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Debt Control with a Non-linear Dynamical Model: An Application to the Peruvian Economy for the Period 1991-2014

TESIS PARA OPTAR EL GRADO ACADÉMICO DE MAGÍSTER EN ECONOMÍA

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Resumen

La crisis financiera del 2008 en los Estados Unidos de América impulsó a que los economistas se enfocaran en sus determinantes y políticas (Crotty, 2009) para salir de ella. A partir de este esfuerzo, las ideas y el trabajo de (Minsky, 1982) surgieron nuevamente. Steven Keen en (Keen, 1995) presenta un modelo que cristaliza estas ideas en un sistema de ecuaciones diferenciales donde la participación de la deuda es una de las variables e indicadores de una posible crisis. Este modelo es una extensión del modelo en (Goodwin, 1967) donde se añade el sector financiero. El Perú no está exento de una crisis financiera y uno de los motivos es que es un país dependiente del precio del cobre que a su vez depende de grandes economías como China y los Estados Unidos de América que se encuentran en competencia comercial constante. El apalancamiento desmedido de la deuda privada durante un periodo de prosperidad económica debido a la apreciación del sector minero seguido de una caída significante del precio del cobre puede ser un esbozo de la configuración de una crisis financiera en Perú. La contribución del presente documento es la calibración de los modelos de Goodwin y Keen para Perú durante el periodo de 1991 al 2014 y la obtención de políticas económicas de control para la estabilización de la economía peruana en caso haya un crisis prevista. Nos apoyamos en el análisis de estabilidad de (Costa Lima, 2013) y en las técnicas de control no lineal presentadas y desarrolladas en (Gray, 1995) e (Isidori, 2002). Uno de los resultados obtenidos para el caso particular en el que la participación laboral y el ratio de empleo son altos es que la fuerza laboral total debe aumentar, mientras que la productividad laboral y la tasa de interés real deben dismunir. La dismunición de la productividad laboral puede parecer contra intuitivo, pero se traduce en mayor tasa de empleo mientras que el producto es no creciente. Los resultados no son generales y dependen del punto inicial de la economía, pero ayudan guiar las políticas de desarrollo económico.

Acknowledgement

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Contents

Chapter 1

Introduction

> If economy fluctuates for whatever reason, then it is almost always possible to neutralize these cyclic motions by means of anti-cyclic demand policies. (Gabish and Lorentz, 1987)

In 2008, the global economy fell into crisis due to the collapse of the US subprime mortgage market and flaws of the financial system. In particular, deregulation and the globalization of financial markets were important factors for the crisis (Crotty, 2009). In this context, the work of Minsky becomes relevant to analyze this phenomenon. His *financial instability hypothesis* (FIH) (Minsky, 1982) acknowledges the unmoderated issuing of private debt as the causal of instability. Minsky states that there is a cycle of debt which starts with the appreciation of an asset during a prosperous stage of the economy and investors targeting this asset, for example, because of innovation in technology, debt is issued without restrictions to invest in this asset.

An endogenous non-linear cycle model which considers the FIH was provided by Keen (Keen, 1995). This model is based on the Goodwin model (Goodwin, 1967) which is a growth model that aims to represent endogenous cycles between employment rate and wage share, but it disregards the financial sector. Moreover, originally, the model was set up as a Lotka-Volterra dynamical systems that provides a predator-prey dynamics. This was done by setting the Phillip's curve as a linear relationship in the dynamics of the wage share. In (Blatt, 1983), the author assumes a non-linear Phillips curve so that there is a singularity in the dynamics at full employment which translates into the employed people having power over their wages at full employment. That modification, also, restricts the employment rate from zero to one. The empiric analysis has been addressed by works in (Harvie, 2000; Herrera-Medina & Garcia-Molina, 2010; Grasselli & Maheswari, 2018; Moura & Ribeiro, 2012; Dávila-Fernandez & Araujo, 2019) in which the authors provide a methodology to calibrate the model.

A model which includes the private/financial sector was developed by Keen with the intention of reproducing financial instabilities (Keen, 1995, 2013). This model can be seen as an extension to the Goodwin model. The idea behind it is that debt follows a cycle which starts with the appreciation of an asset, then financial entities issuing debt to the private sector without the proper restrictions. The price of the asset keeps increasing until it is too expensive. This leads to a decrease in the demand for the asset. This reduces its price and so it leaves an unpayable debt of a lower value asset (Bernanke, 2012). Such is the case depicted by the transition from the Great Moderation to the financial crisis of 2008, for example. In the model, debt share is added as a third variable with its dynamics being affected by the utilities of the firms and the capital-to-output ratio. Specifically, the increment of debt share is proportional to the product of the difference of current debt share and capital-to-output ratio and the utilities. Roughly, if capital is less than current debt share and there is an increment in the utilities, then firms prefer to pay their debt, if capital is greater than the current debt and there is an increment in utilities, then firms prefer to get more debt to invest. Also, the utilities affect the employment rate postively instead of negatively as the wage share does in the Goodwin model. The Keen model is not the only extension of the Goodwin model, for example the work of (Sordi & Vercelli, 2014) extends the former model in a different way by assuming dissequilibrium in the goods market, but for this manuscript the focus is on the Keen model.

Well known is the fact that Peru is one of the leading countries in copper production [1] and the Peruvian economy depends mainly on mineral exporting activities [2]. Moreover, Peru, which is a small open economy is subjected to the international commodity prices, particularly, the price of copper. Recently the price of copper and its relative price [3] have been decreasing [4]. After an eventual sustained rise it follows the appreciation of the sector and with this an increase in private investment. Credit accompanies react to these changes and the financial sector is more prone to issue debt to investors and firms. As for this context, the price of copper is unstable and a leverage of

[1] Based on data from U.S. Geological Survey, Mineral Commodity Summaries 2018, Peru was the second world producer of copper after Chile.

[2] According to BCRP Annual Report 2017, external sector, in 2017 the share of total exports derived from mining and hydrocarbons was 70% and the share of exports of copper was 30%.

[3] CPI in USD divided by the price of copper in USD per kilogram.

[4] As for 2019. The reasons for this might be the commercial war between the US and China which results in speculation on several markets, particularly, the copper market as China leads the world's demand of copper.

great magnitude, justified by the economic blossom, followed by a slump in the price would lead to reach high unpayable amounts of debt, so depicting a financial crisis.

This work is based on the the analysis of equilibria (Costa Lima, 2013) and the contribution lies in the calibration of the Goodwin and Keen models for Peru for the period of 1991 to 2014 along with the application of control techniques to drive Keen's model states in order to avoid a financial crisis, which otherwise would be inevitable. It is assumed that the initial state of the economy lies in the instability region of the model, this is, out of the significant and stable equilibrium region. The trajectory soon leads to rapid increments of debt share with shrinkage of employment and wage share. To be able to steer the state of the economy from this undesirable situation, first, feedback linearization (Kailath, 1980) is used to obtain a linear version of the model through a coordinate change known as the Byrne-Isidori transformation (Isidori, 2002). Then, the control techniques are applied to this linear model. These are pole placement and optimal control.

The first technique does not offer a way to execute constraints on the control policy, which are of interest, because the results may show abrupt and unreal jumps, but it is much simpler computationally so results are obtained faster. Results show that a desirable condition on the real interest rate such as being positive is not met in general. Given this, optimal control is applied which guarantees the satisfaction of the constraints when there is feasibility. One step ahead prediction is used for this setting. For this, Ode45 from MatLab was modified to obtain the value of the sampling time which is not fixed in its algorithm. It is shown that by modifying the model and adding the proper dynamics (linear) of a set of former parameters, the controllability of the system is guaranteed (*exact* linearization of the model). With this modification, the controls of the model are the growth of labor productivity, total labor force and real interest rate. These economics variables are not immediately changeable so the results are interpreted in terms of expected or desirable policies to achieve to get the economy out of financial crisis path.

To steer a point in the financial unstable region to the financial stable region, a reference trajectory has to be followed. In the present work it is designed following the gradient-based planner algorithm (Lavalle, 2006). The procedure consists in modifying the vector field of the Keen model by placing an attractor point inside the stability region of the vector field. Depending on the force of this attractor the algorithm results in different reference paths. It could be the case in which there is no path available, this occurs when the force of the attractor is not strong enough and the trayectory runs into singularities, for example, the singularity of the Phillip's curve when there is full employment.

The manuscript is organized as follows. Chapter 2 and 3, address the Goodwin and Keen models respectively, along with their equilibria analysis. In chapter 4, the models for Peru are calibrated. The control is performed and the main results are stated in chapter 5. Conclusions and discussion are found in chapter 6 and the theoretical aspects of the techniques are found in the appendix.

Chapter 2

The Goodwin Model

2.1 The model

The Goodwin model is a two-dimensional dynamical system based on Lotka-Volterra model in which wages and employment work as predator and prey respectively generating a cycle. This stylized fact has been empirically tested in (Harvie, 2000) and (Herrera-Medina & Garcia-Molina, 2010). Evidence suggests that the model works well qualitatively, this is, the behavior of the variables taken from real data are strongly correlated with that of the trajectories of the simulation. On the other side, at the quantitative level, the model makes inaccurate predictions of the center and direction of the cycles.

The model does not consider the financial and government sectors, but extensions to it have been proposed [1]. In the present section we follow the work of (Costa Lima, 2013) in which a mathematical analysis of the model is provided. In what follows we deduce the Goodwin model from economic assumptions as presented in (Costa Lima, 2013). We assume that capital and employment are complementary so production follows a Leontief behavior determined by capital and labor.

$$Y(t) = \min\left\{\frac{K(t)}{\nu}, a(t)L(t)\right\}$$

The capital-to-output ratio which indicates productivity of capital investment is represented by ν. Technology is assumed exogenous and continually and constantly impacting labor productivity growth. Total labor force is,

[1] Costa Lima (2013), Harvie, Kelmanson and Knapp (2007) and Sordi and Vercelli (2014), for example.

also, growing constantly, this is

$$\frac{\dot{a}(t)}{a(t)} = \alpha, \tag{2.1}$$

$$\frac{\dot{N}(t)}{N(t)} = \beta \tag{2.2}$$

therefore the functions evolve according to

$$a(t) = a_0 e^{\alpha t}, \tag{2.3}$$

$$N(t) = N_0 e^{\beta t} \tag{2.4}$$

for some real numbers a_0 and N_0. Employment rate is denoted by

$$\lambda(t) = \frac{L(t)}{N(t)} \tag{2.5}$$

In order not to depart from the optimal allocation of resources we take on full capital utilization

$$Y(t) = \frac{K(t)}{\nu} = a(t)L(t) \tag{2.6}$$

Next, as it is explained in (Blatt, 1983), the Goodwin model assumes that the fractional rate of change of real wages is a function of the employment rate. As wages have a positive correlation with inflation and employment is easily expressed in terms of unemployment such a function is the Phillips curve. As it is seen in section 4.2, the same author provides an specific form for this function that guarantees that wages grow faster when the economy is near full employment. This is explained by the pressure employers produce due to their appreciation in the labor market in the lapse of economic blossom. Let us denote $\mathrm{w}(t)$ as the real-wage function so that the relationship is expressed as

$$\frac{\dot{\mathrm{w}}(t)}{\mathrm{w}(t)} = \Phi(\lambda(t)),\ \Phi'(\lambda(t)) > 0 \tag{2.7}$$

