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Rabi Hamiltonian and Geometric Phases

Thesis submitted for the degree of Master of Science in Physics
by

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Resumen

Esta tesis estudia fases geométricas que aparecen cuando un átomo de dos niveles interacciona con un campo electromagnético monomodal cuantizado, un modelo descrito por el Hamiltoniano de Rabi (HR). Como se conoce, el HR no tiene una solución cerrada; no obstante, cuando el acoplamiento entre el átomo y campo es débil, la aproximación de onda rotante (RWA) puede ser aplicada. Esto resulta en el Hamiltoniano de Jaynes-Cummings (HJC), el cual es una útil solución analítica aproximada del primero. Cuando la RWA puede ser aplicada, fenómenos físicos predichos en el modelo de Rabi deben también aparecer en el modelo de Jaynes-Cummings; caso contario, la aproximación sería físicamente inconsistente. Esto último generó una controversia después de una reciente afirmación sobre fases de Berry en el HR. De acuerdo a ésta, la RWA no es válida para ningún valor del acoplamiento entre el átomo y campo. Los resultados de esta investigación, cálculos numéricos de la fase de Berry en el HR, muestran que este no es el caso y que afirmaciones contrarias son inconsistentes con un argumento analítico que concierne al modelo de Rabi. Adicionalmente, se muestra que estos resultados convergen a los respectivos para el HJC, concluyendo así que la RWA es consistente al aplicarse a fases de Berry, como era de esperarse. Finalmente, se discute que la aparición de fases de Berry no depende de la condición adiabática; por lo tanto, el marco de estudio apropiado es el cinemático, el cual contiene a la fase de Berry como un caso particular de la fase geométrica. También se discute que el Hamiltoniano no desempeña un rol importante, salvo de proveedor de los autovectores usados en el cálculo de la fase geométrica. Esto manifiesta la característica esencial de la cual depende la fase geométrica, que es la geometría del espacio de rayos. Este espacio depende de los tipos de evolución que sean considerados. Este punto es ilustrado estudiando una diferente transformación unitaria en el modelo de Schwinger.

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Abstract

This thesis addresses geometric phases that appear when a two-level atom interacts with a quantized one-mode electromagnetic field, a model that is described by the Rabi Hamiltonian (RH). As it is known, the RH has no closed-form solution; nevertheless, when the coupling between the atom and field is weak, the rotating-wave approximation (RWA) can be applied. This results in the Jaynes-Cummings Hamiltonian (JCH), which is a useful analytically solvable approximation of the former one. Whenever the RWA can be applied, physical phenomena predicted within the Rabi model should also show up within the Jaynes-Cummings model; otherwise, the approximation would be physically inconsistent. This issue became controversial after a recent claim concerning Berry phases in the RH. According to this claim, the RWA breaks down at all values of the coupling between the atom and field. The results of this research, numerical calculations of Berry phases in the RH, showed that this is not the case and that claims on the contrary are inconsistent with an analytical argument regarding the Rabi model. Additionally, these results also converge to the corresponding ones obtained with the JCH, concluding that the RWA consistently applies when dealing with Berry phases, as expected. Finally, it is argued that the appearance of Berry phases does not depend on adiabatic conditions, hence the appropriate framework is the kinematic one, which contains Berry's phase as a special case of the geometric phase. It is also argued that the Hamiltonian does not play an essential role in the whole, except as a provider of the eigenvectors used in the calculation of geometric phases. This brings to the fore the essential feature on which the geometric phase depends, which is the geometry of the ray space. This space depends on the types of evolutions being considered. This point is illustrated by addressing a different unitary transformation in the Schwinger model.

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Chapter 1

Introduction

In quantum optics, the Rabi Hamiltonian describes the light-matter interaction; in other words, it governs the dynamics of a two-level atom which interacts with a quantized one-mode electromagnetic field within the dipole approximation. Despite the fact that the Rabi Hamiltonian is one of the simplest models in physics, a closed-form analytical solution has not been found yet. However, applying the rotating-wave approximation (RWA), we can obtain a useful analytically solvable model known as the Jaynes -Cummings Hamiltonian. The two Hamiltonians are given by the following expressions, respectively:

$$H_R = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g\left(\sigma_+ + \sigma_-\right)\left(a^\dagger + a\right) \tag{1.1}$$

and

$$H_{JC} = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{1.2}$$

Here, ω_0 is the transition frequency of the two level-atom; ω is the single-mode frequency of the electromagnetic field that is described through the annihilation and creation operators a and $a^\dagger$, respectively; $\sigma_\pm$ are the raising and lowering Pauli operators, which refer to the atomic transition; and g is the coupling constant between the atom and field.

The conditions that need to be satisfied in order to apply the RWA are a near resonance case, $\omega_0 \approx \omega$, and a weak atom-field interaction regime, to be more precise, $g/\omega \ll 1$. Therefore, when these conditions are achieved, the Rabi Hamiltonian is reduced to the Jaynes-Cummings one. A consequence of this is that the results described within the Rabi model must show agreement with the ones obtained within the Jaynes-Cummings model as long as the RWA is valid. Recently, this last issue has generated a controversy.

The debate began when Larson claimed that the Berry phase in the Rabi Hamiltonian vanishes for every value g/ω, including the ones where the RWA is well-founded [2, 3]. These results are in contradiction with a previous work by Fuentes-Guridi *et al.* that states that the Berry phase in the Jaynes-Cummings Hamiltonian is non-vanishing [4]. In summary, the controversy is generated because, apparently, there is an instance where the RWA breaks down, even when the needed conditions are fulfilled.

Larson's findings have been refuted by several authors, either through approximate solutions [5, 6] or through the discussion of his theoretical approach [7]. These authors have found that the Berry phase in the Rabi Hamiltonian is non-vanishing; nevertheless, this controversy has not been resolved yet. In the present thesis we address this controversy. We present both numerical and analytical evidence of non-vanishing Berry (geometric) phases related to the Rabi model. To achieve this, we present numerical calculations which manifest that, indeed, this phase is non-vanishing. Also, these calculations are in accordance with an analytical argument regarding the Rabi model, which is part of this thesis. Lastly, we show that these numerical results converge to the Jaynes-Cummings ones when g/ω is sufficiently small. With this, we conclude that the RWA applies properly in the instance discussed previously.

Other theoretical aspects of this controversy are examined. One of these aspects is that the phase we are dealing with is not, strictly speaking, a Berry phase, but rather, a geometric phase. This geometric phase is a more general concept and includes Berry's phase as a particular case. Likewise, we discuss the role that the Hamiltonian performs in this matter, which is just a provider of the initial eigenvector used in the calculation of the respective phase. In order to illustrate these points, we calculate the geometric phase acquired with a different unitary evolution in the Schwinger model.

This thesis is further organized into four chapters. In Chapter 2, we briefly explain some preliminary concepts which are used throughout this work. In Chapter 3, we summarize the controversy and present the results, which allow us to take a stand in this debate. In Chapter 4, we discuss with more attention some theoretical aspects which we considered before. Lastly, in Chapter 5, we draw the conclusions of this work.

Chapter 2

Preliminary Concepts

2.1 Rabi and Jaynes-Cummings Hamiltonians

2.1.1 Rabi Hamiltonian

The Rabi Hamiltonian was first introduced around 80 years ago to describe the interaction between nuclear spins with magnetic fields [8]. In quantum optics, it describes the interaction of a two-level atom with a quantized one-mode electromagnetic field. Thus, we can write the Hamiltonian as:

$$\hat{H}_R = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int}. \tag{2.1}$$

Following [9], we discuss each term of Eq.(2.1).

Atomic Hamiltonian

For a two-level atom that has a ground state $|g\rangle$ and an excited state $|e\rangle$ with eigenenergies E_g and E_e, respectively, the Hamiltonian in the energy representation is as follows:

$$\hat{H}_{atom} = E_e|e\rangle\langle e| + E_g|g\rangle\langle g|. \tag{2.2}$$

In this 2×2 Hilbert space, every operator can be written as a linear combination of the identity operator $\hat{I}$ and the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$. Nevertheless, a possible better choice for an operator basis in this space is the following:

$$\hat{I} = |e\rangle\langle e| + |g\rangle\langle g|, \tag{2.3}$$

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|, \tag{2.4}$$

$$\hat{\sigma}_+ = |e\rangle\langle g|, \tag{2.5}$$

$$\hat{\sigma}_- = |g\rangle\langle e|. \tag{2.6}$$

The usefulness of this choice becomes obvious if we look at the action of $\hat{\sigma}_+$ and $\hat{\sigma}_-$ when applied to an arbitrary state in this Hilbert space. The operator $\hat{\sigma}_+$ produces a transition from the ground to the excited state, and $\hat{\sigma}_-$ does the opposite. These properties would be helpful later when we analyze the interaction between the atom and field.

From Eqs. (2.3-2.4) we can write the atomic Hamiltonian in the following form:

$$\hat{H}_{atom} = \frac{E_e + E_g}{2}\hat{I} + \frac{E_e - E_g}{2}\hat{\sigma}_z. \tag{2.7}$$

However, the first term on the right side of Eq. (2.7) can be eliminated because terms proportional to the $\hat{I}$ have no influence on the dynamics.

Finally, taking into account the fact that $E_e - E_g = \hbar\omega_0$, where ω_0 is the transition frequency between the two atomic levels, Eq (2.7) can be written as:

$$\hat{H}_{atom} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z. \tag{2.8}$$

Field Hamiltonian

In free space, the atom would interact with infinite modes of the electromagnetic field, so in order to restrict the interaction with only one mode, we confine the field into a cavity (e.g. Cavity QED experiments).

We assume the electric field is polarized in the x-direction, which has a unit vector $\vec{e}_x$, as follows:

$$\vec{E}(\vec{r}, t) = \vec{e}_x E_x(z, t). \tag{2.9}$$

In Eq. (2.9), we assume the term $E_x(z, t)$ has the following monomodal character:

$$E_x(z, t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} q(t) \sin kz. \tag{2.10}$$

Here, V is the volume of the cavity; ε_0 is the vaccum permittivity; k is the wavenumber (module of the wavevector $\vec{k}$); and ω is the single-mode frequency of the field. The term $q(t)$ carries the time dependence.

In order to calculate the magnetic field, we replace Eq. (2.10) in the Maxwell equation (no sources case) $\nabla \times \vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$, and obtain:

$$\vec{B}(\vec{r}, t) = \vec{e}_y B_y(z, t), \tag{2.11}$$

with

$$B_y(z,t) = \left(\frac{\mu_0\varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t)\cos kz. \tag{2.12}$$

If we define $p(t) \equiv \dot{q}(t)$, we have for the energy (Hamiltonian) of the electromagnetic field:

$$H_{field} = \frac{1}{2}\int dV \left[\varepsilon_0 \vec{E}^2(\vec{r},t) + \frac{1}{\mu_0}\vec{B}^2(\vec{r},t)\right] = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2(z,t) + \frac{1}{\mu_0}B_y^2(z,t)\right] \tag{2.13}$$

Replacing Eqs. (2.10-2.12), we obtain:

$$H_{field} = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \tag{2.14}$$

This expression is equivalent to the harmonic oscillator Hamiltonian. Thus, in order to achieve the quantization of this problem, we identify q and p not as numbers, but instead as operators $\hat{q}$ and $\hat{p}$, respectively, which obey the conmutation relation $[\hat{q},\hat{p}] = i\hbar$.

In analogy to the harmonic oscillator problem, we introduce the operators $\hat{a}$ and $\hat{a}^\dagger$, which are referred to as the annihilation and creation operators, respectively, in the following form:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \tag{2.15}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}). \tag{2.16}$$

Replacing Eqs. (2.15-2.16) in Eq. (2.14), we get the expresion of the Hamiltonian of the electromagnetic field:

$$\hat{H}_{field} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{2.17}$$

Finally, as in the previous section, by eliminating the term proportional to the I, we get:

$$\hat{H}_{field} = \hbar\omega\hat{a}^\dagger\hat{a} \tag{2.18}$$

Interaction Hamiltonian

The interaction between an atom and an external electromagnetic field is classically described by the minimal-coupling Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + eU(\vec{r},t) + V(r) \tag{2.19}$$

Here, $\vec{p}$ is the momentum of the electron of charge e and mass m; $\vec{A}(\vec{r},t)$ and $U(\vec{r},t)$ are the vector and scalar potentials of the external electromagnetic field, respectively; and $V(r)$ is the atomic binding of the electron with the nucleus.

