Note the remarks 5.3 in §5: If $f \in C^*(A, A)$ and $g \in C^*(A, P)$, then we will write the operation of f on g as $f \circ g$ or $[f, g]$, which are the same. Similarly, if $f \in C^*(A, P)$ and $g \in C^*(A, A)$, we will use the same notation.

Formula (8.1) holds either when $f^m \in C^m(A, A)$, $g^n \in C^n(A, P)$ or when $f^m \in C^m(A, P)$, $g^n \in C^n(A, A)$. It follows that $Z^m(A, A) \smile Z^n(A, P)$ and $Z^m(A, P) \smile Z^n(A, A)$ are both contained in $Z^{m+n}(A, P)$ and that $B^m(A, A) \smile Z^n(A, P)$ and $Z^n(A, A) \smile B^m(A, P)$ are both contained in $B^{m+n}(A, P)$. As a consequence of this, we have that $H^*(A, P)$ has a structure of a two sided module over $(H^*(A, A), \smile)$. However, generally we will not have that $Z^*(A, A)$ is not closed

under the pre-Lie multiplication, but it will be seen to be closed under the braked product. If we have that A is associative, then $\pi \in C^2(A, A)$ defined by $\pi(a, b) = ab$ is a 2-cochain, in fact

$$\begin{aligned}\delta\pi(a, b, c) &= a\pi(b, c) - \pi(ab, c) + \pi(a, bc) - \pi(a, b)c \\ &= a(bc) - (ab)c + a(bc) - (ab)c \\ &= 0\end{aligned}$$

If is even a coboundary of the identity cochain $l(a) = a$.

$$\begin{aligned}\delta l(a, b) &= al(b) - l(ab) + l(a)b \\ &= a(b) - ab + a(b) \\ &= ab \\ &= \pi(a, b)\end{aligned}$$

Now we can express the cup product between $f^m \in C^m(A, A)$ and $g^n \in C^n(A, A)$ using the coboundary $\pi \in C^2(A, A)$ with the operators of the pre-Lie system $(C^*(A, A), \circ_i)$.

$$f^m \smile g^n = (\pi \circ_0 f^m) \circ_m g^n$$

In fact,

$$\begin{aligned}(\pi \circ_0 f^m) \circ_m g^n(c_0 \otimes \dots \otimes c_{m-1} \otimes d_0 \otimes \dots \otimes d_{n-1}) &= (\pi \circ_0 f^m)(c_0 \otimes \dots \otimes c_{m-1} \otimes g^n(d_0 \otimes \dots \otimes d_{n-1})) \\ &= \pi(f^m(c_0 \otimes \dots \otimes c_{m-1}) \otimes g^n(d_0 \otimes \dots \otimes d_{n-1})) \\ &= f^m(c_0 \otimes \dots \otimes c_{m-1})g^n(d_0 \otimes \dots \otimes d_{n-1})\end{aligned}$$

But not only the cup product, also the coboundary operator could be expressed using π and the pre-Lie product.

$$\delta f^m = -(f \circ \pi - (-1)^{m-1} \pi \circ f) = (-1)^{m-1} (\pi \circ f - (-1)^{m-1} f \circ \pi)$$

In fact

•

$$\delta f^m(a_0 \otimes a_1 \otimes \dots \otimes a_m) = a_0 f^m(a_1 \otimes \dots \otimes a_m) + \sum_{i=0}^{m-1} (-1)^{i+1} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_m) \\ + (-1)^{m+1} f(a_0 \otimes \dots \otimes a_{m-1}) a_m$$

•

$$\pi \circ f(a_0 \otimes \dots \otimes a_m) = \sum_{i=0}^1 (-1)^{(m-1)i} \pi \circ_i f(a_0 \otimes \dots \otimes a_m) \\ = \pi \circ_0 f(a_0 \otimes \dots \otimes a_m) + (-1)^{m-1} \pi \circ_1 f(a_0 \otimes \dots \otimes a_m) \\ = \pi(f(a_0 \otimes \dots \otimes a_{m-1}) \otimes a_m) + (-1)^{m-1} \pi(a_0 \otimes f(a_1 \otimes \dots \otimes a_m)) \\ = f(a_0 \otimes \dots \otimes a_{m-1}) a_m + (-1)^{m-1} a_0 f(a_1 \otimes \dots \otimes a_m)$$