The model doesn't take financial and government sector into account and embraces Say's law which states that if there is something demanded it is because supply makes it affordable through income. This means that aggregate supply and demand are the same. This is

$$Y(t) = I(t) + C(t) \tag{2.8}$$

because of this the profits and consumption functions adopt the following form:

$$\Pi(t) := Y(t) - C(t) = I(t) \tag{2.9}$$
$$C(t) = \mathrm{w}(t)L(t) \tag{2.10}$$

This implies that capital grows according to the following expression:

$$\dot{K}(t) = (Y(t) - \mathrm{w}(t)L(t)) - \delta K(t) \tag{2.11}$$

with a δ depreciation rate. We define ω as the wage share or labor share of the economy as

$$\omega(t) := \frac{\mathrm{w}(t)L(t)}{Y(t)}$$

From equation 2.6, the following relationship holds:

$$\omega(t) = \frac{\mathrm{w}(t)L(t)}{a(t)L(t)} = \frac{\mathrm{w}(t)}{a(t)}, \tag{2.12}$$

also,

$$\Pi(t) = Y(t) - \mathrm{w}(t)L(t) \tag{2.13}$$
$$= Y(t) - \omega(t)a(t)L(t) \tag{2.14}$$
$$= (1 - \omega(t))Y(t) \tag{2.15}$$

from full capital utilization we have [2]

$$Y(t)\nu = K(t).$$

Replacing the latter identity into equation 2.11, we have

$$\begin{aligned}\nu\dot{Y}(t) &= (1 - \omega(t))Y(t) - \delta\nu Y(t)\\ &= (1 - \omega(t) - \delta\nu)Y(t)\end{aligned}$$

then

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{(1 - \omega(t))}{\nu} - \delta \tag{2.16}$$

[2] This also implies that capital and product have the same rate of growth.

Taking logarithm to equation 2.12, deriving and from the definition of employment rate in equation 2.5 and full capital utilization we obtain the following system in terms of growth

$$\begin{aligned} \frac{\dot{\omega}(t)}{\omega(t)} &= \frac{\dot{\mathrm{w}}(t)}{\mathrm{w}(t)} - \frac{\dot{a}(t)}{a(t)} \\ \frac{\dot{\lambda}(t)}{\lambda(t)} &= \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{N}(t)}{N(t)} - \frac{\dot{a}(t)}{a(t)} \end{aligned} \tag{2.17}$$

taking equation 2.7 and 2.16 into account we finally have the following differential equation system:

$$\begin{aligned} \dot{\omega}(t) &= \omega(t)\left[\Phi(\lambda(t)) - \alpha\right] \\ \dot{\lambda}(t) &= \lambda(t)\left[\frac{1-\omega(t)}{\nu} - \alpha - \beta - \delta\right] \end{aligned} \tag{2.18}$$

2.2 Equilibria

The number of equilibria of the system depends on the functional form of Φ which represents the relationship of employment and the growth of real wages in a similar manner as the Phillips curve. In the original setting of the Goodwin model the Phillips curve is a linear function which reduces the analysis of the system to a Lotka Volterra system. Later, we take a specific non-linear curve.

The trivial equilibrium of the model is the origin with jacobian matrix equal to

$$J(0,0) = \begin{bmatrix} \Phi(0) - \alpha & 0 \\ 0 & \frac{1}{\nu} - \alpha - \beta - \delta \end{bmatrix}$$

the eigenvalues are equal to $\Phi(0) - \alpha$ and $\frac{1}{\nu} - \alpha - \beta - \delta$. In order for this equilibrium point to be a saddle point [3], we need to impose the following conditions:

$$\Phi(0) - \alpha < 0 \tag{2.19}$$

$$\frac{1}{\nu} - \alpha - \beta - \delta > 0 \tag{2.20}$$

The non-trivial equilibria of system 2.18 are equal to

$$(\bar{\omega}, \bar{\lambda}) = (1 - \nu(\alpha + \beta + \delta), \Phi^{-1}(\alpha)) \tag{2.21}$$

[3] Hartman-Grobman theorem is used.

The jacobian matrix at the equilibrium points is the following:

$$J(\bar{\omega}, \bar{\lambda}) = \begin{bmatrix} 0 & \bar{\omega}\Phi'(\bar{\lambda}) \\ -\frac{\bar{\lambda}}{\nu} & 0 \end{bmatrix}$$

the trace of the matrix is zero and the determinant is

$$det J(\bar{\omega}, \bar{\lambda}) = \frac{\bar{\lambda}\bar{\omega}}{\nu}\Phi'(\bar{\lambda})$$

assuming $\Phi'(\bar{\lambda}) > 0$ as we did previously the solutions around the equilibrium are orbits [4] . The eigenvalues are complex with imaginary coefficient equal to

$$\sqrt{\frac{\bar{\lambda}\bar{\omega}}{\nu}\Phi'(\bar{\lambda})},$$

then the length of the cycles very close to the equilibrium is

$$2\pi\left(\frac{\bar{\lambda}\bar{\omega}}{\nu}\Phi'(\bar{\lambda})\right)^{-1/2}$$

The direction of the cycle can be obtain from the phase diagram. For this we take $\omega = 1 - \nu(\alpha + \beta + \delta)$ which is one coordinate of the non-trivial equilibrium. From equation 2.18 one has that for $\Phi(\lambda) - \alpha > 0$ the flow field arrows head to the right and for $\Phi(\lambda) - \alpha < 0$ arrows head to the left. Thus the direction of the trajectories are clockwise.

[4]Structurally speaking, this is because the growth of the labor share is purely in terms of the employment rate. In the same manner for the growth of the employment rate.

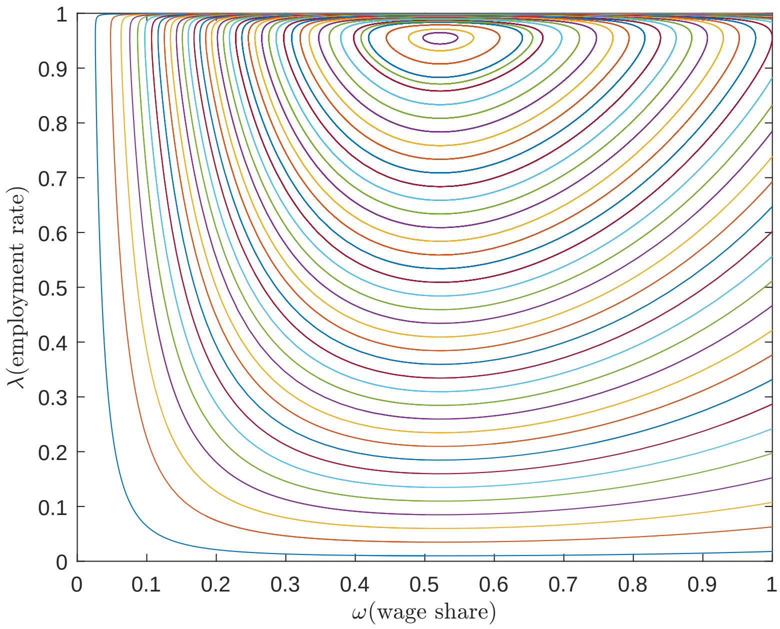

Figure 2.1: The Goodwin model represents the trade-off between wage share and employment rate. In the figure, several cycles of the Goodwin model are shown.

There is no conclusive answer of whether the cycles in the Goodwin model are short-run or a long-run cycles [5] . The cycle length for Peru has been estimated in (Seminario, 2016). The short-run cycle is between 0 and 10 years, the medium-run cycle is between 11 and 49 years and the long-run cycle is greater than 50 years. As mentioned in (Herrera-Medina & Garcia-Molina, 2010): the cycle measurement literature focuses on GDP and not so on labor or wage share as in Goodwin. This leaves the door open for the possibility of uncorrelated length between the measurement and the classical literature.

2.3 Cycle structure

The Goodwin model 2.18 describes cycles around the stable equilibrium. This means that labor share and employment rate are both being limited to grow or decrease and cannot do it indefinitely. We are going to shed light on what is the restriction. From the model, it is observed that the rate of

[5] Harvie (2000)

growth of both variables fluctuate: increasing for some determined regions and decreasing for others. These regions are the phases of the business cycle. To understand the dynamics of the model, we work with the model of growths and infer features of the behavior of the fluctuation of the variables by taking the derivative:

$$\begin{aligned} \frac{d}{dt}\frac{\dot{\omega}(t)}{\omega(t)} &= \Phi'(\lambda(t))\lambda(t)\left[\frac{1-\omega(t)}{\nu} - \alpha - \beta - \delta\right] \\ \frac{d}{dt}\frac{\dot{\lambda}(t)}{\lambda(t)} &= -\frac{\omega(t)}{\nu}\left[\Phi(\lambda(t)) - \alpha\right] \end{aligned} \tag{2.22}$$

As $\Phi'(\lambda(t))\lambda(t) > 0$, the rate of growth of labor share increases if and only if the complement of capital share accumulated due to economic growth with respect to the product is greater than labor share. This is,

$$\frac{d}{dt}\frac{\dot{\omega}(t)}{\omega(t)} > 0 \iff \omega(t) < 1 - \nu(\alpha + \beta + \delta)$$

from the second equation, rate of growth of employment rises if and only if wage inflation is less than productivity growth:

$$\frac{d}{dt}\frac{\dot{\lambda}(t)}{\lambda(t)} > 0 \iff \Phi(\lambda(t)) < \alpha$$

Figure 2.2 describes the phases of the cycle. Labor share rate of growth increases and the growth of employment is positive during the recovery and boom

$$\frac{d}{dt}\frac{\dot{\omega}(t)}{\omega(t)} > 0, \ \frac{\dot{\lambda}(t)}{\lambda(t)} > 0$$

During boom and recession employment rate of growth starts decreasing and labor share growth remains positive

$$\frac{d}{dt}\frac{\dot{\lambda}(t)}{\lambda(t)} < 0, \ \frac{\dot{\omega}(t)}{\omega(t)} > 0$$

As the economy advances, the rate of growth of labor share starts decreasing and employment shrinks. This is what happens in the recession and the later slump

$$\frac{d}{dt}\frac{\dot{\omega}(t)}{\omega(t)} < 0, \ \frac{\dot{\lambda}(t)}{\lambda(t)} < 0$$

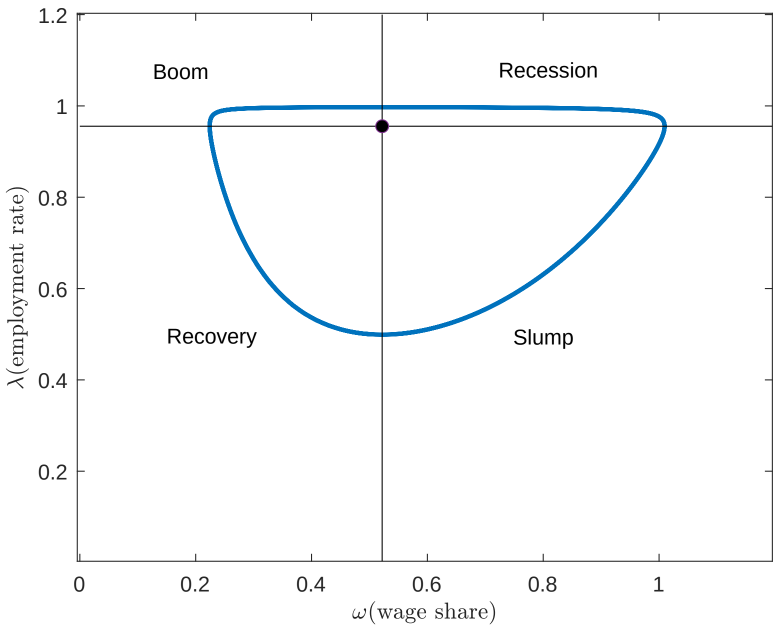

Figure 2.2: For the Goodwin model, the space is split in four regions. During the boom, wage share and employment rate increase, this follows a recession in which employment rate starts deaccelerating, then wage share starts srhinking along with the employment rate during the slump stage and finally the recovery is characterized by the deacceleration of the decrease in wage share and the increase in employment rate.