The minimal-coupling Hamiltonian (Eq. (2.19)) can be simplified by using the *dipole approximation*. In general, the wavelength of the electromagnetic field that induces or is emitted during atomic transitions is much larger than the size of an atom. Therefore, we can consider that all the atom, whose nucleus is located at $\vec{r}_0$, is immersed in a plane electromagnetic wave described by a vector potential $\vec{A}(\vec{r}_0,t)$ in the following form:

$$\vec{A}(\vec{r}_0 + \vec{r},t) = \vec{A}(t)\exp[i\vec{k}.(\vec{r_0} + \vec{r})] \tag{2.20}$$

Based on the fact that $\vec{k}.\vec{r} \ll 1$ (dipole approximation), we can approximately write Eq. (2.20) as:

$$\vec{A}(\vec{r}_0 + \vec{r},t) = \vec{A}(t)\exp(i\vec{k}.\vec{r}_0)(1 + i\vec{k}.\vec{r} + ...) \approx \vec{A}(t)\exp(i\vec{k}.\vec{r}_0) \tag{2.21}$$

In the Coulomb or "radiation" gauge, $U(\vec{r},t) = 0$ and $\nabla.\vec{A}(\vec{r},t) = 0$. As a result, the corresponding Schrödinger equation for the minimal coupling Hamiltonian is as follows:

$$\left\{-\frac{\hbar^2}{2m}\left[\nabla - \frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\right]^2 + V(r)\right\}\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} \tag{2.22}$$

Eq. (2.22) can be simplified by defining the new wavefunction:

$$\psi(\vec{r},t) = \exp\left[\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t).\vec{r}\right]\phi(\vec{r},t) \tag{2.23}$$

Inserting Eq. (2.23) into Eq. (2.22), we get:

$$i\hbar\left[\frac{ie}{\hbar}\dot{\vec{A}}.\vec{r}\phi(\vec{r},t) + \dot{\phi}(\vec{r},t)\right]\exp\left(\frac{ie}{\hbar}\vec{A}.\vec{r}\right) = \exp\left(\frac{ie}{\hbar}\vec{A}.\vec{r}\right)\left[\frac{p^2}{2m} + V(r)\right]\phi(\vec{r},t) \tag{2.24}$$

Rearranging Eq. (2.24) and taking into account that in the gauge we are working $\vec{E} = -\dot{\vec{A}}$, we obtain:

$$i\hbar\frac{\partial\phi(\vec{r},t)}{\partial t} = \left[\left(\frac{p^2}{2m} + V(r)\right) + (-e\vec{r}.\vec{E}(\vec{r}_0,t))\right]\phi(\vec{r},t) \tag{2.25}$$

From Eq. (2.25), we can identify:

$$H_0 = \frac{p^2}{2m} + V(r), \tag{2.26}$$

$$H_{int} = -\vec{d}.\vec{E}(\vec{r}_0, t). \tag{2.27}$$

Here, H_0 is the Hamiltonian of the electron bound to the nucleus; hence, H_{int} is the Hamiltonian ruling the interaction of the electron with the external electromagnetic field in the dipole approximation, where $\vec{d}$ is the electric dipole.

In order to obtain the Hamiltonian operator for the interaction of the atom and field, we replace the electric dipole $\vec{d}$ and the electric field $\vec{E}$ with their respective operators:

$$\hat{H}_{int} = -\hat{\vec{d}}.\hat{\vec{E}} \tag{2.28}$$

The electric field operator can be obtained by replacing Eqs. (2.15-2.16) into Eq. (2.10), thus resulting in:

$$\hat{\vec{E}} = \vec{e}\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2} \sin kz \left(\hat{a} + \hat{a}^\dagger\right), \tag{2.29}$$

where $\vec{e}$ is the polarization vector in general.

Replacing Eq. (2.29) into Eq. (2.28), we get:

$$\hat{H}_{int} = -\hat{\vec{d}}.\hat{\vec{E}} = \lambda\hat{d}\left(\hat{a} + \hat{a}^\dagger\right), \tag{2.30}$$

where $\hat{d} \equiv \hat{\vec{d}}.\vec{e}$ and $\lambda = -\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2} \sin kz$.

Taking into account that the electric dipole operator $\hat{\vec{d}}$ has odd parity and the wavefunctions of a two-level atom have defined parity (either even or odd), we can simplify $\hat{d}$ since $\langle e|\hat{d}|e\rangle = 0 = \langle g|\hat{d}|g\rangle$. Hence, the hermitian operator $\hat{d}$ can be expressed as:

$$\hat{d} = d|g\rangle\langle e| + d^*|e\rangle\langle g|. \tag{2.31}$$

Because the relative phase between $|e\rangle$ and $\langle g|$ can be freely chosen, we can assume that $d = d^*$, i.e., $d \in \Re$. Keeping this in mind and recalling Eqs. (2.5-2.6), we can write Eq. (2.31) as:

$$\hat{d} = d\left(\hat{\sigma}_+ + \hat{\sigma}_-\right). \tag{2.32}$$

Finally, replacing Eq. (2.32) into Eq. (2.30), we obtain the Hamiltonian for the

interaction between the atom and field in the dipole approximation:

$$\hat{H}_{int} = \hbar g \left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right), \tag{2.33}$$

where $g \equiv d\lambda/\hbar$ is known as the coupling constant between the atom and field.

Introducing Eqs. (2.8, 2.18 and 2.33) into Eq. (2.1), we obtain the expression below for the Rabi Hamiltonian:

$$H_R = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g \left(\sigma_+ + \sigma_-\right)\left(a^\dagger + a\right) \tag{2.34}$$

Note that the "hat" notation for operators has been dropped and we keep this notation from now on.

The Rabi Hamiltonian is among the simplest ones in physics; still, it has not been exactly solved. Despite this fact, when some specific conditions are fulfilled, the Rabi Hamiltonian can be reduced to the Jaynes-Cummings Hamiltionian, which we analyze next.

2.1.2 Jaynes-Cummings Hamiltonian

The Jaynes-Cummings Hamiltonian was introduced in 1963 [10] to describe the interaction between a two-level molecule with a quantized field in order to apply it to the beam maser. It is obtained from the Rabi Hamiltonian by applying the so-called rotating-wave approximation (RWA).

Rotating-Wave Approximation (RWA)

The RWA was originally introduced in the context of nuclear magnetic resonance, where a spin-1/2 interacts with two magnetic fields, one being uniform and the other one oscillating in a plane perpendicular to the first one. The former one produces a Larmor precession of the spin. The latter one can be decomposed into the sum of two fields rotating in opposite directions with regard to each other. One of these will rotate in the same direction of the precession of the spin (co-rotating), and the other in the opposite direction (counter-rotating). Each one of these fields yields a different effect: the co-rotating one produces constant torque on the magnetic moment and the counter-rotating one, a time dependent torque that reverses itself with a frequency equal to twice the Larmor one. Thus, we can neglect the counter-rotating term of the oscillating field because its effect is small. This is originally known as the RWA [11].

In quantum optics, this reasoning can be applied because of the fact that every Hamiltonian in a two-state Hilbert space can be shown to be equivalent to a spin-1/2 interacting with a magnetic field. In spite of this, the idea of neglecting counter-rotating terms lacks a physical interpretation in the atom-field interaction context.

The standard justification of the RWA requires that the Rabi Hamiltonian (Eq. (2.34)) be expressed in the interaction picture with reference to $H_0 = (\hbar\omega_0/2)\sigma_z + \hbar\omega a^\dagger a$ and $V = \hbar g\left(\sigma_+ + \sigma_-\right)\left(a^\dagger + a\right)$. We achieve this by applying the unitary transformation:

$$H_I(t) = U_0^\dagger V U_0, \tag{2.35}$$

where

$$U_0 = \exp\left(\frac{-iH_0 t}{\hbar}\right). \tag{2.36}$$

A straightforward calculation yields:

$$\begin{aligned} H_I(t) = \hbar g(&\sigma_+ a \exp\left[i(\omega_0 - \omega)t\right] + \sigma_- a^\dagger \exp\left[-i(\omega_0 - \omega)t\right] \\ &+\sigma_+ a^\dagger \exp\left[i(\omega_0 + \omega)t\right] + a\sigma_- \exp\left[-i(\omega_0 + \omega)t\right]). \end{aligned} \tag{2.37}$$

In analogy with the nuclear magnetic resonance case, the terms $\sigma_+ a$ and $\sigma_- a^\dagger$ are called co-rotating terms and oscillate in time with the phase factors $\exp\left[\pm i(\omega_0 - \omega)\right]$, whereas the terms $\sigma_- a$ and $\sigma_+ a^\dagger$ are called counter-rotating terms, which oscillate with the phase factors $\exp\left[\pm i(\omega_0 + \omega)\right]$.

At near resonance, $\omega_0 \approx \omega$, the co-rotating terms oscillate slowly, while the counter-rotating terms oscillate rapidly; and if g is sufficiently small, $g \ll \min\{\omega_0, \omega\}$, then the time scales can be detached from one another and the counter-rotating terms can be substituted by their vanishing time average [12] regarding only the co-rotating terms.

In a more strict mathematical way, the previous argument can be justified if we consider the Dyson expansion of the evolution operator for the Rabi Hamiltonian instead, which is obtained from the following expression:

$$\begin{aligned} U(t) = I - \left(\frac{-i}{\hbar}\right)\int_0^t dt_1 H_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H_I(t_1)H_I(t_2) + \\ \left(\frac{-i}{\hbar}\right)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 H_I(t_1)H_I(t_2)H_I(t_3) + \ldots \end{aligned} \tag{2.38}$$

Replacing Eq. (2.37) into Eq. (2.38), we obtain expressions consisting of time-independent operators (e.g. $\sigma_+ a$, $\sigma_- a^\dagger$) accompanied by integrals of the following

form:

$$\int_0^t dt_1 \exp(i\alpha t_1) = \frac{-i\left[\exp(i\alpha t) - 1\right]}{\alpha}, \tag{2.39}$$

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \exp(i\alpha t_1)\exp(i\alpha t_2) = \frac{-\left[\exp(i\alpha t) - 1\right]^2}{\alpha^2} \tag{2.40}$$

and so on, where $\alpha = \pm i(\omega_0 \pm \omega)$. We can observe that the co-rotating terms are proportional to powers of $g/(\omega_0 - \omega)$ and the counter rotating terms are proportional to powers of $g/(\omega_0 + \omega)$. Thus, for the case of near resonance, $\omega_0 \approx \omega$, and when the relative coupling is weak, $g/\min\{\omega_0, \omega\} \ll 1$, the counter-rotating terms can be neglected.

Therefore, when the RWA is applied to the Rabi Hamiltonian (Eq. (2.34)), i.e., the counter-rotating terms are eliminated, the Rabi Hamiltonian is reduced to the Jaynes-Cummings Hamiltonian:

$$H_{JC} = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^\dagger a + \hbar g\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{2.41}$$

Discussion of the validity of the RWA

The validity of the Jaynes-Cummings Hamiltonian depends on the fulfillment of some conditions. There are many cases in which these conditions hold true. This has made the JC model a powerhouse of quantum optics, widely used and whose predictions successfully describe diverse physical phenomena, such as, Rabi oscillations, collapse-revivals of entanglement, Schrödinger cat states, and so on. These predictions have also been corroborated with experiments, among which was the first implementation of the Jaynes-Cummings Hamiltonian achieved with the one-atom maser [13]. However, the same cannot be said about the RWA, because, unlike other approximations that are based on mathematical or physical arguments, this one has several unsatisfactory features. One of the first features that was pointed out is that the equation obtained from applying the RWA to the atom-field Hamiltonian from which the master equation for the density operator is derived does not match the one obtained from applying the RWA to the master equation obtained from the original Hamiltonian [14]. This peculiarity of the RWA has prompted physicists to claim that the "generally accepted use of the RWA Hamiltonian of the system has changed into a sort of *symbol of faith*" [15].