•

$$-(-1)^{m-1} f \circ \pi(a_0 \otimes \dots \otimes a_m) = -(-1)^{m-1} \left[\sum_{i=0}^{m-1} (-1)^i f \circ_i \pi(a_0 \otimes \dots \otimes a_m) \right] \\ = -(-1)^{m-1} \left[\sum_{i=0}^{m-1} (-1)^i f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_m) \right]$$

$$= (-1)^{m-1} \left[\sum_{i=0}^{m-1} (-1)^{i+1} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_m) \right]$$

- Finally if we make

$$\begin{aligned} & (-1)^{m-1} [f(a_0 \otimes \dots \otimes a_{m-1}) a_m + (-1)^{m-1} a_0 f^m(a_1 \otimes \dots \otimes a_m) + \\ & \quad (-1)^{m-1} \sum_{i=0}^{m-1} (-1)^{i+1} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_m)] \\ \dots = & \quad a_0 f^m(a_1 \otimes \dots \otimes a_m) + \sum_{i=0}^{m-1} (-1)^{i+1} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_m) \\ & \quad + (-1)^{m-1} f(a_0 \otimes \dots \otimes a_{m-1}) a_m \\ = & \quad \delta f(a_0 \otimes \dots \otimes a_m) \end{aligned}$$

In terms of the bracket multiplication, since the degree of π is one.

$$\delta f^m = [f, -\pi] = (-1)^{m-1} [\pi, f]$$

The operator δ is therefore, a right inner derivation of degree 1 of the graded Lie ring $(C^*(A, A), [,])$.

However δ is generally not a derivation of any degree of the ring $(C^*(A, A), \circ)$.

The following result is transcendental in order to see how the derivations behaves on the pre-Lie ring $(C^*(A, A), \circ)$.

Theorem 9.1 *If A is an associative ring and if $f^m, g^n \in C^m(A, A), C^n(A, A)$, respectively. Then*

$$f^m \circ \delta g^n - \delta(f^m \circ g^n) + (-1)^{n-1} \delta f^m \circ g^n = (-1)^{n-1} [g^n \smile f^m - (-1)^{mn} f^m \smile g^n]$$

Proof. We are going to use the expression that we have just seen $\delta f^m = (-1)^{m-1}(\pi \circ f - (-1)^{m-1} f \circ \pi)$.

A.

$$\begin{aligned} f^m \circ \delta g^n &= f^m \circ ((-1)^{n-1}(\pi \circ g - (-1)^{n-1} g \circ \pi)) \\ &= (-1)^{n-1} [f \circ (\pi \circ g) - f \circ (g \circ \pi)] \\ &= -f \circ (g \circ \pi) + (-1)^{n-1} f \circ (\pi \circ g) \end{aligned}$$

B.

$$\begin{aligned} \delta(f^m \circ g^n) &= (-1)^{m+n} [\pi \circ (f \circ g) - (-1)^{m+n} (f \circ g) \circ \pi] \\ &= -(f \circ g) \circ \pi + (-1)^{m+n} \pi \circ (f \circ g) \end{aligned}$$

C.

$$\begin{aligned} (-1)^{n-1} \delta f \circ g &= (-1)^{n-1} [(-1)^{m-1}(\pi \circ f - (-1)^{m-1} f \circ \pi)] \circ g \\ &= (-1)^{m+n} [\pi \circ f - (-1)^{m-1} f \circ \pi] \circ g \\ &= (-1)^{m+n} ((\pi \circ f) \circ g - (-1)^{m-1} (f \circ \pi) \circ g) \end{aligned}$$

Applying the definition of pre-Lie ring to (A), we get.