In the slump and recovery phases rate of growth of employment increases and the growth of labor share dwindles

$$\frac{d}{dt}\frac{\dot{\lambda}(t)}{\lambda(t)} > 0, \; \frac{\dot{\omega}(t)}{\omega(t)} < 0$$

We also observe that during the boom labor share growth is positive and so does employment rate

$$\frac{\dot{\omega}(t)}{\omega(t)} > 0, \; \frac{\dot{\lambda}(t)}{\lambda(t)} > 0$$

during the recession the growth of labor share still increases, but employment rate of growth starts decreasing

$$\frac{\dot{\omega}(t)}{\omega(t)} > 0, \; \frac{\dot{\lambda}(t)}{\lambda(t)} < 0$$

during the slump the growth of labor share turns negative and the growth of employment keeps decreasing

$$\frac{\dot{\omega}(t)}{\omega(t)} < 0, \ \frac{\dot{\lambda}(t)}{\lambda(t)} < 0$$

Recovery starts by making the growth of employment positive, the the growth of labor share is still negative

$$\frac{\dot{\omega}(t)}{\omega(t)} < 0, \ \frac{\dot{\lambda}(t)}{\lambda(t)} > 0$$

2.4 Interpretation

The system in terms of growth provides a better grasp of the intuition the model carries through the equations. Following system 2.17, we have

$$\begin{aligned}\frac{\dot{\omega}(t)}{\omega(t)} &= \Phi(\lambda(t)) - \alpha \\ \frac{\dot{\lambda}(t)}{\lambda(t)} &= \frac{1-\omega(t)}{\nu} - \alpha - \beta - \delta\end{aligned}$$

If product growth rate rises faster than the sum of constant rates of labor productivity and total labor force (equations 2.3 and 2.4) then employment starts rising. This will pull the growth of labor share to stop decaying and start increasing as employment reaches its stationary value. Now labor share is growing. This implies that product growth decreases so slowing employment growth [6] . As labor share reaches its stationary value employment starts decaying. It does so until again it reaches its stationary value and the trade-off between the variables repeats again. The figure 2.1 illustrates the previous assertion [7] .

Now that we have a thorough understanding of the Goodwin model, we study the Keen model which extends Goodwin on the basis of the Financial Instability Hypothesis of Minsky.

[6]In Hamilton (1979), the author provides the example of business cycles in a small township analyzing the demand of employment. The example is attributed to M.B Hamilton

[7]We also note in the figure the asymmetry of the fluctuation.

Chapter 3

The Keen Model

3.1 The model

The Keen model is presented as in (Costa Lima, 2013). We abandon the neoclassical idea of Say's law and relax the assumption that capitalists invest the total of their profits. Firms can now borrow credit from banks at a fixed interest rate. Debt is paid when investment is lower than profits and firms leverage when higher investment is required. In this way we introduce the finance sector in Goodwin's model. The time lapse is infinite so that debt is not necessarily entirely paid and credit can increase boundlessly leading the economy to a financial crisis.

The variable D stands for the amount of debt in real terms. Profits, as we had before, are represented by the following equation:

$$\Pi(t) = (1 - \omega(t))Y(t)$$

we define the share of debt on product

$$d(t) := \frac{D(t)}{Y(t)}$$

and the net profits share

$$\pi(t) := 1 - \omega(t) - rd(t)$$

where r is the interest rate. Then we have

$$\begin{aligned}\Pi(t) &= (1 - \omega(t))Y(t) - rD(t) \\ &= (1 - \omega(t) - rd(t))Y(t) \\ &= \pi(t)Y(t).\end{aligned}$$

According to (Keen, 1995) in which the author models the ideas of (Minsky, 1982), new investment is an increasing function of the net profit share, called here π. Then investment is represented as:

$$I(t) = \kappa(\pi(t))Y(t)$$

and capital accumulation follows

$$\dot{K}(t) = \kappa(\pi(t))Y(t) - \delta K(t)$$

from the Leontief production function and the full capital utilization assumptions, capital and product have the same rate of growth [1] , this is

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\kappa(\pi(t))}{\nu} - \delta$$

and from the definition of λ we obtain

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{\kappa(\pi(t))}{\nu} - \alpha - \beta - \delta$$

where the rate of new investment is a nonlinear increasing function of the net profit. The model assumes three economic agents which are households, firms and banks. The varying behavior of debt is led by the return on investment. If ROI [2] rises considerably, debt is being paid to the banks, on the contrary if it lowers, firms are leveraging and so financing the increase in investment with bank credit. This pattern is mathematically crystallized by the following differential equation:

$$\dot{D}(t) = \kappa(\pi(t))Y(t) - \pi(t)Y(t)$$

we translate the behavioral equation to

$$\begin{aligned}\frac{\dot{D}(t)}{D(t)} &= [\kappa(\pi(t)) - \pi(t)]\,\frac{Y(t)}{D(t)}\\ &= [\kappa(\pi(t)) - \pi(t)]\,d^{-1}(t)\end{aligned}$$

then we have the following rate of growth relationship:

$$\begin{aligned}\frac{\dot{d}(t)}{d(t)} &= \frac{\dot{D}(t)}{D(t)} - \frac{\dot{Y}(t)}{Y(t)}\\ &= [\kappa(\pi(t)) - \pi(t)]\,d^{-1}(t) - \frac{\kappa(\pi(t))}{\nu} + \delta\\ &= [\kappa(\pi(t)) - (1 - \omega(t))]\,d^{-1}(t) + r - \frac{\kappa(\pi(t))}{\nu} + \delta\end{aligned}$$

[1] See equation 2.6

[2] $ROI = Y/I$

as in 2.17 we have the growth system:

$$
\begin{aligned}
\frac{\dot{\omega}(t)}{\omega(t)} &= \frac{\dot{\mathrm{w}}(t)}{\mathrm{w}(t)} - \frac{\dot{a}(t)}{a(t)} \\
\frac{\dot{\lambda}(t)}{\lambda(t)} &= \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{N}(t)}{N(t)} - \frac{\dot{a}(t)}{a(t)} \\
\frac{\dot{d}(t)}{d(t)} &= \frac{\dot{D}(t)}{D(t)} - \frac{\dot{Y}(t)}{Y(t)}
\end{aligned}
\tag{3.1}
$$

which yields following system:

$$
\begin{aligned}
\dot{\omega}(t) &= \omega(t)\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda}(t) &= \lambda(t)\left[\frac{\kappa(\pi(t))}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d}(t) &= d(t)\left[r - \frac{\kappa(\pi(t))}{\nu} + \delta\right] + \left[\kappa(\pi(t)) - (1 - \omega(t))\right]
\end{aligned}
\tag{3.2}
$$

3.2 Equilibria

We are going to calculate the different equilibria of the model. In total there are five equilibrium points, but only two of them appear always without any restriction. The existence of the three others equilibrium points are subjected to the compliance of very strict equations. Thus, most probably these points will not exist so the stability analysis will focus only on the economically meaningful points.

We proceed to find the equilibrium points by setting

$$
\begin{aligned}
\omega(t) &= \overline{\omega} \\
\lambda(t) &= \overline{\lambda} \\
d(t) &= \overline{d}
\end{aligned}
$$

and defining

$$\overline{\pi} := 1 - \overline{\omega} - r\overline{d} = \pi(t),$$

we have

$$
\begin{aligned}
0 &= \overline{\omega}\left[\Phi(\overline{\lambda}) - \alpha\right] \\
0 &= \overline{\lambda}\left[\frac{\kappa(\overline{\pi})}{\nu} - \alpha - \beta - \delta\right] \\
0 &= \overline{d}\left[r - \frac{\kappa(\overline{\pi})}{\nu} + \delta\right] + \left[\kappa(\overline{\pi}) - (1 - \overline{\omega})\right]
\end{aligned}
\tag{3.3}
$$

which is a system with four solutions. The first type of equilibrium has the following form:

$$(0, 0, \overline{d})$$

where $\overline{d}$ is a solution of

$$\begin{aligned}
&\overline{d}\left[r - \frac{\kappa(\overline{\pi})}{\nu} + \delta\right] + [\kappa(\overline{\pi}) - (1 - \overline{\omega})] = 0 \\
&\overline{\pi} = 1 - r\overline{d}
\end{aligned}$$

The second type of equilibrium has the following form:

$$(0, \overline{\lambda}, \overline{d})$$

where $\overline{\lambda} \in \mathbb{R}$, $\kappa(\overline{\pi}) = \nu(\alpha + \beta + \delta)$ satisfying

$$\begin{aligned}
&\overline{d} = \frac{1 - \nu(\alpha + \beta + \delta)}{r - \alpha - \beta}, \\
&\kappa^{-1}(\nu(\alpha + \beta + \delta)) = 1 - r\overline{d}
\end{aligned}$$

The third type of equilibrium must satisfy $\Phi(0) = \alpha$ and has the following form:

$$(\overline{\omega}, 0, \overline{d})$$

where $\overline{\omega}$ and $\overline{d}$ satisfy

$$\overline{d}\left[r - \frac{\kappa(\overline{\pi})}{\nu} + \delta\right] + [\kappa(\overline{\pi}) - (1 - \overline{\omega})] = 0.$$