Thus, the neglecting of the counter-rotating terms turns out to be an unsatisfactory argument. This point can be asserted by applying the unitary transformation

$S = \exp\left[-ixa^\dagger a - i\frac{\theta}{2}\sigma_z\right]$ to the Rabi Hamiltonian (Eq. (2.34)), which results in the exchange of the slowly and rapidly oscillating terms. Hence, the terms that are eliminated are the co-rotating ones.

Another debatable argument often used in textbooks is that the counter-rotating terms do not conserve energy because they correspond to non-physical processes. This occurs because $\sigma_- a$ corresponds to a transition from the ground to the excited state under emission of a photon; and $\sigma_+ a^\dagger$, to a transition from the excited to the ground state under absorption of a photon. The problem of this argument is the use of the word "energy". In the processes aforementioned, the system is considered as a *non-interacting* two-level atom with a quantized electromagnetic field; however, this is clearly not the case because the two bipartite systems are interacting with each other, so the energies are not proportional to $\hbar\omega_0$ and/or $\hbar\omega$.

Hence, the most suitable argument to justify the RWA is the one based on the work of Swain [16], which consists in expanding the interaction term H_{int} of the Rabi Hamiltonian in powers of g/ω. The near resonance condition, $\omega_0 \approx \omega$, is to ensure the two-level atom condition and the small relative coupling, $g/\omega \ll 1$, which allows us to disregard second order and higher terms, thereby obtaining the rotating terms and subsequently, the Jaynes-Cummings Hamiltonian (Eq. (2.41)).

2.1.3 Rabi vs Jaynes-Cummings Hamiltonian

The previous argument of the power series expansion of g/ω allows us to discuss the analytical solvability of the Jaynes-Cummings Hamiltonian against the Rabi Hamiltonian.

At the first order of g/ω, i.e., the Jaynes-Cummings Hamiltonian, the possible transitions of the initial state $|e, n\rangle$ are shown in the following figure:

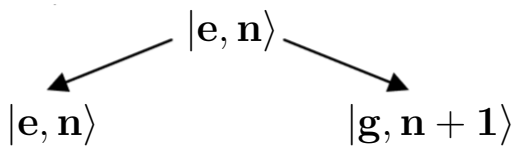

Figure 2.1: Possible transitions of the initial state $|e, n\rangle$ in the Jaynes-Cummings Hamiltonian.

Instead, if we go to the second order of g/ω, i.e., the Rabi Hamiltonian (up to the second order only), the possible transitions of the initial state $|e, n\rangle$ are shown in the following figure:

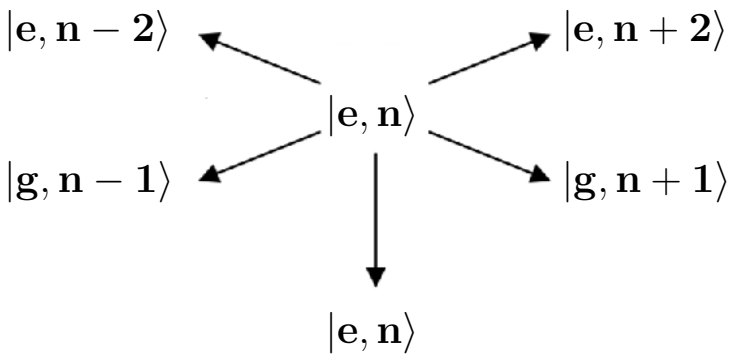

Figure 2.2: Possible transitions of the initial state $|e, n\rangle$ in the Rabi Hamiltonian (up to the second order). Adapted from [17].

From Fig. (2.1), we can deduce that the state $|e, n\rangle$ only couples with the state $|g, n + 1\rangle$. Therefore, the infinite-dimensional matrix representation of the Rabi Hamiltonian splits into a block-diagonal (two-dimensional) in the Jaynes-Cummings Hamiltonian, i.e., when the RWA is invoked. As a result, each (two-dimensional) block can be diagonalized.

On the other hand, from Fig. (2.2), we can deduce that when expanded to all orders of g/ω, the Rabi Hamiltonian couples $|e, n\rangle$ to an infinite number of states; as a consequence, the diagonalization of the Rabi Hamiltonian cannot be achieved.

This issue and the fact that the Rabi Hamiltonian has no second conserved quantity besides energy has led physicists to think that the Rabi Hamiltonian is non-integrable; in other words, it has no analytical solution. We will see that this is not exactly the case.

For the Jaynes-Cummings Hamiltonian, the other quantity that is conserved is the total number of excitations of the system $N_q = a^\dagger a + \sigma_+\sigma_-$, which leads to the diagonalization of the Hamiltonian in the subspaces $\{|e, n\rangle, |g, n + 1\rangle\}$. The conserved quantity N_q in the JC model produces a $U(1)$ symmetry, which is broken down into a $Z(2)$ symmetry when taking into account the counter-rotating terms. This symmetry is related to the parity operator $P = \exp(i\pi N_q)$, which is conserved in the Rabi Hamiltonian. This conservation of the parity of the states enables the division of the infinite-dimensional Hilbert space into two unconnected ones, also infinite-dimensional ($H = H_+ + H_-$). This property was used by Braak [18] to obtain an analytical solution to the Rabi Hamiltonian. However, these solutions depend on a transcendental function which in turn depends on the power series of g/ω. Therefore, the Rabi Hamiltonian is analytically solvable, but a closed-form solution has not been found yet.

Despite this fact, many numerical calculations and approximate solutions have been proposed for the Rabi Hamiltonian, some of which will be analyzed in the next

chapter.

2.2 Geometric Phases

The concept of geometric phase was introduced in 1984 by Berry in a somewhat limited context [19]. This was when a quantum system undergoes a unitary, cyclic and adiabatic evolution. Before Berry's work, it was assumed that when a quantum system undergoes a unitary cyclic evolution, it acquires only a dynamical phase which can be gauged away, i.e., eliminated, by redefining the state as $|\psi\rangle \to |\psi'\rangle = e^{i\alpha}|\psi\rangle$. Berry discovered that apart from the dynamical phase, there was another additional phase whose origin was geometrical that could not be eliminated. In the following years, this concept was generalized and redefined. In 1987, Aharonov and Anandan removed the condition of adiabacity [20]. In 1988, Samuel and Bhandari claimed that the evolution neither needed to be cyclic nor unitary [21, 22]. Finally, this concept was redefined by Mukunda and Simon in a purely kinematic approach [23].

Next, we are going to summarize two approaches to the geometric phase based on [24]. First, Berry's original approach to understand what it means when we say that we are calculating Berry's phase, and second, the more general approach to the geometric phase proposed by Simon and Mukunda, which is called the kinematic approach.

2.2.1 Berry's Approach: Berry's phase

Suppose we have a non-conservative system whose evolution is ruled by a time-dependent Hamiltonian $H(t)$. This happens when a system evolves under the influence of a changing environment whose configuration can be specified by a set of time-dependent parameters $\{R_1(t), R_2(t), ...\}$. A consequence of this is that all observables, especially the Hamiltonian $H(R(t)) \equiv H(R_1(t), R_2(t), ...)$, depend on these parameters.

Assuming that there is an orthonormal basis $|n, R(t)\rangle$ for every time t in such a way that:

$$H(R(t))|n, R(t)\rangle = E_n(R(t)|n, R(t)\rangle, \tag{2.42}$$

we can write the following general state:

$$|\psi(t)\rangle = \sum_k c_k(t)|k, R(t)\rangle. \tag{2.43}$$

The adiabatic approximation consists in taking an approximate solution of the Louiville equation:

$$i\hbar\frac{\partial\rho(t)}{\partial t} = [H(R(t)), \rho(t)] \tag{2.44}$$

as follows:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \approx |n, R(t)\rangle\langle n, R(t)|. \tag{2.45}$$

This means that if the system starts as an eigenstate of the Hamiltonian, it remains the same eigenstate of the Hamiltonian. It is important to note that Eq. (2.45) can only be an approximation because if we replace it in Eq. (2.44), we obtain a stationary state as a solution, which would be a contradiction. The adiabatic approximation (Eq. (2.45)) means that Eq. (2.43) can be written as follows:

$$|\psi(t)\rangle \approx c_n(t)|n, R(t)\rangle \tag{2.46}$$

with $c_n(0) = 1$.

In order to obtain the condition to consider an evolution adiabatic, we replace Eq. (2.46) in the Schrödinger equation:

$$i\hbar\frac{\partial|\psi\rangle}{\partial d} = H(R(t))|\psi\rangle, \tag{2.47}$$

thereby obtaining:

$$\frac{dc_n(t)}{dt}|n, R(t)\rangle \approx -c_n\left[iE_n(t)|n, R(t)\rangle + \frac{d}{dt}|n, R(t)\rangle\right]. \tag{2.48}$$

If we multiply Eq. (2.48) by $\langle k, R(t)|$, it reduces to:

$$\langle k, R(t)|\frac{d}{dt}|n, R(t)\rangle \approx 0, \text{ for all } k \neq n. \tag{2.49}$$

Deriving Eq. (2.42) with respect to t, we can write Eq. (2.49) in the following form:

$$\frac{\langle k, R(t)|dH(t)/dt|n, R(t)\rangle}{E_n(R(t)) - E_k(R(t))} \approx 0, \text{ for all } k \neq n. \tag{2.50}$$

This expression sets the time scale for $H(t)$ to be considered an adiabatic evolution; thus, the adiabatic approximation is correctly applied. This depends on the transition frequencies of the evolving system $(E_n(R(t)) - E_k(R(t)))/\hbar$.

In order to obtain the equation that the coefficient $c_n(t)$ of Eq. (2.46) must satisfy, we multiply Eq. (2.48) by $\langle n, R(t)|$, which results in:

$$\frac{dc_n(t)}{dt} = -c_n(t)\left[iE_n(t) + \langle n, R(t)|\frac{d}{dt}|n, R(t)\rangle\right], \tag{2.51}$$

whose solution is

$$c_n(t) = \exp\left[-i\int_0^t E_n(s)ds\right]\exp\left[-\int_0^t \langle n, R(s)|\frac{d}{ds}|n, R(s)\rangle ds\right]. \tag{2.52}$$

From Eq. (2.52), we define the following terms:

$$\Phi_{dyn}(t) \equiv \int_0^t E_n(s)ds, \tag{2.53}$$

$$\gamma_n(t) \equiv i\int_0^t \langle n, R(s)|\frac{d}{ds}|n, R(s)\rangle ds. \tag{2.54}$$

Finally, replacing Eq. (2.52) into Eq. (2.46), we obtain the final state after the evolution:

$$|\Phi(t)\rangle \approx \exp\left[-i\Phi_{dyn}(t)\right]\exp\left[i\gamma_n(t)\right]|n, R(t)\rangle. \tag{2.55}$$

Here, we see that besides the usual dynamical phase Φ_{dyn}, which can be gauged-away, another phase γ_n appears which is called the geometric phase. The choice of this name becomes obvious after realizing that γ_n can be written in the following way by using the chain rule for derivation:

$$\gamma_n(t) = i\int_{R(0)}^{R(t)} \langle n, R|\frac{\partial}{\partial R_k}|n, R\rangle dR_k \equiv \int_{R(0)}^{R(t)} \boldsymbol{A}^{(n)}.d\boldsymbol{R}, \tag{2.56}$$

where $\boldsymbol{A}^{(n)} \equiv i\langle n, R|\nabla|n, R\rangle$ is the vector potential known as the *Mead-Berry vector potential*. Eq. (2.56) makes it clear that γ_n does depend only on the path connecting the points $R(0)$ and $R(t)$ in the parameter space, showing us the geometric nature of γ_n.