$$\begin{aligned}
-f \circ (g \circ \pi) + (-1)^{n-1} f \circ (\pi \circ g) &= -f \circ (g \circ \pi - (-1)^{n-1} \pi \circ g) \\
&= -((f \circ g) \circ \pi - (-1)^{n-1} (f \circ \pi) \circ g) \\
&= -(f \circ g) \circ \pi + (-1)^{n-1} (f \circ \pi) \circ g
\end{aligned}$$

From (C) we have $(-1)^{m+n}(\pi \circ f) \circ g - (-1)^{n-1}(f \circ \pi) \circ g$. Therefore,

$$\begin{aligned}
f^m \circ \delta g^n - \delta(f^m \circ g^n) + (-1)^{n-1} \delta f^m \circ g^n &= -(f \circ g) \circ \pi + (-1)^{n-1} (f \circ \pi) \circ g - (-f \circ g) \circ \pi + (-1)^{m+n} \pi \circ (f \circ g) \\
&\quad + (-1)^{m+n} (\pi \circ f) \circ g - (-1)^{n-1} (f \circ \pi) \circ g \\
&= (-1)^{m+n} (\pi \circ f) \circ g - (-1)^{m+n} \pi \circ (f \circ g) \\
&= (-1)^{m+n} ((\pi \circ f) \circ g - \pi \circ (f \circ g))
\end{aligned}$$

And by Theorem 8.1 ii), we have

$$\begin{aligned}
(\pi \circ f) \circ g - \pi \circ (f \circ g) &= \sum (-1)^{(m-1)i+(n-1)j} (\pi \circ_i f) \circ_j g \\
&= (-1)^{(n-1)m} (\pi \circ_0 f^m) \circ_m g^n + (-1)^{m-1} (\pi \circ_1 f^m) \circ_0 g^n \\
&= (-1)^{(n-1)m} (\pi \circ_0 f^m) \circ_m g^n + (-1)^{m-1} (\pi \circ_0 g^n) \circ_{(n-1)+1} f^m \\
&= (-1)^{(n-1)m} (\pi \circ_0 f^m) \circ_m g^n - (-1)^m (\pi \circ_0 g^n) \circ_{(n-1)+1} f^m \\
&= (-1)^{(n-1)m} f^m \smile g^n - (-1)^m g^n \smile f^m
\end{aligned}$$

Finally,

$$\begin{aligned}
(-1)^{m+n}[(\pi \circ f) \circ g - \pi \circ (f \circ g)] &= (-1)^{m+n}[(-1)^{(n-1)m} f^m \smile g^n - (-1)^m g^n \smile f^m] \\
&= (-1)^{m+n+mn-m} f^m \smile g^n - (-1)^{m+n+m} g^n \smile f^m \\
&= (-1)^{n(m+1)} f^m \smile g^n - (-1)^n g^n \smile f^m \\
&= (-1)^{n-1} g^n \smile f^m - (-1)^{n(m+1)-1} f^m \smile g^n \\
&= (-1)^{n-1} g^n \smile f^m - (-1)^{mn+(n-1)} f^m \smile g^n \\
&= (-1)^{n-1} [g^n \smile f^m - (-1)^{mn} f^m \smile g^n]
\end{aligned}$$

If f^m and g^n are co cycles, that is, $\delta f^m = 0$, $\delta g^n = 0$. Then from the last result we have.

$$(-1)^n [g^n \smile f^m - (-1)^{mn} f^m \smile g^n] = \delta(f^m \circ g^n)$$

Therefore

Corollary 9.2 *If A is an associative ring, then the ring $\{H^*(A,A), \smile\}$ is a graded commutative ring, where dimensions gives its grading. i.e. if $\eta^m \in H^m(A,A)$, $\zeta^n \in H^n(A,A)$ Then*

$$\eta^m \smile \zeta^n = (-1)^{mn} \zeta^n \smile \eta^m$$

When using cup product, we shall tacitly assume any gradation is by dimension, and when we refer to bracket product, we use the degree unless otherwise stated.