The forth and desirable type of equilibrium is generated by setting $\lambda = \Phi^{-1}(\alpha)$, $\kappa(\overline{\pi}) = \nu(\alpha + \beta + \delta)$ and

$$\overline{d} = \frac{\nu(\alpha + \beta + \delta) - 1 + \overline{\omega}}{\alpha + \beta - r}$$

then we have the point $(\overline{\omega}, \overline{\lambda}, \overline{d})$ with

$$\overline{\omega} = 1 - \overline{\pi} - r\overline{d}, \tag{3.4a}$$

$$\overline{\lambda} = \Phi^{-1}(\alpha), \tag{3.4b}$$

$$\overline{d} = \frac{\nu(\alpha + \beta + \delta) - \overline{\pi}}{\alpha + \beta} \tag{3.4c}$$

associated to this point is its stability region. Figure 3.2 shows this region for the case presented in (Costa Lima, 2013) and figure 3.1 shows a stable trajectory. Next, by making the following change of variable $u(t) = d^{-1}(t)$ the system is obtained:

$$\begin{aligned}
\dot{\omega}(t) &= \omega(t)\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda}(t) &= \lambda(t)\left[\frac{\kappa(\pi(t))}{\nu} - \alpha - \beta - \delta\right] \\
\dot{u}(t) &= u(t)\left[\frac{\kappa(\pi(t))}{\nu} - \delta - r\right] - u^2(t)\left[\kappa(\pi(t)) - (1 - \omega(t))\right]
\end{aligned}$$

then a fifth equilibrium appears:

$$(0, 0, 0)$$

We analyze the stability of these equilibria by the Hartman-Grobman theorem. The jacobian matrix of the system is the following:

$$J(\omega, \lambda, d) = \begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 \\ -\lambda\dfrac{\kappa'(\pi)}{\nu} & \dfrac{\kappa(\pi)}{\nu} - \alpha - \beta - \delta & -r\lambda\dfrac{\kappa(\pi)}{\nu} \\ \kappa'(\pi)\left[\dfrac{d}{\nu} - 1\right] + 1 & 0 & \gamma \end{bmatrix}$$

with

$$\gamma = r - \frac{\kappa(\pi)}{\nu} + \delta + r\kappa'(\pi)\left[\frac{d}{\nu} - 1\right],$$

we focus on determining the stability of the fourth and fifth equilibrium as both are meaningful points. For the fourth point we have the following jacobian matrix:

$$J(\omega, \lambda, d) = \begin{bmatrix} 0 & K_0 & 0 \\ -K_1 & 0 & -rK_1 \\ K_2 & 0 & rK_2 - (\alpha + \beta) \end{bmatrix}$$

with

$$
\begin{aligned}
K_0 &= \overline{\omega}_1 \Phi'(\overline{\lambda}_1) \\
K_1 &= \frac{\overline{\lambda}_1 \kappa'(\overline{\pi}_1)}{\nu} \\
K_2 &= \frac{(\overline{d}_1 - \nu)\kappa'(\overline{\pi}_1) + \nu}{\nu}
\end{aligned}
$$

we now compute the characteristic polynomial and then applied the Routh-Hurwitz criterion for third degree polynomials to give a sufficient condition for the point to be stable.

$$p_3(y) = y^3 + [(\alpha + \beta) - rK_2]y^2 + K_0 K_1 y + K_0 K_1 (\alpha + \beta)$$

by the alleged criterion we have that the following condition establishes the stability of the point:

$$r[\frac{\kappa'(\overline{\pi}_1)}{\nu}(\overline{\pi}_1 - \nu\delta) - (\alpha + \beta)] > 0 \tag{3.5}$$

assuming a positive real interest rate $r > 0$, this implies

$$\kappa'(\overline{\pi}_1)(\overline{\pi}_1 - \nu\delta) > \nu(\alpha + \beta) \tag{3.6}$$

under the conditions of $\alpha > 0$, $\beta > 0$ and $\kappa'(\overline{\pi}_1) > 0$, this implies that

$$\overline{\pi}_1 > \nu\delta. \tag{3.7}$$

For the point $(0, 0, 0)$ we have

$$
J(0,0,0) = \begin{bmatrix}
\Phi(0) - \alpha & 0 & 0 \\
0 & \frac{\kappa(-\infty)}{\nu} - \alpha - \beta - \delta & 0 \\
0 & 0 & \frac{\kappa(-\infty)}{\nu} - (r + \delta)
\end{bmatrix}
$$

Under regular conditions, by assuming $\kappa(-\infty) = 0$ we have that the point $(0, 0, 0)$ is stable. In chapter 4, both models are calibrated with Peruvian data. Then the significant equilibirum is computed and some trajectories are show to analyze the economy.

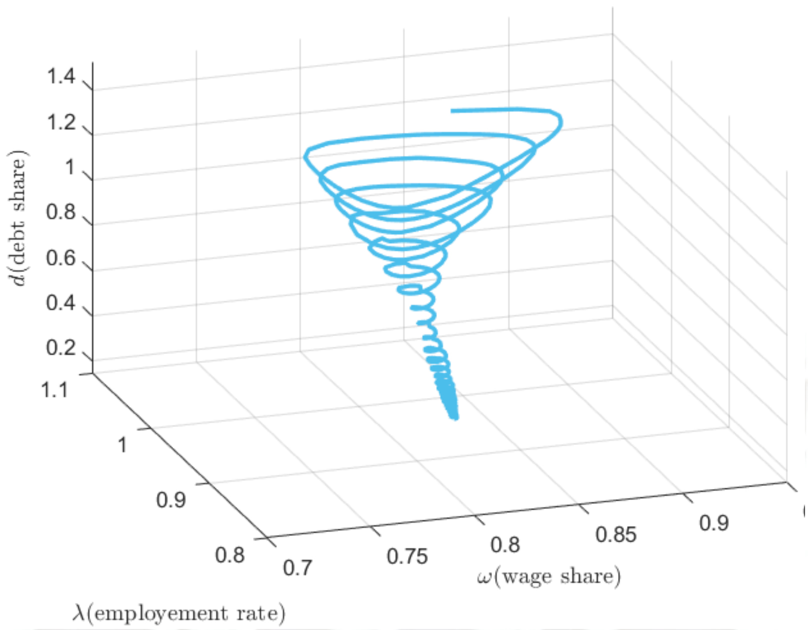

Figure 3.1: A trajectory in the stable region of the Keen model. It starts with a debt share of 1.2 and converges to the equilibrium point with a lower debt share.

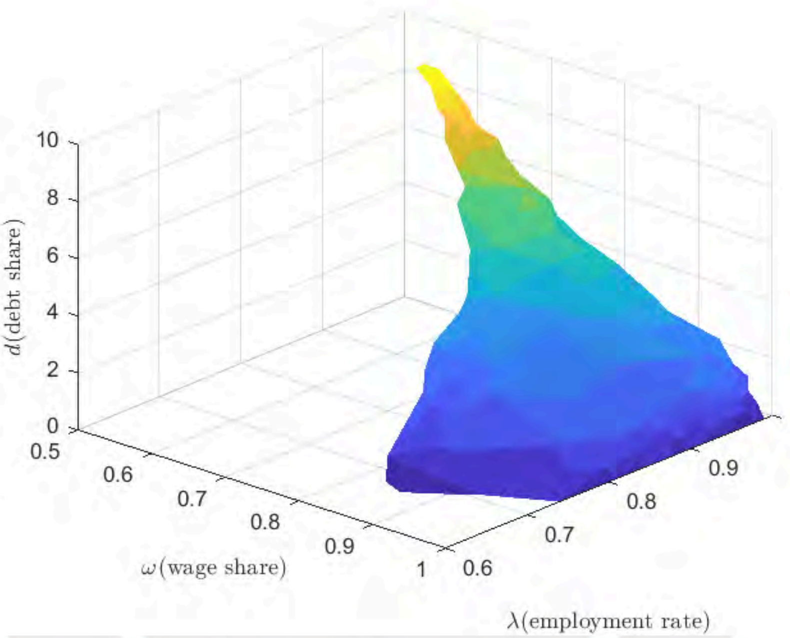

Figure 3.2: The graph shows the stability region of a Keen model. Points outside this region may lead to a rapid increment in debt or to trivial equilibrium points, whereas points inside the region converge to a finite debt, wage share and employment rate.

Chapter 4

The case of Peru

4.1 Stylized facts

The Goodwin model rests upon one main stylized fact: the trade-off between labor share and employment rate. Figure 4.1 shows labor share and employment rate for Peru for the period 1991-2014. We observe part of that behavior between the two variables. When labor share dwindles employment rate rises. In figure 4.2 we observe the two variables joint together. The graphic unveils what seems to be part of a clockwise cycle. Precisely, a quarter of a cycle and, although, the short horizon of the data available curbs our gains to conclude the fit of the model with certainty, the observed direction is the same the one predicted by the model and it seems that we are in part of the dynamics.

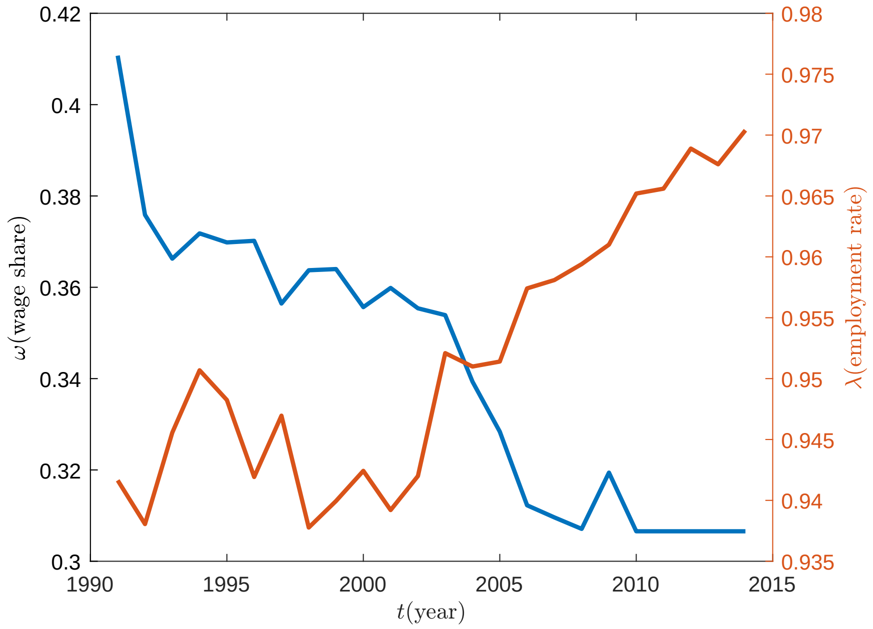

Figure 4.1: Source: WDI and PWT, own elaboration.The graph shows the trade-off between wage share and employment rate. While the first has a decreasing tendency, the second has an increasing tendency with a small exception at the beginning of the sample between 1990 and 1998.

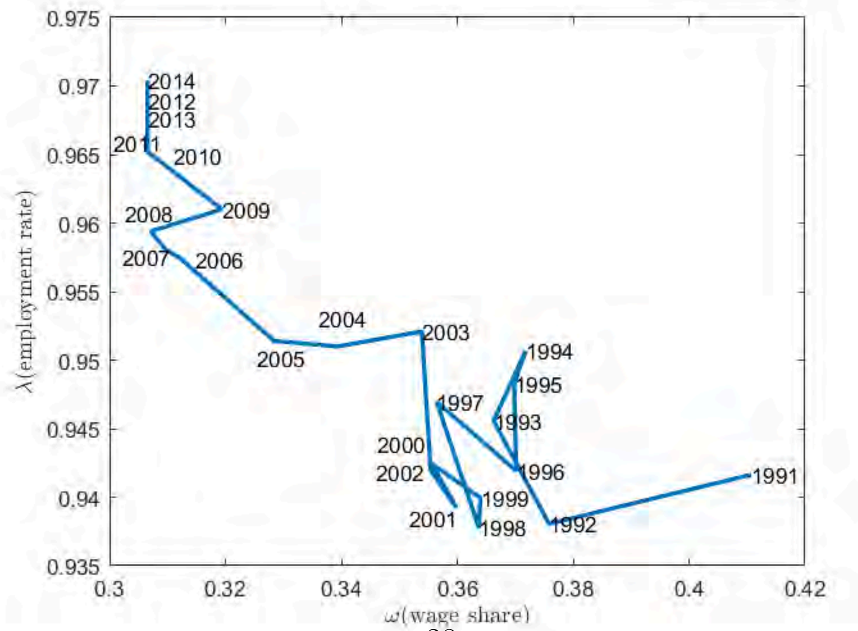

Figure 4.2: Source: WDI, own elaboration. It is observed that the data collected resembles a quarter of a cycle located in the third quadrant.