If we do the gauge transformation $|n, R\rangle \to |n, R\rangle' = e^{i\alpha_n(R)}|n, R\rangle$, the vector potential and, as a consequence, the geometric phase, both transform, respectively, as follows:

$$\boldsymbol{A}^{(n)} \to \boldsymbol{A'}^{(n)} = \boldsymbol{A}^{(n)} - \nabla\alpha_n(R), \tag{2.57}$$

$$\gamma_n(t) \to \gamma_n'(t) = \gamma_n(t) - \left[\alpha_n(R(t)) - \alpha_n(R(0))\right]. \tag{2.58}$$

From Eq. (2.58), we can use the gauge freedom of $\alpha_n(R(t))$ in order to eliminate the phase factor γ_n. Indeed, if we repeat all the calculation up to Eq. (2.55) with the aforementioned gauge transformation $|n, R\rangle'$, we obtain Eq. (2.55) but with prime

quantities. Therefore, choosing $\alpha_n(R(t)) = -\gamma_n(t)$, we obtain:

$$|\psi(t)\rangle \approx \exp\left[-i\Phi_{dyn}(t)\right]|n, R(t)\rangle. \tag{2.59}$$

These results were already obtained by Fock when addressing adiabatic processes [25]. However, Berry first discovered that when we are dealing with cyclic evolution (with period T), i.e., $R(T) = R(0)$, we cannot use this gauge freedom in order to remove the geometric phase anymore. In fact, if $R(T) = R(0)$, then $\exp\left[i\alpha_n(T)\right] = \exp\left[i\alpha(0)\right]$, and $\alpha_n(T) = \alpha_n(0) + 2\pi m$ with m integer. Consequently, Eq. (2.58) becomes:

$$\gamma_n'(t) = \gamma_n(t) - 2\pi m. \tag{2.60}$$

From Eq. (2.60), we see that there is no gauge freedom, so the geometric phase cannot be eliminated.

In summary, whenever it is claimed that the Berry phase is being calculated, this means that Eq. (2.54) is being used and that the adiabatic condition $|\psi(t)\rangle\langle\psi(t)| \approx |n, R(t)\rangle\langle n, R(t)|$ is being assumed.

2.2.2 Kinematic Approach: Geometric phase

Next, we are going to see how all the important features of the geometric phase develop from a purely kinematic approach by using parallel transport.

First, we define a subset of the Hilbert space, $\mathcal{H}$, whose elements are non-null normalized vectors $|\psi\rangle$. A curve $\mathcal{C}_0$ in $\mathcal{H}_0$ is defined by a set of vectors $|\psi(s)\rangle$ which are a continuous function of a parameter $s \in [s_1, s_2]$. Because $|\psi(s)\rangle$ is normalized, i.e., $\langle\psi(s)|\psi(s)\rangle = 1$, $\langle\psi(s)|\dot{\psi}(s)\rangle$ has a null real part. This allows us to write:

$$\langle\psi(s)|\dot{\psi}(s)\rangle = i\mathrm{Im}\langle\psi(s)|\dot{\psi}(s)\rangle. \tag{2.61}$$

Additionally, we define the phase between the initial state $|\psi(s_1)\rangle$ and the final state $|\psi(s_2)\rangle$ of a curve $\mathcal{C}_0$, the so-called *total phase*, in the following way:

$$\Phi_{tot}(\mathcal{C}_0) = \arg\langle\psi(s_1)|\psi(s_2)\rangle \tag{2.62}$$

This phase is also called the Pancharatnam phase, which Pancharatnam himself defined in the context of polarized states of light [26]. The total phase is generally defined for any two non-orthogonal states, without reference to a curve that possibly joins them with one another.

Under a gauge transformation in the form of $|\psi(s)\rangle \rightarrow |\psi'(s)\rangle = e^{i\alpha(s)}|\psi(s)\rangle$,

the total phase $\Phi_{tot}(\mathcal{C}_0)$ and the quantity $\mathrm{Im}\langle\psi(s)|\dot{\psi}(s)\rangle$ transform, respectively, as follows:

$$\Phi_{tot}(\mathcal{C}_0) \to \Phi'_{tot}(\mathcal{C}_0) = \arg\langle\psi(s_1)|\psi(s_2)\rangle + \alpha(s_2) - \alpha(s_1), \tag{2.63}$$

$$\mathrm{Im}\langle\psi(s)|\dot{\psi}(s)\rangle \to \mathrm{Im}\langle\psi'(s)|\dot{\psi}'(s)\rangle = \mathrm{Im}\langle\psi(s)|\dot{\psi}(s)\rangle + \dot{\alpha(s)}. \tag{2.64}$$

Finally, from Eqs. (2.63-2.64), we can construct the following gauge invariant quantity as a functional of $\mathcal{C}_0$:

$$\Phi_g(\mathcal{C}_0) = arg\langle\psi(s_1)|\psi(s_2)\rangle - Im\int_{s_1}^{s_2}\langle\psi(s)|\dot{\psi}(s)\rangle ds. \tag{2.65}$$

This quantity is not only gauge invariant, but also re-parametrization invariant (change in the parameter s). Gauge and re-parametrization invariance means that although we define Φ_g as a functional of $|\psi(s)\rangle$, hence, of $\mathcal{C}_0$, it is actually a functional of some equivalent class of $|\psi(s)\rangle$. As a matter of fact, the set $\{|\psi'\rangle = e^{i\alpha}|\psi\rangle\}$ is known as the ray space $\mathcal{R}_0$. In our case, it is useful to work with the projector $|\psi\rangle\langle\psi|$, which is defined by means of a projection map $\pi : \mathcal{H}_0 \to \mathcal{R}_0$. Under this projection, the curve $\mathcal{C}_0 \in \mathcal{H}_0$ projects onto $C_0 \in \mathcal{R}_0$, which is the curve defined by $|\psi(s)\rangle\langle\psi(s)|, s \in [s_1, s_2]$. Therefore, we write the geometric phase as a functional of C_0 as follows:

$$\Phi_g(C_0) = \Phi_{tot}(\mathcal{C}_0) - \Phi_{dyn}(\mathcal{C}_0) \tag{2.66}$$

with

$$\Phi_{dyn}(\mathcal{C}_0) = Im\int_{s_1}^{s_2}\langle\psi(s)|\dot{\psi}(s)\rangle ds. \tag{2.67}$$

Lastly, we emphasize the fact that this approach is called kinematic because it only depends on the curve C_0 described during the evolution, not on the cause of the evolution. Also, we want to highlight that this approach does not depend on the assumptions made by Berry; therefore, the curve does not have to be closed, and the evolution does not have to be adiabatic or unitary.

Chapter 3

Berry Phase in Rabi Hamiltonian: Controversy and Results

3.1 The Controversy

Since Berry's discovery [19], geometric phases in quantum theory have drawn a lot of attention. However, they were studied as a quantum system that undergoes an evolution produced by an interaction with a classical field until 2002 with the work of Fuentes-Guridi *et al.* [4], in which they calculated the Berry phase of a spin-1/2 interacting with a magnetic field, while addressing the quantum nature of the field. In more details, the Berry phase studied is the one that is acquired when an eigenstate of the Jaynes-Cummings Hamiltonian performs an adiabatic and cyclic evolution ruled by the unitary operator $U(\varphi) = exp(-i\varphi a^\dagger a)$. To achieve the adiabatic and cyclic conditions, φ varies slowly from 0 to 2π.

The reason why the authors chose the particular evolution $U(\varphi)$ is because when applied to the Jaynes-Cummings Hamiltonian (in units of $\hbar = 1$),

$$H_{JC} = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g\left(\sigma_+ a + \sigma_- a^\dagger\right), \tag{3.1}$$

we obtain the following expression:

$$U(\varphi)H_{JC}U^\dagger(\varphi) = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g\left(\sigma_+ a e^{i\varphi} + \sigma_- a^\dagger e^{-i\varphi}\right). \tag{3.2}$$

Thus, the effect of U is to phase-shift the operators a and $a^\dagger$. If we replace a and $a^\dagger$ by their corresponding classical amplitude in Eq. (3.2), we obtain the semiclassical Jaynes-Cummings Hamiltonian, i.e., a classical field interacting with a two-level atom. This is the reason for the choice of $U(\varphi)$.

The results obtained by Fuentes-Guridi *et al.* for the Berry phases $\gamma_{n,\pm}$ are the

following:

$$\gamma_{n,+} = \pi(1 - \cos\theta_n) + 2\pi n, \tag{3.3}$$

$$\gamma_{n,-} = -\pi(1 - \cos\theta_n) + 2\pi(n+1). \tag{3.4}$$

Here, $\cos\theta_n = (\omega_0 - \omega)/\sqrt{(\omega_0 - \omega)^2 + 4g^2(n+1)}$. From these results, we can see that the Berry phase is different from zero, even in the case of vacuum, i.e., $n = 0$.

On the other hand, in 2012, Larson claimed that the non-vanishing Berry phase (Eqs. (3.3-3.4)) is an artifact of the RWA based upon the fact that in his work, he discovered that when not invoking the RWA, i.e., when addressing the Rabi Hamitlonian, the Berry phase is null regardless of the parameter choices [2]. In order to arrive at this conclusion, Larson addressed the semiclassical case and calculated the corresponding energy surfaces in phase-space for the Jaynes-Cummings and Rabi Hamiltonian, obtaining the following plots:

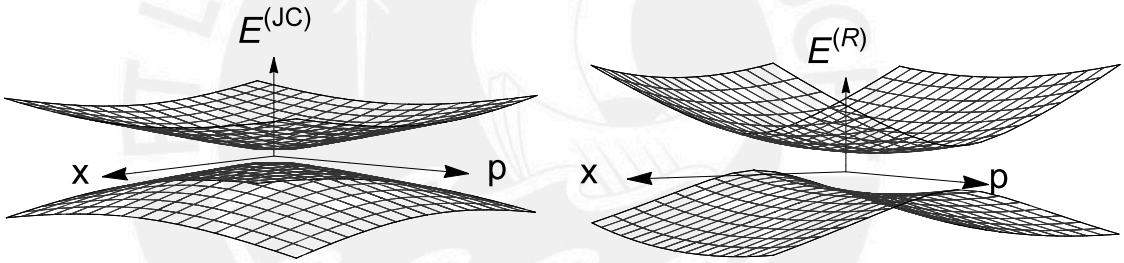

Figure 3.1: Left figure: semiclassical energy surface for the Jaynes-Cummings Hamiltonian shows a CI; thus, there is a non-vanishing Berry phase in this model. Right figure: semiclassical energy surface for the Rabi Hamiltonian does not show a CI; thus, there is a vanishing Berry phase in this model. Adapted from [2].

From Fig. (3.1), we can see that for the Rabi's semiclassical Hamiltonian, the energy surface does not contain a conical intersection (CI), whereas the one for the Jaynes-Cummings's semiclassical Hamiltonian does for (x,p)=(0,0). Trajectories in the above mentioned surfaces that encircle CIs produce non-vanishing Berry phases [25, 27]. Based on this fact, Larson allegated that the Rabi model gives rise to null Berry phases, but in the Jaynes-Cummings model, i.e., when the RWA is applied, there are non-vanishing Berry phases.

At this point, we can argue that this occurs only in the semiclassical case; however, when we address the fully quantum case, the conclusions could be different. Nevertheless, Larson asserted that numerical calculations for the Rabi Hamiltonian,

show that the Berry phase vanishes independently from the parameter choices [2]. This was shown a year later in 2013, when Larson presented the numerical diagonalization of the transformed Rabi Hamiltonian $H_R(\varphi) \equiv U(\varphi)H_R U^\dagger(\varphi)$, which manifests that regardless the value of g, the Berry phase nullifies when $\varphi = 2\pi$ [3].

All the above mentioned results led Larson to claim that "imposing the RWA in certain cavity QED systems can impart incorrect results regardless of system parameters" [2]. This statement is very controversial because Larson claims that there is a physical phenomenon, in this case the Berry phase, for which the RWA does not apply even when the conditions $\omega_0 \approx \omega$ and $g/\omega \ll 1$ are satisfied. Even though the justification of the RWA is not very clear, this approximation has predicted many physical phenomena seen in experiments, which leads us to the conclusion that there is something incongruent in Larson's claim.

The conclusions Larson reached have been disputed by several authors. For instance, Wang *et al.* [7] object the way that Larson performs his semiclassical approximation, in which the operators a and $a^\dagger$ are replaced by the corresponding complex numbers α and α^*. Wang *et al.* claim that the correct way to do the semiclassical approximation consists in applying the variational method to the effective Hamiltonian $H_{eff}(\alpha) \equiv \langle\alpha|H_R|\alpha\rangle$, in which $|\alpha\rangle$ is a coherent state, i.e., $|\alpha\rangle$ satisfies the equation $a|\alpha\rangle = \alpha|\alpha\rangle$. By doing this, a non-vanishing Berry phase appears in the Rabi model. Other authors calculate approximate solutions for the Berry phase in the Rabi Hamiltonian that compute non-vanishing Berry phases [5, 6]. In addition, very recently, semi-analytical results have shown that the energy landscape of the Rabi Hamiltonian (full quantum case) does indeed present conical intersections [28].