Now we can give the module direct sum $A + P$, an associative ring structure over S , with A as a subring and P as an ideal with $P^2 = 0$ by setting

$$(a, x)(b, y) = (ab, ay + xb) \quad a, b \in A, x, y \in P$$

There exist natural inclusions $C^m(A, A), C^m(A, P) \subset C^m(A + P, A + P)$ defined by setting for any f^m in $C^m(A, A)$ or $C^m(A, P)$, $f(\alpha_1 \otimes \dots \otimes \alpha_m) = 0$ if any α_i is in P . Under this definition if we take $f^m \in C^m(A, A)$ or $f^m \in C^m(A, P)$ and $\alpha_0 \dots \alpha_{m-1} \in A, \alpha_m \in P$. It is straightforward to see $\delta f^m(\alpha_0 \otimes \dots \otimes \alpha_m) = 0$, because $\alpha_m \in P$. (i.e. f is a cocycle) but

$$\begin{aligned} \delta f^m(\alpha_0 \otimes \dots \otimes \alpha_m) = & \alpha_0 f^m(\alpha_1 \otimes \dots \otimes \alpha_m) + \sum_{i=1}^{m-1} (-1)^i f^m(\alpha_0 \otimes \dots \otimes \alpha_i \alpha_{i+1} \otimes \dots \otimes \alpha_m) \\ & + (-1)^m f^m(\alpha_0 \otimes \dots \otimes \alpha_{m-1}) \alpha_m \end{aligned}$$

All the summands vanish by definition. However the last term belongs to A or P . Because $f^m(\alpha_0 \otimes \dots \otimes \alpha_{m-1}) \neq 0$, belongs to A or P , therefore $\delta f^m \neq 0$ that means that the image of a cocycle is not necessarily a cocycle.

With this we can extend Theorem 9.1 to a statement about the module structure of $H^*(A, P)$ over $H^*(A, A)$.

Corollary 9.3 *Let A be an associative ring and P be a two-sided A module. If either $\eta^m \in H^m(A, A)$, $\zeta^n \in H^*(A, P)$ or $\eta^m \in H^m(A, P)$, $\zeta^n \in H^*(A, A)$, then*

$$\eta^m \smile \zeta^n = (-1)^{mn} \zeta^n \smile \eta^m$$

Chapter 10

Obstructions are cocycles

While we have only considered the case of an algebra over a field thus far, the definitions of the obstruction cochain given in the preceding sections do not require the existence of a ground field. Under the assumption that A is an associative ring, we show in this section that the obstruction cochains are cocycles.

First, consider the case of derivations. Given a one-parameter family of linear automorphisms of A as in (2.5).

$$\Phi_t(a) = a + t\phi_1(a) + t^2\phi_2(a) + \dots$$

The condition (2.7) is a necessary and sufficient condition, that is, a one parameter family of multiplicative automorphisms, may be written.

$$\sum_{\substack{\lambda+\mu=v \\ \lambda,\mu>0}} \phi_\lambda \smile \phi_\mu = -\delta\phi_v \quad (10.1)$$

Suppose given $\phi_1, \dots, \phi_{m-1}$ satisfying (10.1) for $v = 1, \dots, m-1$. The cohomology class of $F = \sum_{\substack{\lambda+\mu=m \\ \lambda,\mu>0}} \phi_\lambda \smile \phi_\mu$, may be consider as the obstruction to extending the sequence $\phi_1, \dots, \phi_{m-1}$ to a sequence $\phi_1, \dots, \phi_{m-1}, \phi_m$ satisfying (10.1) for $v = 1, \dots, m$.

Proposition 10.1 Let A be an associative ring and $\phi_1, \dots, \phi_{m-1}$ be elements of $C^1(A, A)$ such that $\sum_{\lambda+\mu=\nu} \phi_\lambda \smile \phi_\mu = -\delta\phi_\nu$, for $\nu = 1, \dots, m-1$. Set $F = \sum_{\lambda+\mu=m} \phi_\lambda \smile \phi_\mu$ then $\delta F = 0$, i.e. F is in $Z^2(A, A)$.