	Real GDP	λ	ω	I
Real GDP	1	0.925622	-0.911305	0.59338
λ		1	-0.871394	0.539893
ω			1	-0.485033
I				1

Table 4.1: Correlation between variables

Parameters

We verify the assumptions of the model. We check if capital-to-ouput ratio is constant. From figure 4.3, it is clear that the ratio is not constant, but the mean captures the symmetric fluctuations. As for productivity growth and total labor force growth, neither of them is constant. Figure 4.4 and 4.5 suggests that there is an upward and downward trend respectively.

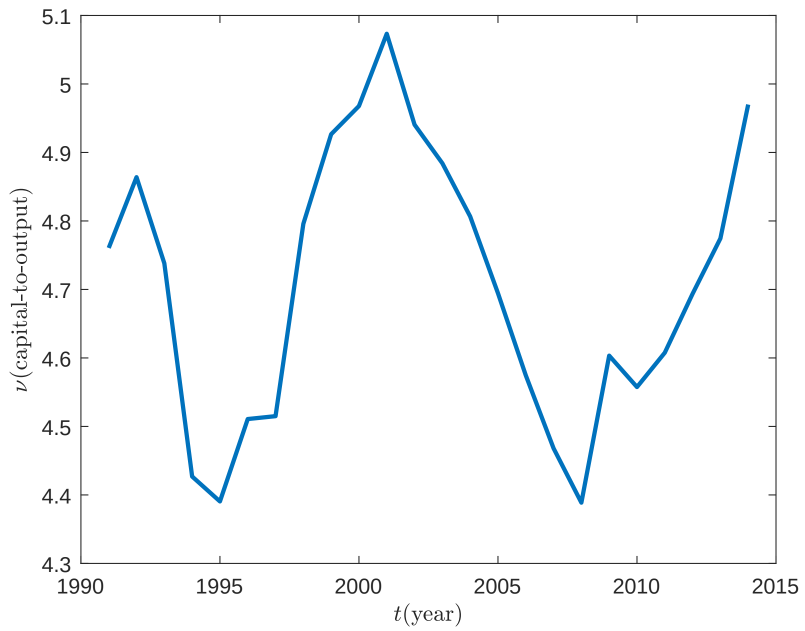

Figure 4.3: Source: WDI, own elaboration. The graph shows variation in the capital-to-output values. This variation seems to respond to a cycle.

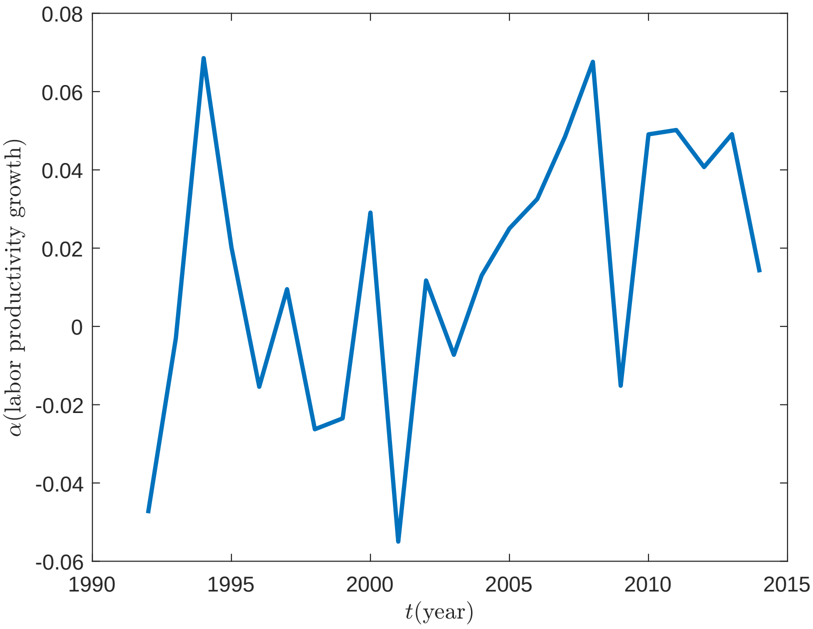

Figure 4.4: Source: WDI, own elaboration. It shows a small positive tendency.

4.2 Calibration of the Goodwin model

The purpose of this section is to provide details on how we handled data to estimate the parameters of the model and how we validated it.

We worked with the World Development Indicator (WDI) database from the World Bank, the Penn World Table (PWT) and the statistical series of BCRP. There is data available from 1950 to 2019 in the case of PWT and from 1960 to 2018 for WDI for many variables, but data for employment rate is only available from 1991 to 2014 so this is our time horizon for all the involved variables in the models. The description of the variables and its source are detailed in Table 4.2.

The parameters of the Goodwin model we need to estimate are the following: growth rate of productivity, growth rate of total labor force, capital-to-output ratio and deprecation rate of capital. To estimate the growth of, both, productivity and total labor force α and β we perform the following

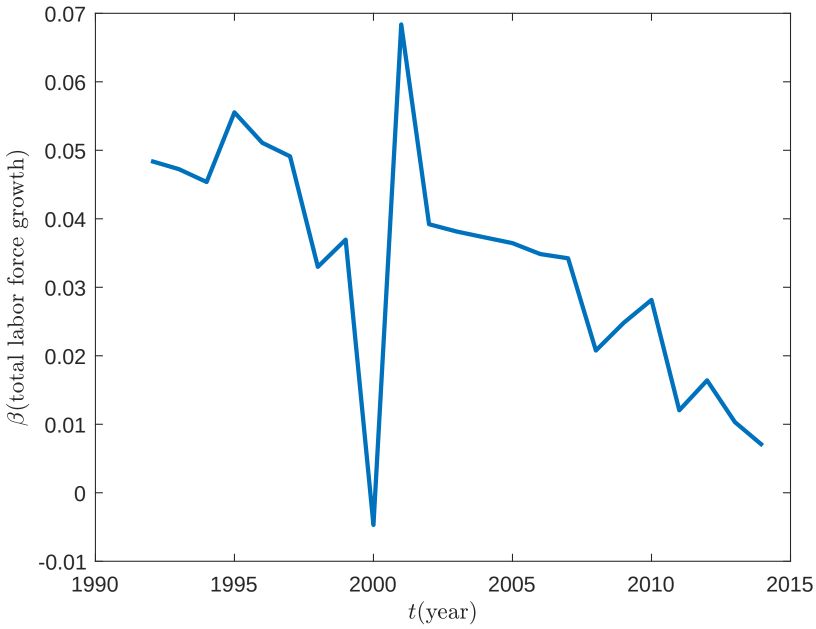

Figure 4.5: Source: WDI, own elaboration. The graph shows a clear negative tendency.

linear regressions:

$$\ln(a_t) = \ln a_0 + \alpha t + \varepsilon, \ \ \varepsilon \sim N(0, \sigma_1)$$
$$\ln(N_t) = \ln N_0 + \beta t + \varepsilon, \ \ \varepsilon \sim N(0, \sigma_2)$$

productivity a_t is calculated by dividing real GDP by employment. For the latter we need to multiply employment rate with the population older than 14 years old.

$$\text{EMP} = \text{EMPR}/100 \times (100 - \text{POP14})/100 \times \text{POP}$$

so we have

$$\text{PRODTY} = \text{RGDP}/\text{EMP}$$

Labor force N_t is calculated by the product of labor force percentage and the total population

$$\text{LABF} = \text{LABFR}/100 \times (100 - \text{POP14})/100 \times \text{POP}$$

The details of the estimations are found in table $C.1$. We compute capital-to-output ratio ν by taking the mean of CAPITAL/RGDP. Finally, capital depreciation δ is computed by taking the mean of DELTA. As the estimated value from 1950 to 2014 is close to the one used in (Céspedes & Ramírez-Rondán, 2014), this latter value is used. Table 4.3 provides with the estimated values and table C.1 shows the details of the regressions.

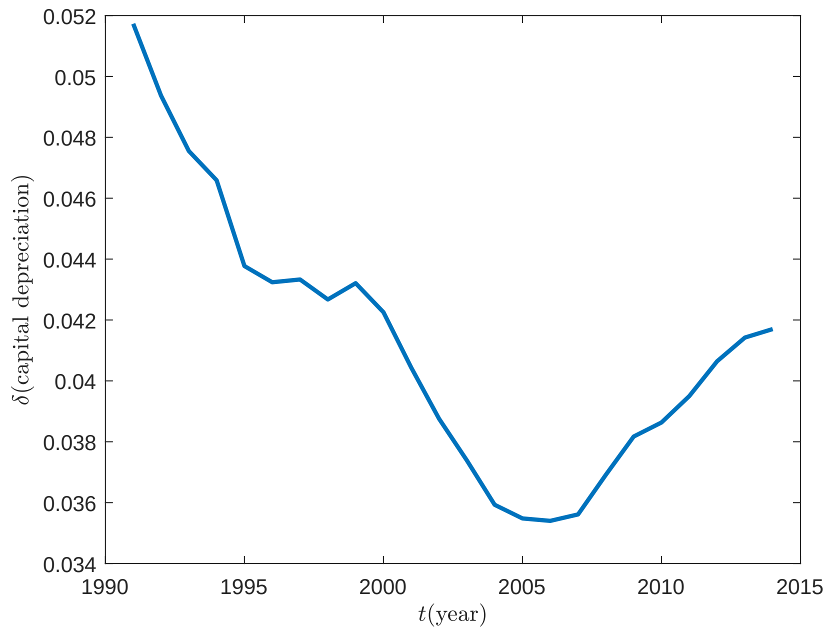

Figure 4.6: Capital depreciation shows a negative tendency until 2005, then a rise is observed.

The variable employment rate λ is obtained as

$$\text{LAMBDA} = \text{EMPR}/\text{LABFR}$$

and labor share ω is obtained from the variable LABSH specified in the table 4.2.