Despite all these refutations, a final word on this controversy has not been said. In order to help to settle this issue, we present analytical and numerical arguments. Before doing this, we present a calculation of the eigenvalues of the Rabi Hamiltonian to highlight the idea that the predictions of the Rabi model and the Jaynes-Cummings model must agree when the RWA applies and to exhibit the accuracy of our numerical results.

3.2 Eigenvalues of the Rabi Hamiltonian

To achieve the eigenvalues of the Rabi Hamiltonian (in units of $\hbar = 1$),

$$H_R = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g\left(\sigma_+ + \sigma_-\right)\left(a^\dagger + a\right), \tag{3.5}$$

we numerically diagonalyzed the truncated Rabi matrix 22×22, obtaining the six first eigenvalues with an accuracy of 99.9% for three relative detunings $\Delta' \equiv (\omega_0 -$

$\omega)/\omega$, one corresponding to the resonant case ($\Delta' = 0$) and the other two, to non-resonant cases ($\Delta' = \pm 0.5$). This was attained by using an $N \times N$ Rabi matrix and fixed N so that the results would not change by more than 0.1% if N were increased to $N + 1$.

In these results, we also included the first six eigenvalues for the Jaynes-Cummings Hamiltonian, which are written (in units of $\hbar = 1$) as follows:

$$E_{n,\pm} = \omega \left(n + \frac{1}{2} \right) \pm \frac{1}{2} \sqrt{\Delta^2 + 4g^2(n+1)} \tag{3.6}$$

From the results in Fig. (3.2), we can see that for sufficiently small values of g/ω the eigenvalues of the Rabi Hamiltonian converge to the corresponding ones of the Jaynes-Cummings Hamiltonian. This is proof that the RWA correctly applies when dealing with eigenvalues as it should be in the case of any other physical observable described within the Rabi and Jaynes-Cummings model.

In addition, in order to show the accuracy of these numerical results, we compared the numerically calculated eigenvalues of the Rabi Hamiltonian with analytically approximated solutions given by Irish [29] and by Zhang *et al.* [30]. The former one is based on the generalized rotating-wave approximation (GRWA), which consists in performing a change of basis before eliminating the counter-rotating terms and reducing the matrix to a 2×2 block-diagonal form; and the latter one, on a coherent-state approach.

From Fig. (3.3), we can observe that with a negligible difference, the numerical results agree with the results of Irish, but in the case of Zhang *et al.*, small differences are observed for some values of g/ω. A hypothesis that can explain the aforementioned differences is that in the corresponding deduction of their approximated solution, Zhang *et al.* claim that some coefficients D_{mn} for $m \neq n$ can be neglected if $\lambda/\omega \gg 1$ [30]; however, this claim has not been correctly justified because $D_{mn}(x)$ is a damped oscillatory function of x.

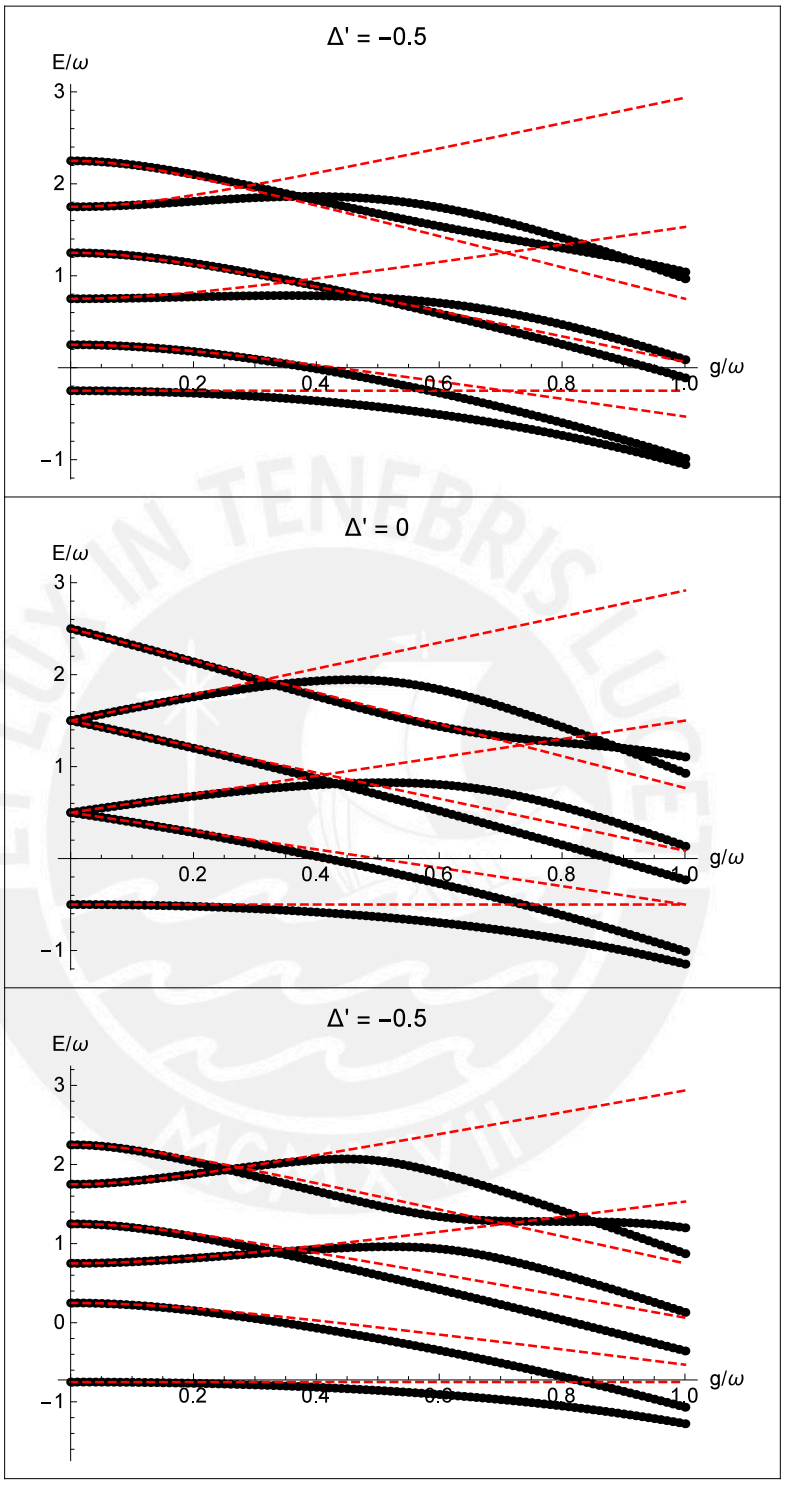

Figure 3.2: Eigenvalues (in units of ω) of the Rabi and Jaynes-Cummings Hamiltonian as a function of g/ω for three different detunings $\Delta' \equiv (\omega_0 - \omega)/\omega$. The solid black lines correspond to the numerical results of the Rabi Hamiltonian, and the dashed red lines correspond to the analytical results of the Jaynes-Cummings Hamiltonian.

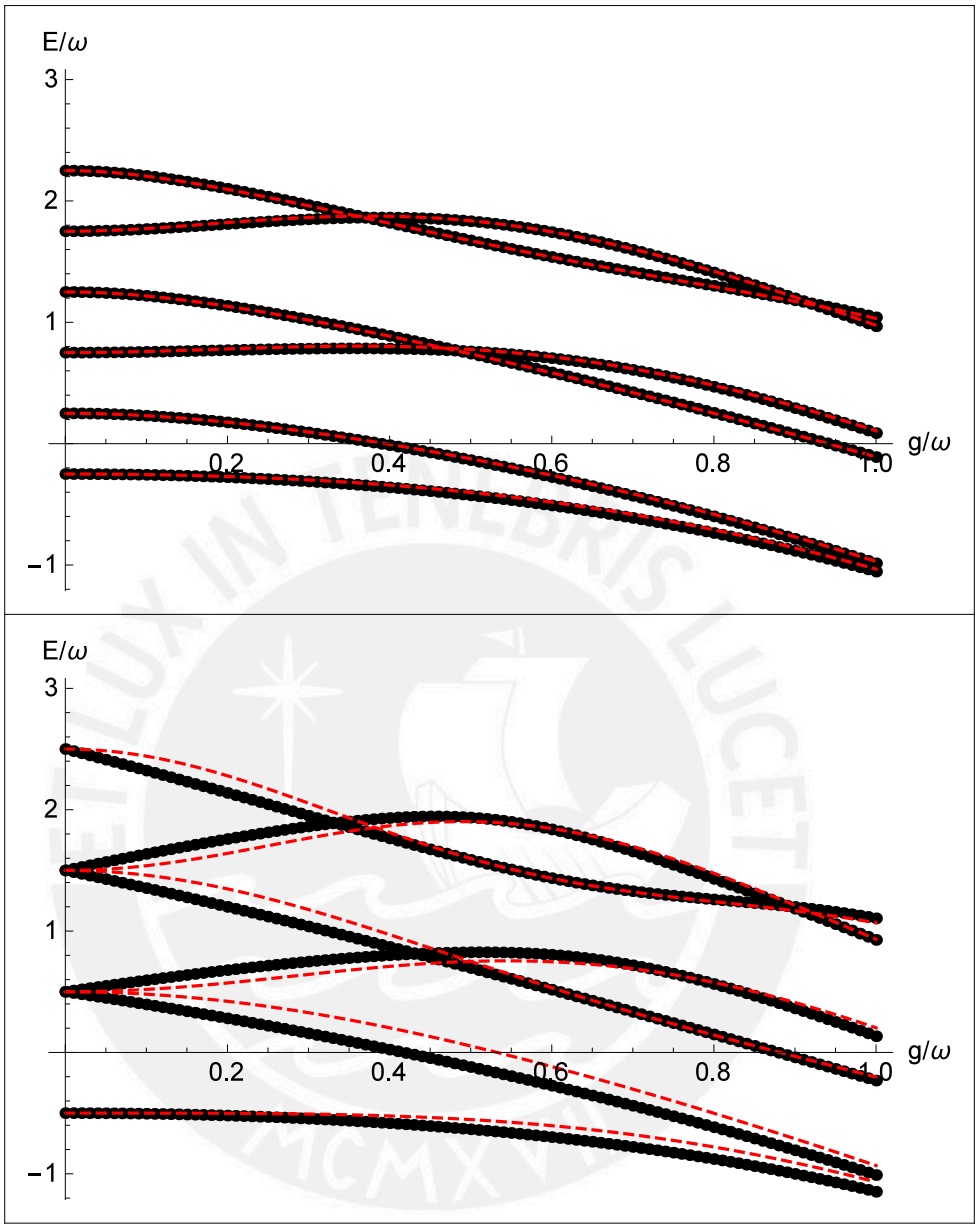

Figure 3.3: Eigenvalues (in units of ω) of the Rabi and Jaynes-Cummings Hamiltonian as a function of g/ω. The solid black lines correspond to the numerical results of the Rabi Hamiltonian, while the dashed red lines correspond to analytical approximations provided by Irish (left panel) and by Zhang *et al.*(right panel). In the case of Irish, we consider the non-resonant case $\Delta' = -0.5$ and for Zhang *et al.*, the resonant case $\Delta' = 0$.

3.3 Berry Phase of the Rabi Hamiltonian

We first present an analytical argument regarding Berry phases in the Rabi Hamiltonian which shows that there is a non-vanishing Berry phase in the Rabi case. This argument is in conflict with Larson's findings [2, 3].