Proof. If $\phi_\lambda, \phi_\mu \dots$ in $C^1(A, A)$, then from equation (9.1) we have $\delta(\phi_\lambda \smile \phi_\mu) = \delta\phi_\lambda \smile \phi_\mu - \phi_\lambda \smile \delta\phi_\mu$, then computing δF explicitly, one has

$$\begin{aligned}
\delta F &= \sum_{\lambda+\mu=m} \delta(\phi_\lambda \smile \phi_\mu) \\
&= \sum_{\lambda+\mu=m} (\delta\phi_\lambda \smile \phi_\mu - \phi_\lambda \smile \delta\phi_\mu) \\
&= \sum_{\lambda+\mu=m} \delta\phi_\lambda \smile \phi_\mu - \sum_{\lambda+\mu=m} \phi_\lambda \smile \delta\phi_\mu \\
&= \sum_{\lambda+\mu=m} \left(- \sum_{\alpha+\beta=\lambda} \phi_\alpha \smile \delta\beta \right) \smile \phi_\mu - \sum_{\lambda+\mu=m} \phi_\lambda \smile \left(- \sum_{\alpha+\beta=\mu} \phi_\alpha \smile \delta\beta \right)
\end{aligned}$$

Since the cup product is associative and that $\alpha + \beta + \mu = \lambda + \alpha + \beta = m$ we have in the last expression.

$$- \sum \phi_\alpha \smile \phi_\beta \smile \phi_\gamma + \sum \phi_\gamma \smile \delta_\alpha \smile \delta_\beta = 0$$

This ends the proof.

Consider now the case of deformations. Let A an algebra and, on the underlying vector space of it, a one parameter family of multiplications as in (1.3).

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \dots$$

the necessary and sufficient conditions (1.4) which means that this is a family of associative

multiplications may in the notation of part §2, be expressed in the form.

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu = \delta F_\nu \quad (10.2)$$

Suppose given F_1, \dots, F_{n-1} satisfying (10.2) for $\nu = 1, \dots, m-1$. Setting $G = \sum_{\lambda+\mu=m} F_\lambda \circ F_\mu$, the cohomology class of G may be viewed as the obstruction of an F_m such that F_1, \dots, F_m satisfy (10.2) for $\nu = 1, \dots, m$ and it is this element G of $C^3(A, A)$ which will be shown to be a cocycle.

Now we are going to apply the result from Theorem 8.1 ii) of §8, which states that in a right graded pre-Lie ring B .

$$(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np} [(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$$

In particular

- If we suppose that B has no elements of order 2, i.e. $x \in B$ and $2x = 0$, implies $x = 0$

$$(f \circ g) \circ g - f \circ (g \circ g) = -[(f \circ g) \circ g - f \circ (g \circ g)]$$

$$(f \circ g) \circ g - f \circ (g \circ g) = -(f \circ g) \circ g + f \circ (g \circ g)$$

$$2(f \circ g) \circ g = 2f \circ (g \circ g)$$

Then $(f \circ g) \circ g = f \circ (g \circ g)$.

- If we suppose that B is a pre-Lie ring of a pre-Lie system (V_m, \circ_i) . Also from Theorem 8.1, i) of §8

$$(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p$$

where $0 \leq j \leq i-1$ or $n+i+1 \leq j \leq m+n$.

In particular, if g is homogeneous of odd degree.

$$(f \circ g) \circ g - f \circ (g \circ g) = \sum (-1)^{i+j} (f \circ_i g) \circ_j g$$

for $0 \leq j \leq i-1$ or $n+i+1 \leq j \leq m+n$.