We replace the values of the parameters in the set of equations 2.18 and obtain the following model for Peru:

$$\begin{aligned}\dot{\omega}(t) &= \omega(t)\left[\Phi(\lambda(t)) - 0.0129\right] \\ \dot{\lambda}(t) &= \lambda(t)\left[\frac{1-\omega(t)}{4.7055} - 0.1016\right],\end{aligned}$$

Varible	Description	Source	Code
NGDP	GDP (current US$)	WDI	NY.GDP.MKTP.CD
RGDP	GDP (constant 2010 US$)	WDI	NY.GDP.MKTP.KD
CAPITAL	Capital Stock at Constant National Prices for Peru	FRED	RKNANPPEA666NRUG
EMPR	Employment to population ratio, 15+, total (%) (modeled ILO estimate)	WDI	SL.EMP.TOTL.SP.ZS
LABFR	Labor force participation rate, total (% of total population ages 15+) (modeled ILO estimate)	WDI	SL.TLF.ACTI.ZS
POP	Population, total	WDI	SP.POP.TOTL
POP14	Population ages 0-14 (% of total)	WDI	SP.POP.0014.TO.ZS
LABSH	Share of Labour Compensation in GDP at Current National Prices for Peru, Ratio, Annual, Not Seasonally Adjusted	FRED	LABSHPPEA156NRUG
DELTA	Constant of capital depreciation	PWT90	delta
INV	Gross fixed capital formation, private sector (% of GDP)	WDI	NE.GDI.FPRV.ZS
RIR	Real interest rate	WDI	FR.INR.RINR
DEBT	Domestic credit to private sector by banks (% of GDP)	WDI	FS.AST.PRVT.GD.ZS
REV	Revenue, excluding grants (% of GDP)	WDI	GC.REV.XGRT.GD.ZS
TAXREV	Taxes on income, profits and capital gains (% of revenue)	WDI	GC.TAX.YPKG.RV.ZS
TAXPROF	Profit tax (% of commercial profits)	WDI	IC.TAX.PRFT.CP.ZS

Table 4.2: Variables used for calibrations

Parameter	value	Methodology
$\hat{\alpha}$	0.0129	Ordinary Least square
$\hat{\beta}$	0.0337	Ordinary Least square
$\overline{\nu}$	4.7055	Mean
$\overline{\delta}$	0.055	Mean

Table 4.3: Estimated parameters

wage share at the equilibrium is computed from equation 2.2

$$\overline{\omega} = 0.5216 \tag{4.1}$$

to compute the employment rate at the equilibrium, a specific form of the

Phillips curve needs to be assumed. Also, the regularity conditions of the saddle point provided by equations 2.19 and 2.20 are verified:

$$\frac{1}{\nu} - \hat{\alpha} - \hat{\beta} - \overline{\delta} = 0.111 > 0$$

equation 2.19 is verified assuming a specific Phillips curve form. Two approaches are addressed here considering the discretized version of the model. The first functional form (I) of the Phillips curve considered is the following:

$$\frac{\omega[k+h] - \omega[k]}{\omega[k]} + \alpha = \Phi[k], \tag{4.2}$$

and the second functional form (II) of the Phillips curve is the following:

$$\nu \frac{\lambda[k+h]^2 - \lambda[k]\lambda[k+2h]}{\lambda[k+h]\lambda[k]} = \Phi[k] \tag{4.3}$$

where h is the sampling time of the system. Here we consider $h = 1$. The first approach (I) comes along straightforward by solving for the Phillips curve in equation 2.18 of the dynamics of the wage share. For the second approach (II), labor share is expressed in terms of the employment rate in the second equation of system 2.18:

$$\omega[k] = 1 - \nu \frac{\lambda[k+1] - \lambda[k]}{\lambda[k]} - \alpha \tag{4.4}$$

and then replace it into the first equation.

We test the fit of two types of Phillips curve. The first is the original linear curve which makes the Goodwin system a Lotka-Volterra system. Then, in order to provide a limit for the employment rate and for the cycles, we consider, as in (Blatt, 1983), a non-linear function. This form also represents the relationship of high wages when employment rate is near full employment. The linear Phillips curve is the following:

$$\Phi[k] = \phi_1 \lambda[k] - \phi_0, \tag{4.5}$$

and the non-linear Phillips curve is especified as follows:

$$\Phi[k] = \frac{\phi_1}{(1 - \lambda[k])^2} - \phi_0 \tag{4.6}$$

Table C.2 provides the technical details of the estimations and table 4.4 summarizes the results. In all the estimations the assumption that the Phillips

Approached $\Phi(.)$	$\hat{\phi}_1$	$\hat{\phi}_0$	$\overline{\lambda}$	Cycle length (years)	$\Phi(0) - \alpha$
Linear I	0.4313	0.4238	0.9827	28.9881	-0.4109
Linear II	2.5588	2.4948	0.9716	11.9175	-2.4948
Non-linear I	$1.9286 \cdot 10^{-5}$	0.0231	0.09711	15.1248	-0.0102
Non-linear II	$9.7709 \cdot 10^{-5}$	0.1084	0.9801	11.9175	-0.1083
Non-linear II adjusted	$4.8855 \cdot 10^{-5}$	0.5548	0.9703	3.1333	-0.5543

Table 4.4: Summary of estimated parameters, cycle length and the regularity condition of the origin.

curve is increasing with respect to employment rate is satisfied. The estimated curves assuming a linear model of the Phillips curve for the first and second approaches, respectively, are the following:

$$\Phi_1[k] = 0.4313\lambda[k] - 0.4109 \tag{4.7}$$

$$\Phi_2[k] = 2.5588\lambda[k] - 2.4948 \tag{4.8}$$

for the non-linear case of the Phillips curve in the first and second approaches, respectively, we obtained:

$$\Phi_3[k] = \frac{1.9286 \cdot 10^{-5}}{(1 - \lambda[k])^2} - 0.0102 \tag{4.9}$$

$$\Phi_4[k] = \frac{9.7709 \cdot 10^{-5}}{(1 - \lambda[k])^2} - 0.1084 \tag{4.10}$$

Last fitting is improved by multiplying by a factor. In this case we choose $k = 5$ to affect the coefficients of the Phillips curve. The result of the simulation is observed in figure 4.2. This provides the best fitting from all the cases.

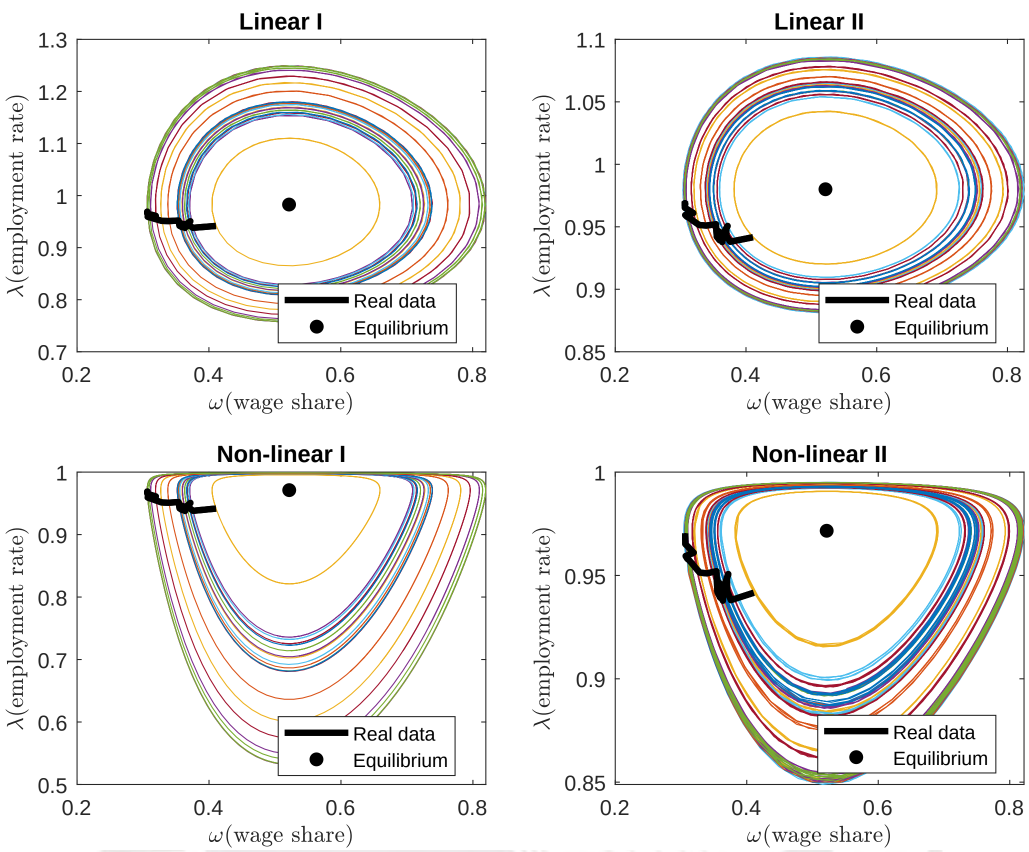

Figure 4.7: The graph shows the estimations of the Phillips curve from data of Peru along with several simulations starting at the points of the curve of real data. The different fittings are shown. It is observed that for the linear models I and II the employment rate surpasses the unity and that is a drawback. Both non-linear models seem to have a greater fluctuation than the real one.

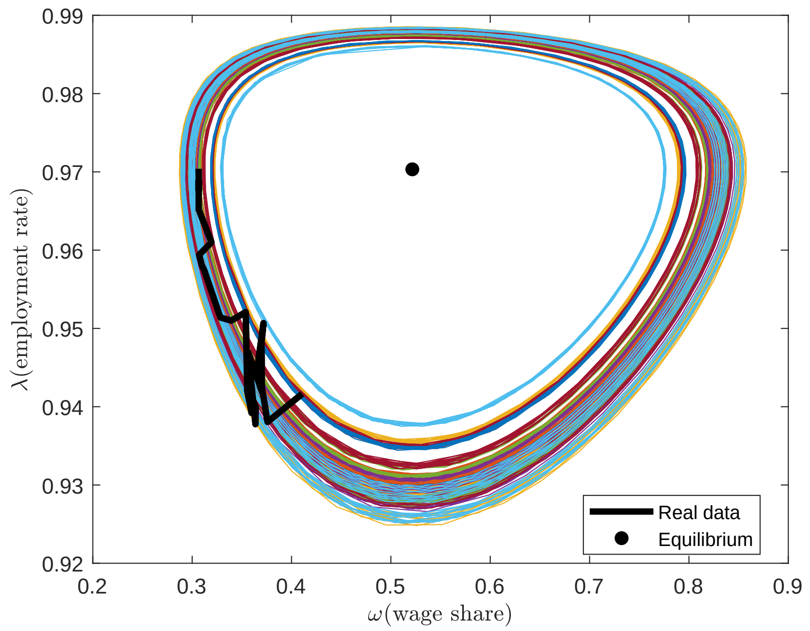

Figure 4.8: The graph shows the curve of real data, the equilibrium point is at $(0.5216, 0.9703)$ and several simulations starting at each point of the real data. This model was fit by the second approach (equation 4.3) with an adjustment in the coefficients of the Phillips curve. This adjustment consists in multiplying the coefficients by a factor of $k = 5$. This factor shrinks the orbits in the vertical axis without altering significantly the displacement on the horizontal axis. This estimation shows the best fitting of all cases.