First, in order to calculate the Berry phase in the Rabi model, we use the expression from Sec. (2.2.1) [2, 4–7]:

$$\gamma_n(t) = i \int_0^t \langle \Psi_n(s)| \frac{d}{ds} |\Psi_n(s)\rangle ds. \tag{3.7}$$

In this case, the parameter s is $\varphi \in [0, 2\pi]$ and the state $|\Psi_n(s)\rangle$ is $U(\varphi)|\psi_n\rangle$, where $U(\varphi) = exp(-i\varphi a^\dagger a)$ and ψ_n is the n-th eigenstate of the considered Hamiltonian. As a result, Eq. (3.7) can be written as follows:

$$\gamma_n = i \oint \langle \psi_n| U^\dagger(\varphi) \frac{d}{d\varphi} U(\varphi) |\psi_n\rangle d\varphi. \tag{3.8}$$

A straightforward calculation yields:

$$U^\dagger(\varphi)\frac{d}{d\varphi}U(\varphi) = U^\dagger(\varphi)\left(-ia^\dagger a\right)U(\varphi) = U^\dagger(\varphi)U(\varphi)\left(-ia^\dagger a\right) = -ia^\dagger a \tag{3.9}$$

Replacing Eq. (3.9) in Eq. (3.8), we obtain:

$$\gamma_n = 2\pi \langle \psi_n| a^\dagger a |\psi_n\rangle. \tag{3.10}$$

Using this expression, we can calculate the Berry phase for the Rabi or Jaynes-Cummings Hamiltonian by substituting the respective eigenvector $|\psi_n\rangle$.

In general, we can write an eigenstate of the Rabi Hamiltonian in the following form:

$$|\psi_n\rangle = \sum_{m=0}^{\infty} \left(A_m^n |e, m\rangle + B_n^m |g, m\rangle\right), \tag{3.11}$$

where $\{|e, m\rangle, |g, m\rangle\}_{m=0}^{\infty}$ is a basis of the Hilbert space on which the Rabi Hamiltonian acts upon. Here, $|e, m\rangle \equiv |e\rangle \otimes |m\rangle$ and $|g, m\rangle \equiv |g\rangle \otimes |m\rangle$, in which the field is in a Fock state $|m\rangle$ with m photons.

Replacing Eq. (3.11) in Eq. (3.10), we obtain:

$$\gamma_n = 2\pi \sum_{m=0}^{\infty} m \left(|A_m^n|^2 + |B_m^n|^2\right). \tag{3.12}$$

From Eq. (3.12), we can deduce that the only instance when the Berry phase vanishes is when $A_m^n = 0 = B_m^n, \forall m > 0$, which clearly does not correspond to an eigenvector of the Rabi Hamiltonian. This result is in discrepancy with Larson's findings; however, he has reported numerical results endorsing his conclusions. Even though we are not able to determine what is wrong in Larson's numerical calculation, we are confident that our numerical results in Fig. (3.4) are in agreement with Eq. (3.12) and with other authors' findings [5–7].

We present four Berry phases associated with eigenvectors of the Rabi Hamiltonian using numerical calculations with the 20×20 Rabi matrix with an accuracy of 99.9%, defined in the same way as in Sec. (3.2). In this case, we need a smaller matrix to achieve this accuracy because we only calculated four Berry phases instead of the six eigenvalues of Sec. (3.2).

From Fig. (3.4), we can observe that the Berry phases in the Rabi Hamiltonian are non-vanishing, indeed. These results are in agreement with the approximated solutions of Liu *et al.* and Deng *et al.*. Also, and most importantly, we can see that the Berry phases for the Rabi Hamiltonian converge to the corresponding ones in the Jaynes-Cummings Hamiltonian when g/ω is sufficiently small, i.e., when the RWA can be applied. Therefore, we can conclude that the RWA is also consistent in the instance of Berry phases as it was expected to be the case.

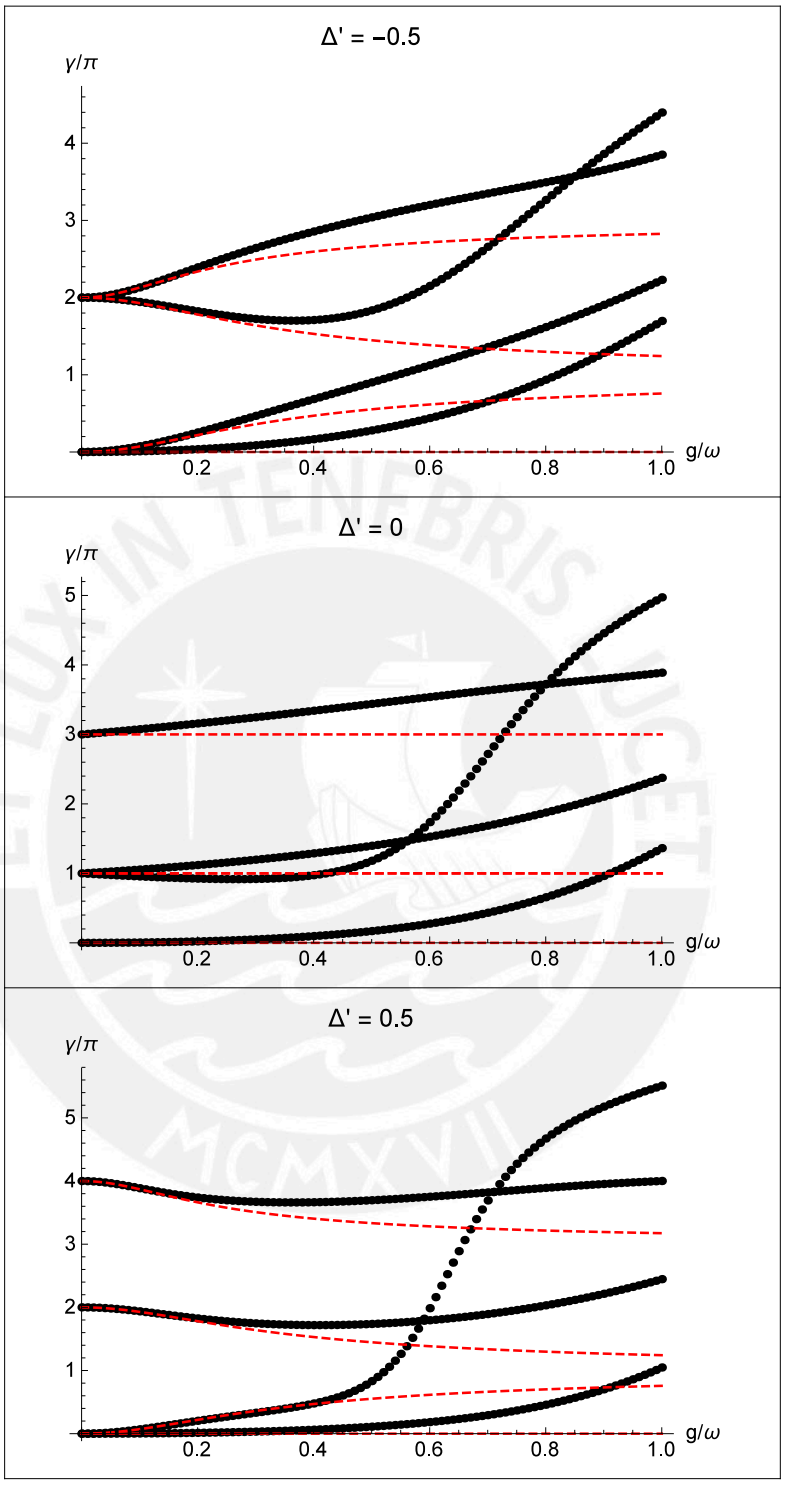

Figure 3.4: Berry phases γ_n (in units of π) of the Rabi and Jaynes-Cummings Hamiltonian as a function of g/ω for three different detunings $\Delta' \equiv (\omega_0 - \omega)/\omega$. The solid black lines correspond to the numerical results of the Rabi Hamiltonian, while the dashed red lines correspond to the analytical solutions of the Jaynes-Cummings Hamiltonian.

Chapter 4

Other Theoretical Aspects of the Controversy

4.1 Misconceptions Related to the Controversy

In spite of what we have concluded in Sec. (3.3), this controversy goes beyond whether there is a vanishing or non-vanishing Berry phase in the Rabi Hamiltonian. There are two points regarding this issue we would like to discuss next.

The first issue is that what we are dealing with in this controversy is not Berry's phase, but a geometric phase instead. Let us recall from Sec. (2.2) that the geometric phase is a general concept which has the Berry phase as a particular case. In order to explain this in more detail, we need to remember that in Sec. (2.2) the geometric phase was defined as:

$$\gamma(\mathcal{C}) = \arg\langle\psi(0)|\psi(s)\rangle + i\int_0^s \langle\psi(s')|\frac{d}{ds'}|\psi(s')\rangle ds'. \tag{4.1}$$

One of the properties of the geometric phase $\gamma(\mathcal{C})$ is that it is invariant under local gauge transformations $|\psi(s)\rangle \rightarrow |\psi'(s)\rangle = \exp[i\alpha(s)]|\psi(s)\rangle$. This property allows us to nullify either one of the contributions, the total phase or the dynamic one. In particular, if we use the gauge transformation $|\psi(s')\rangle \rightarrow |\psi'(s')\rangle = \exp[-i\arg\langle\psi(0)|\psi(s')\rangle]|\psi(s')\rangle$, we nullify the total phase [31]. Thus, using this newly gauge-transformed state and reverting to unprimed notation, Eq. (4.1) reduces to:

$$\gamma(\mathcal{C}) = i\int_0^s \langle\psi(s')|\frac{d}{ds'}|\psi(s')\rangle ds'. \tag{4.2}$$

Therefore, whenever $s' = t$ and $|\psi(t)\rangle$ satisfies the (gauge-transformed) Schrödinger equation with the initial condition $|\psi(t)\rangle = |n, R(0)\rangle$, Eq. (4.2) coincides with the expression deduced by Berry (Eq. (2.54)). If we also assume the adiabatic condi-

tion $|\psi(t)\rangle\langle\psi(t)| \approx |n, R(t)\rangle\langle n, R(t)|$, then $\gamma(\mathcal{C})$ is called Berry phase according to common parlance.

In spite of the fact that Fuentes-Guridi *et al.* [4] and Larson [2] mentioned the adiabatic approximation in their respective research, this approximation was not used in any of their calculations. Although the experimental implementation of the evolution $U(\varphi) = \exp[-i\varphi a^\dagger a]$ may require the adiabatic condition, this does not mean that the geometric phase acquired is linked to the adiabaticity of the Hamiltonian evolution.

Delving into this issue, the Rabi and Jaynes-Cummings Hamiltonian are time-independent, so the evolution generated by these does not generate a geometric phase. Indeed, when $|\psi_n\rangle$ is an eigenvector of the Rabi or Jaynes-Cummings Hamiltonian (H), which is the case that we are dealing with in this controversy, it only acquires a trivial phase in the form of $\exp[-iHt]|\psi_n\rangle = \exp[-iE_nt]|\psi_n\rangle$. In order to produce a geometric phase, the authors propose the unitary transformation $H \to U(\varphi)HU^\dagger(\varphi)$ and invoke it to be applied "adiabatically", i.e., φ has to vary slowly [2]. For this to happen, φ and, as a consequence, $U(\varphi)$ must be time-dependent. Therefore, the corresponding Schrödinger equation when the system evolves under the time-dependent unitary transformation $U(\varphi(t))$ reads:

$$i\frac{\partial|\psi'\rangle}{\partial t} = \left(U(\varphi)HU^\dagger(\varphi) + i\frac{\partial U(\varphi)}{\partial t}U^\dagger(\varphi)\right)|\psi'\rangle \equiv H'(t)|\psi'\rangle. \tag{4.3}$$

From Eq. (4.3), we can observe that a well-defined Berry phase must be related to the evolution produced by the time-dependent Hamiltonian $H'(t)$. With this, we can conclude that any phase acquired by the evolution $U(\varphi) = \exp[-i\varphi a^\dagger a]$ cannot be a Berry phase. On the other hand, we can properly define a geometric phase from Eq. (4.1) by setting $s' = \varphi$ as well as $|\psi(\varphi)\rangle = U(\varphi)|\psi_n\rangle$ and choosing to work on a closed trajectory, which is the case addressed in the controversy, or by using the gauge which imposes that $\arg\langle\psi(s=0)|\psi(s=2\pi)\rangle$.