Then we can express

$$(f \circ g) \circ g - f \circ (g \circ g) = \sum_{0 \leq j \leq i-1} (-1)^{i+j} (f \circ_i g) \circ_j g + \sum_{n+i+1 \leq j \leq m+n} (-1)^{i+j} (f \circ_i g) \circ_j g$$

By definition of a pre-Lie system, if $0 \leq j \leq i-1$, then $(f \circ_i g) \circ_j g = (f \circ_j g) \circ_{i+n} g$ and if $n+i+1 \leq j \leq m+n$, then $(f \circ_i g) \circ_j g = (f \circ_{j-n} g) \circ_i g$. Then the last expression is equal to

$$\begin{aligned} &= \sum_{0 \leq j \leq i-1} (-1)^{i+j} (f \circ_j g) \circ_{i+n} g + \sum_{n+i+1 \leq j \leq m+n} (-1)^{i+j} (f \circ_{j-n} g) \circ_i g \\ &= \sum_{0 \leq j \leq i-1} (-1)^{i+j} (f \circ_j g) \circ_{i+n} g + \sum_{i+1 \leq j' \leq m} (-1)^{i+j'+n} (f \circ_{j'} g) \circ_i g \\ &= \sum_{0 \leq j \leq i-1} (-1)^{i+j} (f \circ_j g) (a_0 \otimes \dots \otimes a_{i+n-1} \otimes g(b_0 \otimes \dots \otimes b_n) \otimes \tilde{a}_{i+n+1} \otimes \dots \otimes \tilde{a}_{m+n}) \\ &\quad - \sum_{i+1 \leq j' \leq m} (-1)^{i+j'} (f \circ_{j'} g) (a_0 \otimes \dots \otimes a_{i-1} \otimes g(c_0 \otimes \dots \otimes c_n) \otimes \bar{a}_{i+1} \otimes \dots \otimes \bar{a}_{m+n}) \\ &= \sum_{0 \leq j \leq i-1} (-1)^{i+j} f(a_0 \otimes \dots \otimes a_{j-1} \otimes g(c_0 \otimes \dots \otimes c_n) \otimes \tilde{a}_{j+1} \otimes \dots \otimes \tilde{a}_{i-1} \otimes \\ &\quad \quad \quad g(b_0 \otimes \dots \otimes b_n) \otimes \tilde{\bar{a}}_{i+1} \otimes \dots \otimes \tilde{\bar{a}}_m) \\ &\quad - \sum_{i+1 \leq j' \leq m} (-1)^{i+j'} f(a_0 \otimes \dots \otimes a_{i-1} \otimes g(c_0 \otimes \dots \otimes c_n) \otimes \bar{a}_{n+1} \otimes \dots \otimes \bar{a}_{j'-1} \otimes \\ &\quad \quad \quad g(b_0 \otimes \dots \otimes b_n) \otimes \bar{\bar{a}}_{j'+1} \otimes \dots \otimes \bar{\bar{a}}_m) \\ &= 0 \end{aligned}$$

Then we also have $(f \circ g) \circ g = f \circ (g \circ g)$.

The last observation provides a necessary condition for a graded right pre-Lie ring to be associated with a pre-Lie system. Note that the degree of an element in $C^2(A, A)$ is one, and the pre-Lie structure on $C^*(A, A)$ has been obtained from a pre-Lie system.

Proposition 10.2 *Let A be an associative ring, and F_1, \dots, F_{m-1} be elements of $C^2(A, A)$ such that $\sum_{\lambda+\mu=\nu} F_\lambda \circ F_\mu = \delta F_\nu$, for $\nu = 1, \dots, m-1$. Let an element G of $C^2(A, A)$ be defined by*

$$G = \sum_{\lambda+\mu=m} F_\lambda \circ F_\mu$$

Then $\delta G = 0$, i.e. G is in $Z^3(A, A)$.

Proof. Since F_λ and F_μ are in $C^2(A, A)$, one has from Theorem 9.1, of §9. That

$$\delta(F_\lambda \circ F_\mu) = F_\lambda \circ \delta F_\mu - \delta F_\lambda \circ F_\mu + (F_\mu \smile F_\lambda - F_\lambda \smile F_\mu)$$