In all the estimations the origin of coordinates satisfies the conditions to be a saddle equilibrium point. None of the estimated cycle lengths matches to the cycle length of the real data. The best calibration is the one that follows the second estimation approach with an adjustment on the coefficients of the non-linear Phillips curve. The parameters corresponding to this estimation are taken to continue with the estimation of the Keen model which is addressed in the next section.

4.3 Calibration of the Keen model

As the Keen model is an extension of the Goodwin model, it inherits the parameters α, β, δ and ν, then the only new parameters to estimate are the real interest rate r and the investment share function κ. For this, as in

(Costa Lima, 2013), the following functional form is assumed:

$$\kappa(x) = \kappa_0 + \kappa_1 \arctan(\kappa_2 x + \kappa_3).$$

The data to estimate the real interest rate is taken from the WDI as presented in table 4.2. Although data from the BCRP is the first option the available data starts on 2005. As the period of work in the present manuscript is from 1991 to 2014 and the period from 1991 to 2001 represents the stabilization efforts of the peruvian economy, then taking account of data only from 2005 on would be misleading. The real interest rate is computed by taking the mean of the variable RIR described in table 4.2.

$$\overline{r} = 0.245902.$$

The investment share function κ is estimated according to the following criteria:

$$\kappa(-\infty) = 0, \tag{4.11}$$
$$\kappa(+\infty) = 1, \tag{4.12}$$
$$\kappa'(\overline{\pi}) = 1.5, \tag{4.13}$$
$$\kappa(\overline{\pi}) = \nu(\alpha + \beta + \delta), \tag{4.14}$$
$$\overline{\pi} = 0.46 \tag{4.15}$$

where $\pi = 1 - \omega - \overline{r}d$. From the data in table 4.2 it would be $\pi = 1 - LABSH - \overline{RIR}\ DEBT$. This results in the mean $\overline{\pi} = 0.5992$, but from equation 3.4, which describes the form of the non-trivial equilibrium for the Keen model, in order to obtain a non-trivial equilibrium with all positive coordinates, the condition $\nu(\alpha + \beta + \delta) > \overline{\pi}$ must be satisfied. This is, $\overline{\pi} < 0.4784$. The value $\overline{\pi} = 0.46$ is chosen as it is located in the confidence interval $[\overline{\pi} - 2\sigma; \overline{\pi} + 2\sigma] = [0.3914; 0.5286]$ of $\overline{\pi}$ where $\sigma = 0.0343$ is the estandard deviation of the utility share π. The following parameters of the investment share function are obtained: From this and equation 3.4, the good

Parameter	value
$\hat{\kappa_0}$	0.5
$\hat{\kappa_1}$	0.31831
$\hat{\kappa_2}$	4.7341
$\hat{\kappa_3}$	-2.24556
$\overline{r}$	0.24592

Table 4.5: Estimated parameters

equilibrium point is the following:

$$(\overline{\omega}, \overline{\lambda}, \overline{d}) = (0.4429, 0.9703, 0.3948) \tag{4.16}$$

the stability of this equilibrium is guaranteed by means of equation 3.2:

$$\overline{\pi} - \nu\delta = 0.2012 > 0. \tag{4.17}$$

Figure 4.9 shows simulations of the trajectory of the Keen model calibrated for Peru. It shows that under this parameters the trajectory lies inside the stability region of the system. This means if the parameters remain the same, the natural path of the economy is led to a finite stable equilibrium. Figure 4.10 shows the trajectory of a point inside the stability region for Peru and the stabiliry region is presented in figure 4.11. Figure 4.12 presents the trajectory for Peru between 1991 and 2014 along with the equilibrium point and figure 4.13 shows the region from the z axis.

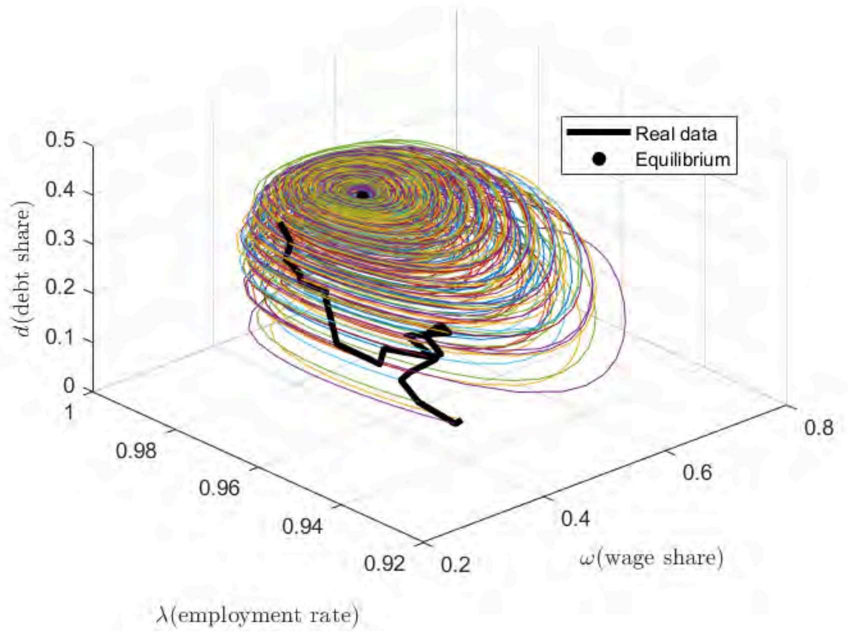

Figure 4.9: The graph shows the curve of real data and the equilibrium at $(0.4429, 0.9703, 0.3948)$ along with several simulations starting at the points of the curve of real data. It is observed that the trajectory of the real data is placed inside the stability region of the Keen model for Peru.

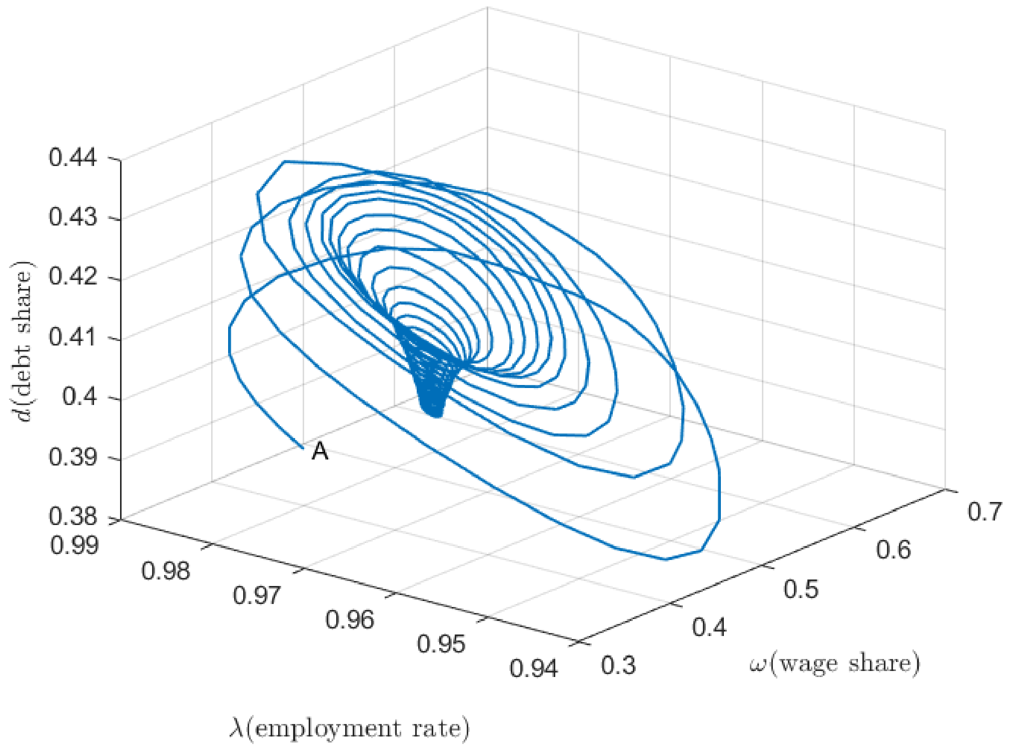

Figure 4.10: The graph shows the trajectory of the simulation for the starting point $A = (0.3, 0.97, 0.4)$.

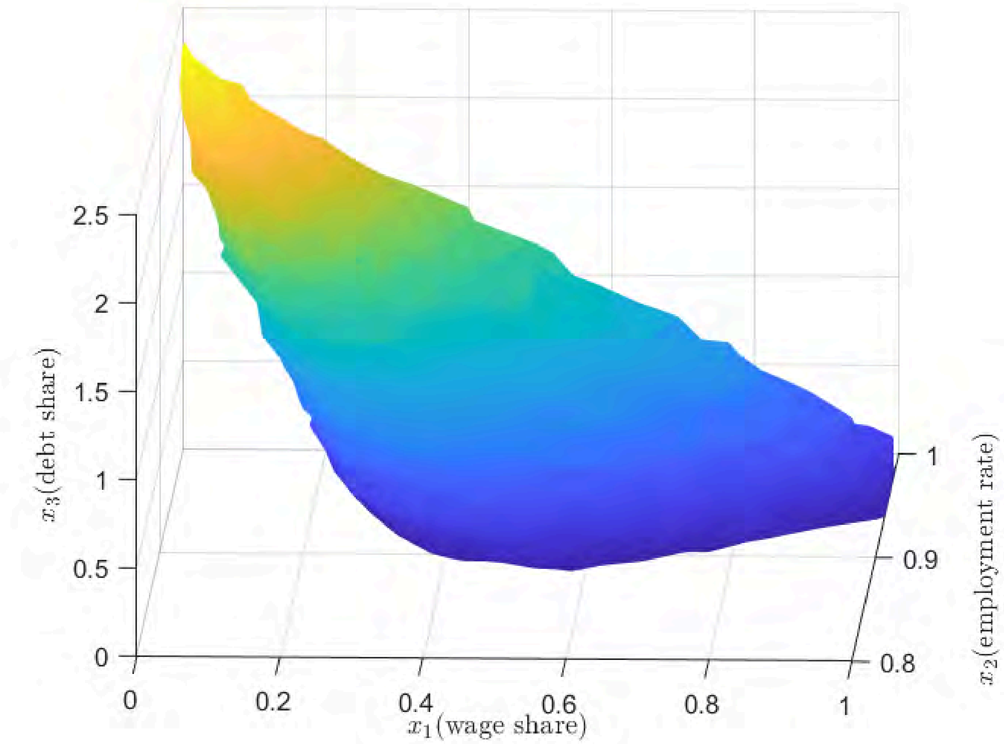

Figure 4.11: Stability region for Peru.