The second issue of our discussion is the role that the Hamiltonian plays in this controversy. As we have seen before, the geometric phase associated to the evolution of the respective eigenvector is not ruled by the Rabi or Jaynes-Cummings Hamiltonian, but instead only by the unitary operator $U(\varphi)$. The Hamiltonians are only providers of the respective initial eigenvectors $|\psi_n\rangle$; once they are fixed, their evolution is ruled by $U(\varphi)$. Instead of using the Hamiltonians as providers, we can use other unitary or even non-unitary evolutions not related in any way to the Hamiltonians. Thus, the point we want to put under debate is how valid it is to associate the geometric phase to the Rabi or Jaynes-Cummings Hamiltonian when these do not rule the evolution linked to this phase.

These two points previously discussed bring to the fore the nature of the geometric phase, which is linked only to the geometry of the ray space. This fact was summarized by Rakhecha and Wagh, who defined the geometry phase as "the Hamiltonian-independent, nonintegrable component of the total phase, depending exclusively on the geometry in the ray space" [32]. Taking this into account, we find it more suitable to adopt another formulation for the calculation of the geometric phase.

In this formulation, the parameters R^μ are no longer related to the Hamiltonian, but instead only to the state vectors $|\psi\rangle = |\psi(R)\rangle$. If the parameters depend on s, i.e., $R^\mu = R^\mu(s)$, then $\langle\psi(s)|\dot{\psi}(s)\rangle = \langle\psi|\partial_\mu\psi\rangle\dot{R}^\mu(s)$, where the dot notation stands for the derivative respect to s and $\partial_\mu \equiv \partial/\partial R^\mu$. For a closed trajectory or for the gauge that nullifies the total phase, the geometric phase can be written as:

$$\gamma(\mathcal{C}) = i\oint_{\mathcal{C}}\langle\psi(s)|\dot{\psi}(s)\rangle ds \equiv -\oint_{\mathcal{C}}\mathcal{A}. \tag{4.4}$$

Here, $\mathcal{A} \equiv \mathcal{A}_\mu dR^\mu$ with $\mathcal{A}_\mu = \langle\psi|\partial_\mu\psi\rangle$. If we define $\mathcal{F} \equiv d\mathcal{A} = (\mathcal{F}_{\mu\nu}/2)dR^\mu \wedge dR^\nu$ and $\mathcal{S}$ as the surface bounded by the closed path $\mathcal{C}$, using Stoke's theorem, we can write Eq. (4.4) as follows [33, 34]:

$$\gamma(\mathcal{C}) = -\int_{\mathcal{S}}\mathcal{F}, \tag{4.5}$$

where

$$\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu} = 2\mathrm{Im}\langle\partial_\mu\psi|\partial_\nu\psi\rangle. \tag{4.6}$$

When the R^μ dependence of the state vectors comes from a unitary operator $U(R)$, i.e., $|\psi(R)\rangle = U(R)|\psi(0)\rangle$, we can alternatively write Eq. (4.6) as:

$$\mathcal{F}_{\mu\nu} = 2\mathrm{Im}\langle\psi|(\partial_\mu U)^\dagger\partial_\nu U|\psi\rangle. \tag{4.7}$$

As we can see, the geometric phase depends on the types of evolutions U considered. This issue will be dealt within the next section.

4.2 Evolution $U(\theta, \phi)$ in the Schwinger Model

In order to illustrate how the types of evolution affect the geometric phase, we are going to address an evolution based on Schwinger's approach to angular momentum.

First, we consider a Hilbert space $\mathcal{H}_T = \mathcal{H}_2 \otimes \mathcal{H}_a \otimes \mathcal{H}_b$, which is the direct product of three Hilbert spaces whose orthonormal bases are $\{|+\rangle, |-\rangle\}$, $\{|n\rangle_a\}_{n=0}^{\infty}$, and $\{|n'\rangle_b\}_{n'=0}^{\infty}$, respectively. The two last ones are Fock bases associated to two

different modes. We then choose the following evolution expressed as the unitary operator:

$$U(\theta,\phi) = \exp\left(-i\phi J_z\right)\exp\left(-i\theta J_y\right). \tag{4.8}$$

Here, the angular momentum operators are defined in terms of the annihilation and creation operators of modes a and b according to Schwinger's approach:

$$J_x = \frac{1}{2}\left(a^\dagger b + a b^\dagger\right), \tag{4.9}$$

$$J_y = \frac{1}{2i}\left(a^\dagger b - a b^\dagger\right), \tag{4.10}$$

$$J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right). \tag{4.11}$$

Next, we apply $U(\theta,\phi)$ to the following initial state in which we obviate the identity operator of $\mathcal{H}_2$:

$$|\Psi^{\pm}_{n,n'}\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle|n\rangle_a \pm |-\rangle|n+1\rangle_a\right]\otimes|n'\rangle_b \equiv \frac{1}{\sqrt{2}}\left[|+,n\rangle \pm |-,n+1\rangle\right]|n'\rangle. \tag{4.12}$$

Replacing Eqs. (4.8-4.12) into Eq. (4.7), we observe that the only non-vanishing terms are $\mathcal{F}_{\theta\phi} = -\mathcal{F}_{\phi\theta}$. Also, taking into consideration that $\partial U(\theta,\phi)/\partial\theta = -iU(\theta,\phi)J_y$ and $\partial U(\theta,\phi)/\partial\phi = -iJ_z U(\theta,\phi)$, we obtain:

$$\mathcal{F}_{\phi\theta} = 2\mathrm{Im}\langle\Psi^{\pm}_{n,n'}|U^\dagger(\theta,\phi)J_z U(\theta,\phi)J_y|\Psi^{\pm}_{n,n'}\rangle. \tag{4.13}$$

From the previous equation, we observe that in order to calculate the geometric phase, we need to calculate first the quantity $\langle U(\theta,\phi)^\dagger J_z U(\theta,\phi)J_y\rangle$. Recalling that $U(\theta,\phi)$ is a rotation operator, we get:

$$\langle U^\dagger J_z U J_y\rangle = \frac{1}{2i}\sin\theta\langle J_z\rangle, \tag{4.14}$$

This means that in order to calculate $\langle U^\dagger(\theta,\phi)J_z U(\theta,\phi)J_y\rangle$, we need to compute the mean value of J_z in the state $|\Psi^{\pm}_{n,n'}\rangle$. The action of J_z on $|\Psi^{\pm}_{n,n'}\rangle$ is as follows:

$$J_z|\Psi^{\pm}_{n,n'}\rangle = \frac{1}{2\sqrt{2}}\left[(n-n')|+,n\rangle \pm (n+1-n')|-,n+1\rangle\right]|n'\rangle. \tag{4.15}$$

Replacing Eqs. (4.14-4.15) in Eq. (4.13), we obtain:

$$2\mathrm{Im}\langle\Psi^{\pm}_{n,n'}|U^\dagger(\theta,\phi)J_z U(\theta,\phi)J_y|\Psi^{\pm}_{n,n'}\rangle = -\sin\theta\left[(n-n')/2 + 1/2\right]/2. \tag{4.16}$$

Inserting this expression in Eq. (4.5), we get:

$$\gamma_{nn'} = -\int_S \mathcal{F} = -\iint \mathcal{F}_{\phi\theta} d\theta d\phi = \frac{1}{2}\left[(n-n') + \frac{1}{2}\right] \iint \sin\theta d\theta d\phi. \tag{4.17}$$

If we define Ω as the solid angle subtended by the closed path $\mathcal{C}$, we finally obtain:

$$\gamma_{nn'} = \left[(n-n') + \frac{1}{2}\right] \frac{\Omega}{2}. \tag{4.18}$$

This result was first found by Fuentes-Guridi *et al.* [4]. As we have seen previously, neither the Hamiltonian nor the adiabatic approximation needed to be invoked in order to obtain the geometric phase $\gamma_{nn'}$. Even more, referring to a Hamiltonian may sometimes lead to incorrect physical interpretations regarding geometric phases, one instance being the case of Fuentes-Guridi *et al.*. We need to recall that their result was interpreted in terms of a Hamiltonian H_0^{2q}, which describes a scheme where a two-level atom interacts with two modes of an electromagnetic field. This Hamiltonian is obtained from the resonant Jaynes-Cummings Hamiltonian plus an additional term $\omega b^\dagger b$ that is related to the second field mode, i.e., $H_0^{2q} = H_{JC}(\omega_0 = \omega) + \omega b^\dagger b$. Initially, there is no interaction between the two-level atom and the field; however, when the $U(\theta, \phi)$ is applied, the interaction is produced [4, 35]. This led Fuentes-Guridi *et al.* to claim that the non-vanishing geometric phase $\gamma_0 = \Omega/4$, which arises when the field is in the vacuum state, is a feature that has "no semiclassical correspondence on account of the absence of a classical interpretation of a vacuum state" [4]. Nevertheless, as we shall see next, this conclusion is not correct.

4.2.1 Connection Between Schwinger's Approach and Angular Momentum Algebra

First, we will give a brief review of how Schwinger's approach connects to standard angular momentum algebra. The standard basis of the latter one $\{|j,m\rangle, j = 0, 1/2, 1, \ldots; m = -j, \ldots, j\}$ consists of common eigenvectors of the commuting operators $\mathbf{J}^2$ and J_z, i.e., $\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle$ and $J_z|j,m\rangle = m|j,m\rangle$. It can be proved that there is a one-to-one correspondence between Schwinger's states $|n,n'\rangle$ and the standard ones, $|j,m\rangle$, which is given by $n = j + m$, $n' = j - m$. The Hilbert space $\mathcal{H}_T = \mathcal{H}_2 \otimes \mathcal{H}_a \otimes \mathcal{H}_b$ is isomorph to $\mathcal{H}_T = \mathcal{H}_2 \otimes \mathcal{H}_J$, where $\mathcal{H}_2 = \text{Span}\{|+\rangle, |-\rangle\}$ and $\mathcal{H}_J = \text{Span}\{|j,m\rangle\}$. Hence, $\mathcal{H}_T = \text{Span}\{|\pm, j, m\rangle\}$.

The initial states addressed in Eq. (4.12) have the feature of being entangled

states, which can be written using the basis $|j,m\rangle$ in the following form:

$$|\Psi^{\pm}_{n,n'}\rangle = \frac{1}{\sqrt{2}}\left[|+,j,m\rangle \pm |-,j+1/2,m+1/2\rangle\right]. \tag{4.19}$$

Here, we can eliminate $|\pm\rangle \in \mathcal{H}_2$ because explicitly the evolution $U(\theta,\phi)$ reads $I_2 \otimes U(\theta,\phi)$; thus, it does not affect the subspace $\mathcal{H}_2$. As a result, we consider that the entangled state $|\Psi^{\pm}_{n,n'}\rangle$ can be rewritten in the following superposition state:

$$|\Psi^{\pm}_{n,n'}\rangle \equiv \frac{1}{\sqrt{2}}\left[|j,m\rangle \pm |j+1/2,m+1/2\rangle\right]. \tag{4.20}$$

Making the same calculations as the ones in Sec .(4.2), we obtain the following geometric phase for the superposition state:

$$\gamma_m = \left(m+\frac{1}{4}\right)\Omega. \tag{4.21}$$

As we can see from Eq. (4.21), the particular case where $\gamma_0 = \Omega/4$ is not related to the quantum vacuum because it was obtained in an angular momentum context, which can be exhibited in both a classical and a quantum-mechanical framework. Therefore, the claim made by Fuentes-Guridi *et al.* that states that the non-vanishing geometric phase $\gamma_0 = \Omega/4$ has "no semicassical correspondence" [4] is incorrect. Indeed, a classical framework in which this "vacuum" geometric phase can be achieved is when the following initial states are considered:

$$|\Psi^{\pm}_{j,m}\rangle = \frac{1}{\sqrt{2}}\left[|j,m\rangle \pm |j+1,m+1\rangle\right]. \tag{4.22}$$

Under the evolution of $U(\theta,\phi)$, these states produce the following geometric phase:

$$\gamma_m = \left(m+\frac{1}{2}\right)\Omega. \tag{4.23}$$

The states given in Eq. (4.22) can be generated with classical light beams carrying orbital angular momentum by setting j integer, and then be submitted under the application of $U(\theta,\phi)$. In this physical realization, the quantum nature of the phenomenon does not need to be invoked, thus, having a classical correspondence.