It follows that $\delta G = \sum_{\lambda+\mu=m} \delta(F_\lambda \circ F_\mu) = \sum_{\lambda+\mu=m} [F_\lambda \circ \delta F_\mu - (\delta F_\lambda) \circ F_\mu]$. By the hypothesis, this equals

$$\begin{aligned} &= \sum_{\alpha+\beta+\gamma=m} [F_\alpha \circ (F_\beta \circ F_\gamma) - (F_\alpha \circ F_\beta) \circ F_\gamma] \\ &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta < \gamma}} (F_\alpha \circ (F_\beta \circ F_\gamma) - (F_\alpha \circ F_\beta) \circ F_\gamma) \\ &\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma < \beta}} (F_\alpha \circ (F_\beta \circ F_\gamma) - (F_\alpha \circ F_\beta) \circ F_\gamma) \end{aligned}$$

but

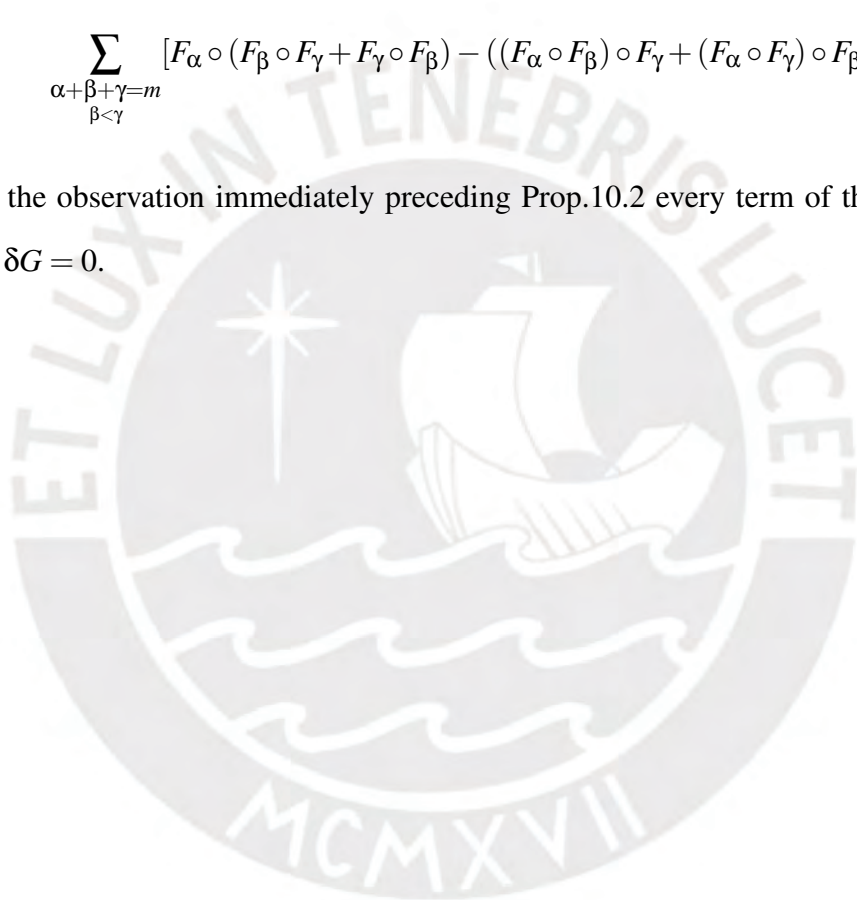
$$\sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma<\beta}} (F_\alpha \circ (F_\beta \circ F_\gamma) - (F_\alpha \circ F_\beta) \circ F_\gamma) = \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta<\gamma}} (F_\alpha \circ (F_\gamma \circ F_\beta) - (F_\alpha \circ F_\gamma) \circ F_\beta)$$

Therefore the last expression is equal to

$$\sum_{\substack{\alpha+\beta+\gamma=m \\ \beta<\gamma}} [F_\alpha \circ (F_\beta \circ F_\gamma + F_\gamma \circ F_\beta) - ((F_\alpha \circ F_\beta) \circ F_\gamma + (F_\alpha \circ F_\gamma) \circ F_\beta)]$$

But by the observation immediately preceding Prop.10.2 every term of the sum vanishes.

Therefore $\delta G = 0$.



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