4.2.2 Generalization for Open Paths

Lastly, we would like to point out the fact that the geometric phase is not restricted only to closed paths, but to any general path $\mathcal{C}$. To stress this fact, we are going to address the geometric phase generated by the evolution $U(\theta,\phi)$ for an open trajec-

tory, so unlike before, this time we need to take into account the two contributions of the geometric phase, the dynamic and the total (Pancharatnam) phase. To achieve this, we are going to address the formal definition of the geometric phase given in Eq. (4.1).

First, the calculation of the dynamic contribution requires the evaluation of the following quantities:

$$\langle U^\dagger \partial_\theta U\rangle_{\Psi^\pm} = -i\langle J_y\rangle_{\Psi^\pm} = 0, \tag{4.24}$$

$$\langle U^\dagger \partial_\phi U\rangle_{\Psi^\pm} = -i\langle J_z\cos\theta - J_x\sin\theta\rangle_{\Psi^\pm} = -i\cos\theta\langle J_z\rangle_{\Psi^\pm} = -i\left(m+\frac{1}{4}\right)\cos\theta. \tag{4.25}$$

Using the parameter-invariance of the geometric phase, we can assume $\phi = \phi(\theta)$, i.e., θ is the parameter which describes the curve $\mathcal{C}$. Hence, the dynamic contribution to the geometric phase reads:

$$i\int_0^\theta \langle\psi(\theta')|\dot{\psi}(\theta')\rangle d\theta' = i\int_0^\theta \langle U^\dagger\partial_\phi U\rangle_{\Psi^\pm} d\theta' = -\left(m+\frac{1}{4}\right)\sin\theta. \tag{4.26}$$

The Pancharatnam contribution is computed as follows:

$$\arg\langle\Psi^\pm_{j,m}|U(\theta,\phi)|\Psi^\pm_{j,m}\rangle = \arg\langle\Psi^\pm_{j,m}|e^{-i\phi J_z}e^{-i\theta J_y}|\Psi^\pm_{j,m}\rangle. \tag{4.27}$$

If we make use of the Wigner coefficients $d^j_{m',m} \equiv \langle j,m'|e^{-i\theta J_y}|j,m\rangle \in \mathbb{R}$ [36], Eq. (4.27) can be written in the following form:

$$\arg\langle\Psi^\pm_{j,m}|U(\theta,\phi)|\Psi^\pm_{j,m}\rangle = \arg\left[\frac{e^{-im\phi}}{2}\left(d^j_{m,m} + e^{-i\phi/2}d^{j+1/2}_{m+1/2,m+1/2}\right)\right], \tag{4.28}$$

which results in

$$\arg\langle\Psi^\pm_{j,m}|U(\theta,\phi)|\Psi^\pm_{j,m}\rangle = -m\phi - \arctan\left(\frac{d^{j+1/2}_{m+1/2,m+1/2}\sin\phi/2}{d^j_{m,m} + d^{j+1/2}_{m+1/2,m+1/2}\cos\phi/2}\right). \tag{4.29}$$

Finally, replacing Eqs. (4.27-4.29) into Eq. (4.1), we obtain:

$$\gamma_{j,m} = -m\phi - \arctan\left(\frac{d^{j+1/2}_{m+1/2,m+1/2}\sin\phi/2}{d^j_{m,m} + d^{j+1/2}_{m+1/2,m+1/2}\cos\phi/2}\right) - \left(m+\frac{1}{4}\right)\sin\theta, \tag{4.30}$$

which is the geometric phase acquired under the evolution $U(\theta,\phi)$ when addressing

open paths.

Chapter 5

Summary and Conclusions

In this thesis, we have dealt with an ongoing controversy regarding Berry phases related to the Rabi Hamiltonian. Apparently, the main focus of this controversy is whether or not the Berry phase in the Rabi Hamiltonian is vanishing or non-vanishing. However, the striking feature of this controversy is that the validity of the widely-known RWA is put under debate. Larson's findings [2, 3] show that the Berry phase in the Rabi Hamiltonian vanishes for cases that other authors find it to be non-vanishing [4–7]. If this were to be true, then the RWA would break down when dealing with Berry phases. This statement emphasizes the importance of putting an end to this debate.

In Chapter 3, we presented analytical and numerical results of a non-vanishig Berry phase acquired when an eigenstate of the Rabi Hamiltonian evolves under the unitary transformation $U(\varphi) = \exp\left[-ia^{\dagger}a\right]$ with $\varphi \in [0, 2\pi]$. Even more, our numerical results converge to the corresponding ones obtained with the Jaynes-Cummings Hamiltonian for sufficiently small values of g/ω. In light of these results, we can conclude that the RWA fully holds when dealing with Berry phases.

In Chapter 4, we discussed several theoretical aspects regarding the previously mentioned controversy, which had not been debated before. First, we saw that the adiabatic approximation did not need to be applied in any of the calculations; therefore, we cannot be dealing with Berry's phase. We also presented analytical arguments that lead to the conclusion that the phase obtained after the evolution of $U(\varphi)$ must be a geometric phase, which is a more general concept. Second, we observed that the respective Hamiltonian, H_{JC} or H_R, is only a provider of the initial eigenvector, and that it does not rule the evolution of the eigenvector. With this, we can conclude that the Hamiltonian does not play an essential role in the controversy. Finally, based on these two conclusions, we propose that the most suitable framework to deal with these phases is the kinematic one posed by Mukunda

and Simon [23], which focuses on the essential connection between geometric phases and the geometry of the ray space. In order to illustrate this point, we also studied the geometric phase obtained from the evolution under the unitary operator $U(\theta, \phi) = \exp(-i\phi J_z) \exp(-i\theta J_y)$. Analyzing these results, we saw that when we refer to a Hamiltonian, the physical interpretation of a phenomenon may be obscured rather than clarified, as when relating the "vacuum" geometric phase with the quantum nature of the field. We have proven that this statement is not true by addressing the connection between Schwinger's approach with the standard angular momentum algebra. Finally, we generalyzed this evolution for the cases of open paths $\mathcal{C}$ to show that the condition of closed paths is not essential.

In summary, with this work, we believe to have contributed to settle the aforementioned controversy and dealt with some important misconceptions regarding Berry's/geometric phases. Future work regarding this thesis could be the experimental realization of the geometric phase acquired under the evolution of $U(\theta, \phi)$ and the corresponding test of our results related to the "vacuum" geometric phase and to open paths.

Bibliography

[1] J. Calderón and F. De Zela, *Geometric phase and the Rabi Hamiltonian*, (Accepted for Publication in Phys. Rev. A).

[2] J. Larson, *Absence of Vacuum Induced Berry Phases without the Rotating Wave Approximation in Cavity QED*, Phys. Rev. Lett. **108** (2012), 033601.

[3] J. Larson, *On the rotating wave approximation in the adiabatic limit*, Phys. Scr. **T153** (2013), 014040.

[4] I. Fuentes-Guridi, A. Carollo, S. Bose, and V. Vedral, *Vaccum Induced Spin-1/2 Berry's Phase*, Phys. Rev. Lett. **89** (2002), 220404.

[5] T. Liu, M. Feng, and K. Wang, *Vaccum-induced Berry phase beyond the rotating-wave approximation*, Phys. Rev. A **84** (2011), 062109.

[6] W.-W. Deng and G. Li, *Berry phase of the Rabi model beyond the rotating-wave approximation*, J. Phys. B **46** (2013), 224018.

[7] M. Wang, L. Wei, and J. Liang, *Does the Berry phase in a quantum optical system originate from the rotating wave approximation?*, Phys. Lett. A **379** (2015), 1087.

[8] I. Rabi, *On the process of space quantization*, Phys. Rev. **49** (1936), 324.

[9] M. Scully and M. Zubairy, *Quantum Optics*, Cambridge University Press, 1997.

[10] E. Jaynes and F. Cummings, *Comparison of Quantum and Semiclassical Radiation Theories with Application to the Beam Maser*, Proc. IEEE **51** (1963), 89.

[11] M. Weissbluth, *Photon-Atom Interactions*, Academic Press, Inc., 1989.

[12] D. Zueco, G. Reuther, S. Kohler, and P. Hänggi, *Qubit-oscillator dynamics in teh dispersive regime: Analytical theory beyond the rotating-wave approximation*, Phys. Rev. A **80** (2009), 033846.

[13] D. Meschede, H. Walther, and G. Müller, *One-Atom Maser*, Phys. Rev. Lett. **54** (1985), 551.

[14] G. Agarwal, *Quantum Statistical Theories of Spontaneous Emission and their Relation to Other Approaches (Quantum Optics 70, Springer Tracts in Modern Physics)*, Springer, 1974.

[15] P. Belobrov, G. Zaslavskii, and G. Tartakovskii, *Stochastic breaking of bound states in a system of atoms interacting with a radiation field*, Sov. Phys. JETP **44** (1976), 945.

[16] S. Swain, *Continued fraction expressions for the eigensolutions of the hamiltonian describing the interaction between a single atom and a single field mode: comparisons with the rotating wave solutions*, J. Phys. A **6** (1973), 1919.

[17] F. De Zela, *Micromaser Dynamics Beyond the Rotating-Wave Approximation (Modern Challenges in Quantum Optics: Selected Papers of the First International Meeting in Quantum Optics Held in Santiago, Chile,13-16 August 2000 , M. Orszag and J.C. Retamal)*, Springer, 2001.

[18] D. Braak, *Integrability of the Rabi Model*, Phys. Rev. Lett. **107** (2011), 100401.

[19] M. Berry, *Quantal Phase factors accompanying adiabatic changes*, Proc. R. Soc. Lond. A **392** (1984), 45.

[20] Y. Aharonov and J. Anandan, *Phase change during a Cyclic Quantum Evolution*, Phys. Rev. Lett. **58** (1987), 1593.

[21] J. Samuel and R. Bhandari, *General Setting for Berry's Phase*, Phys. Rev. Lett. **60** (1988), 2339.

[22] R. Bhandari and J. Samuel, *Observation of Topological Phase by Use of a Laser Interferometer*, Phys. Rev. Lett. **60** (1988), 1211.

[23] N. Mukuda and R. Simon, *Quantum Kinematic Approach to the Geometric Phase: General Formalism*, Ann. Phys. **228** (1993), 205.

[24] F. De Zela, *The Pancharatnam-Berry Phase: Theoretical and Experimental Aspects (Theoretical Concepts of Quantum Mechanics, M. Reza)*, InTech, 2012.

[25] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanziger, *The Geometric Phase in Quantum Systems: Foundations, Mathematical Concepts, and Applications in Molecular and Condensed Matter Physics* , Springer, 2003.

[26] S. Pancharatnam, *Generalized Theory of Interference and Its Applications* , Proc. Ind. Acad. Sci. A **44** (1956), 247.

[27] J. Schön and H. Köppel, *Geometric phases and quantum dynamics in spin-orbit coupled systems*, J. Chem. Phys. **108** (1998), 1503.

[28] M. Batchelor, Z.-M. Li, and H.-Q. Zhou, *Energy landscape and conical intersection points of the driven Rabi model*, arXiv:1509.06572v2.

[29] E. Irish, *Generalized Rotating Wave-Approximation for Arbitrarily Large Coupling*, Phys. Rev. Lett. **99** (2007), 173601.

[30] Y.-Y. Zhang, Q.-H. Chen, and S.-Y. Zhu, *Vacuum Rabi Splitting and Dynamics of the Jaynes-Cummings Model for Arbitrary Coupling*, Chin. Phys. Lett. **30** (2013), 114203.

[31] E. Sjöqvist, *Geometric phase in quantum information*, Int. J. Quantum Chem. **115** (2015), 1311.

[32] V. Rakhecha and A. Wagh, *Geometric phase a la Pancharatnam*, Pramana, J. Phys. **46** (1996), 315.

[33] J. Budich and B. Trauzettel, *From the adiabatic theorem of quantum mechanics to topological states of matter*, Phys. Status Solidi RRL **7** (2013), 109.

[34] R. Resta, *Manifestation of Berry's phase in molecules and condensed matter*, J. Phys.: Conden. Matter **12** (2000), R107.

[35] A. Carollo, M. França Santos, and V. Vedra, *Berry's phase in cavity QED: Proposal for observing an effect of field quantization*, Phys. Rev A. **67** (2003), 063804.

[36] V. Devanathan, *Angular Momentum Techniques in Quantum Mechanics*, Kluwer Academic Publishers, 2002.