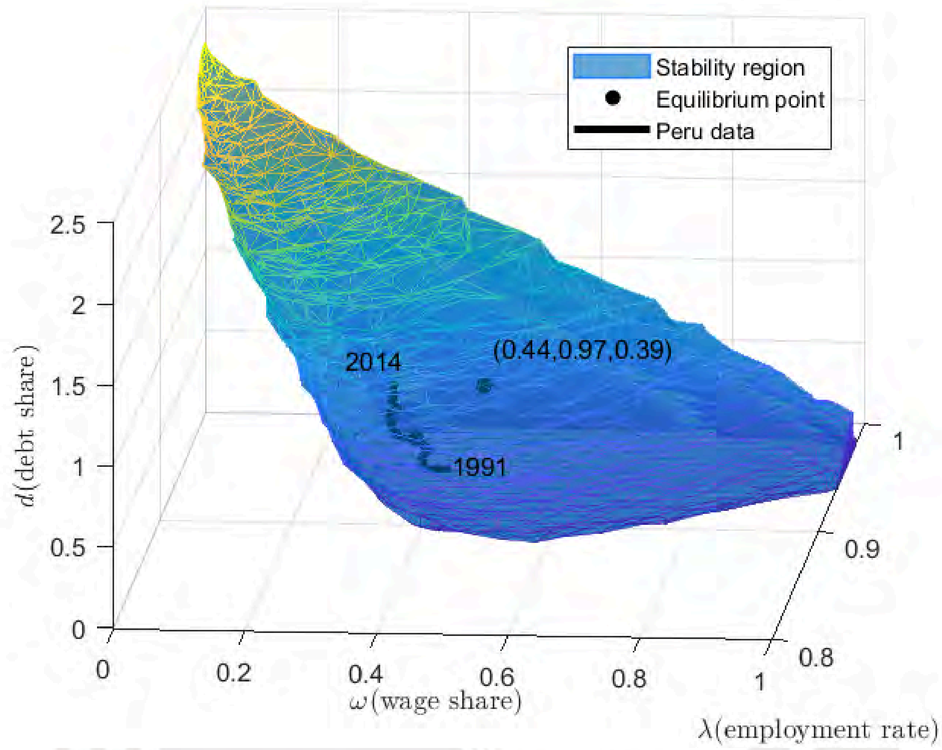

Figure 4.12: The graph shows the stability region and the trajectory for Peru along with the equilibrium point inside the region.

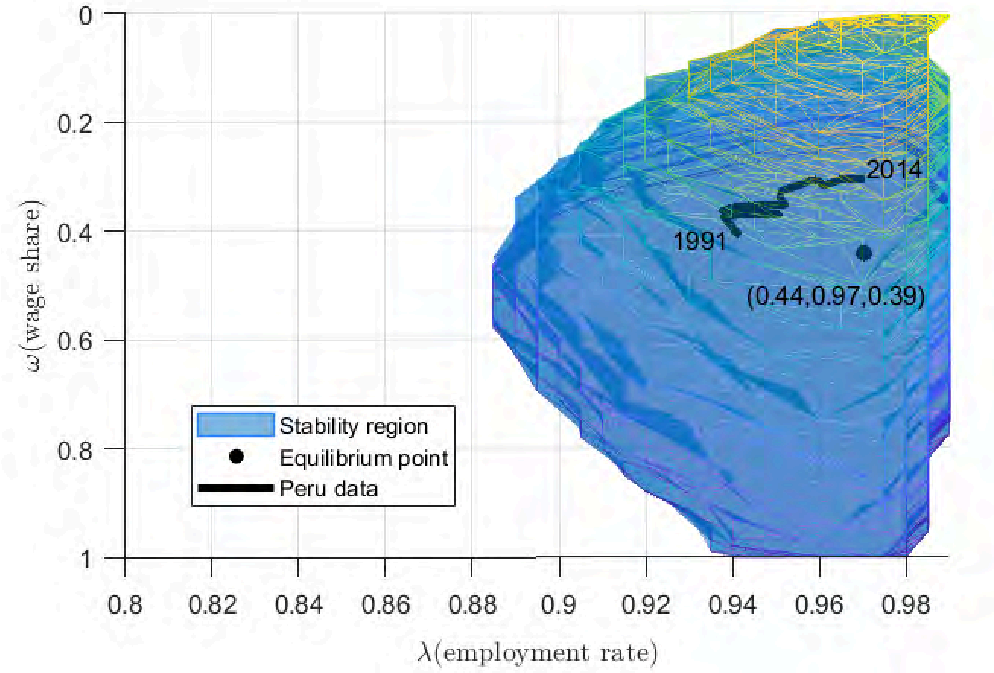

Figure 4.13: The graph shows the projection of the stability region.

With the stability of the equilibria analyzed, next the control technique is explained. The idea behind the application is that for a given point outside the stability region we would like to force it to enter the region. To understand how we are going to do this in the next chapter we are going to explain the method and apply it to avoid a financial crisis.

Chapter 5

Control

In the crisis of 2008, the lack of propper restrictions on credit issueing led to a constant increase on the demand of houses which raised their prices until the real value stopped being reflected. A drop in prices was followed, decreasing the demand and households not being able nor willing to pay their debts of the, now, less valuable assets. In the Keen model this situation is represented by the state of the system being in the unstable region of it. In order to avoid unbounded debt accumulation, a control policy is designed to steer the point in the unstable region towards a point within the stable region, where convergence is towards the *stable* equilibrium. This policy is stated in terms of the following rate of growths: labor productivity, total labor force and interest rate which are denominated u_1, u_2 and u_3 respectively in this chapter. Also, in this section the variables wage share ω, employment rate λ and debt share d are denominated in the more general notation x_1, x_2 and x_3 respectively, following the system theory approach.

5.1 Problem formulation

As mentioned in the introduction of this chapter, in abstract, an intial point in the unstable region is considered and it needs to be steered to the stable region. As in definition 2, consider the stable region of system 3.2 associated to point IV in the table of equilibria 4.2. Denote this E_s. Formally, the *unstable region* of the system is the stable region associated to point V and it is denoted E_u. Let $x_0 \in E_u$ and $x_f \in E_s$. The output y of the initial value problem associated to the Keen system with initial value x_0 converges to point V if all the parameters of the system are left constant. The goal is to steer x_0 to x_f by forcing the system to track a pre-designed reference path $y_r : \mathbb{R} \to \mathbb{R}^3$ with intial point $y_r(0) = x_0$ and final point $y_r(f) = x_f$ with the aid of a policy function $u : \mathbb{R} \to \mathbb{R}^3$ in a finite time t_f. In regards of the controllers which take the form of economic policies as set by the problem, both, labor share growth (α) and total labor force growth (β) appear linearly in system 3.2. This is desirable, as later it is seen in section 5.3, to be able to apply control with the techniques of presented in this work (pole placement and optimal control), the system needs to be transformed to a non-linear affine system with respect to the controllers. The technique to get this transformation is feedback linearization and is based on a change of coordinates which comes from the derivatives of the chosen outputs. On the other side, real interest rate r is in part of the argument of the non-linear function $\arctan(\cdot)$ in the system. This makes it impossible for the linearization technique to put the real interest rate affinely in the transformed system. This is part of the reason why the growth of real interest rate $\dot{r}$ is chosen as a controller. To begin with the transformation, the model is dynamically extended and a fourth state equal to the interest rate is added to it. The dynamics is, then, the following:

$$\dot{x}_1 = x_1 \left[\Phi(x_2) - u_1\right], \tag{5.1a}$$

$$\dot{x}_2 = x_2 \left[\frac{\kappa(x_1, x_3, x_4)}{\nu} - u_1 - u_2 - \delta\right], \tag{5.1b}$$

$$\dot{x}_3 = x_3 \left[r - \frac{\kappa(x_1, x_3, x_4)}{\nu} + \delta\right] + \left[\kappa(x_1, x_3, x_4) - (1 - x_1)\right], \tag{5.1c}$$

$$\dot{x}_4 = u_3 \tag{5.1d}$$

$$y = [x_1 \; x_2 \; x_3]^T \tag{5.1e}$$

with

$$\kappa(x_1, x_3, x_4) = \kappa_0 + \kappa_1 \arctan(\kappa_2(1 - x_1 - x_4 x_3)) \tag{5.2}$$

and

$$x(0) = x_0, \tag{5.3}$$

$$u(0) = (\alpha, \beta, 0) \tag{5.4}$$

The initial point is $x_0 = (0.9, 0.9, 0.5)$ with initial parameter values shown in tables 4.3 and 4.5. Figures 5.1 and 5.2 show the initial trajectory with respect to the initial setting of the Keen system, the undesirable debt share accumulation pattern is observed.

The reference path y_r is set following a natural transition between the unstable region E_u and the stable region E_s. It is designed with the gradient based path planner algorithm described in section 5.4. Figure 5.3 shows the built path. Results and description of the control techniques applied are explained and depicted in section 5.2. It is shown that a point in the unstable region is successfully steered into the stable region. This control application does not show good tracking performance, but that is not the concern of this work as long as the controls take the outputs of the model to the desired stability region.

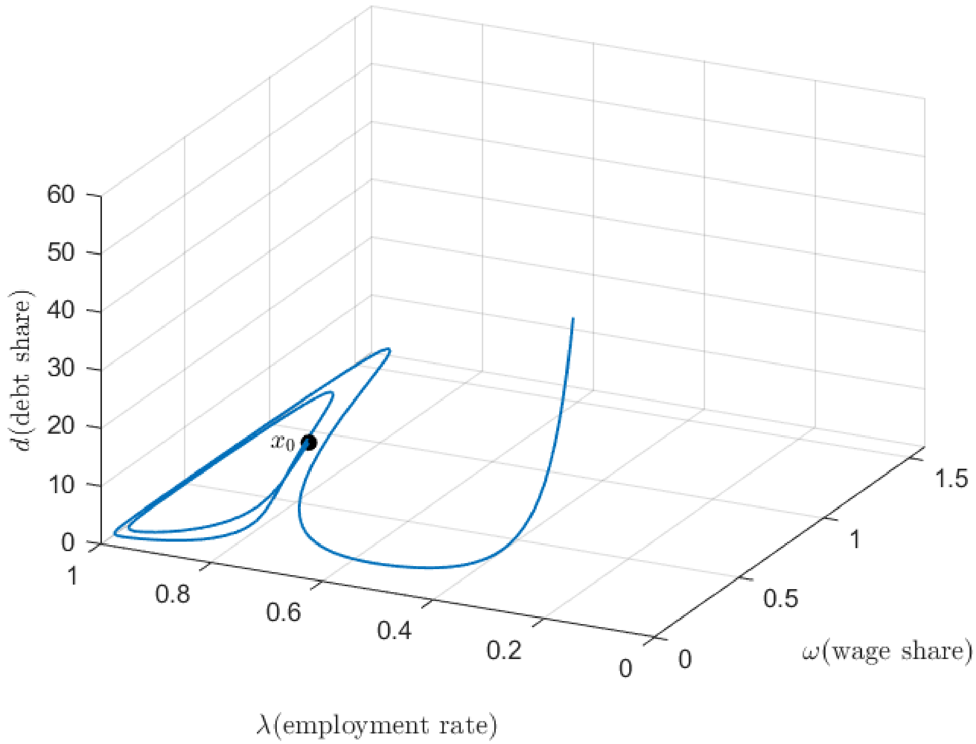

Figure 5.1: Trajectory of the Keen model with initial point $x_0 = (0.9, 0.9, 0.5)$. At first there is a boom of employment rate, followed by a sustained increase of wage share, then debt share starts to accumulate fast.

5.2 Main results

First, stabilization of debt share is done by controlling with the pole placement techinique (see section 5.3.1 and B.2) to the dynamically extended Keen

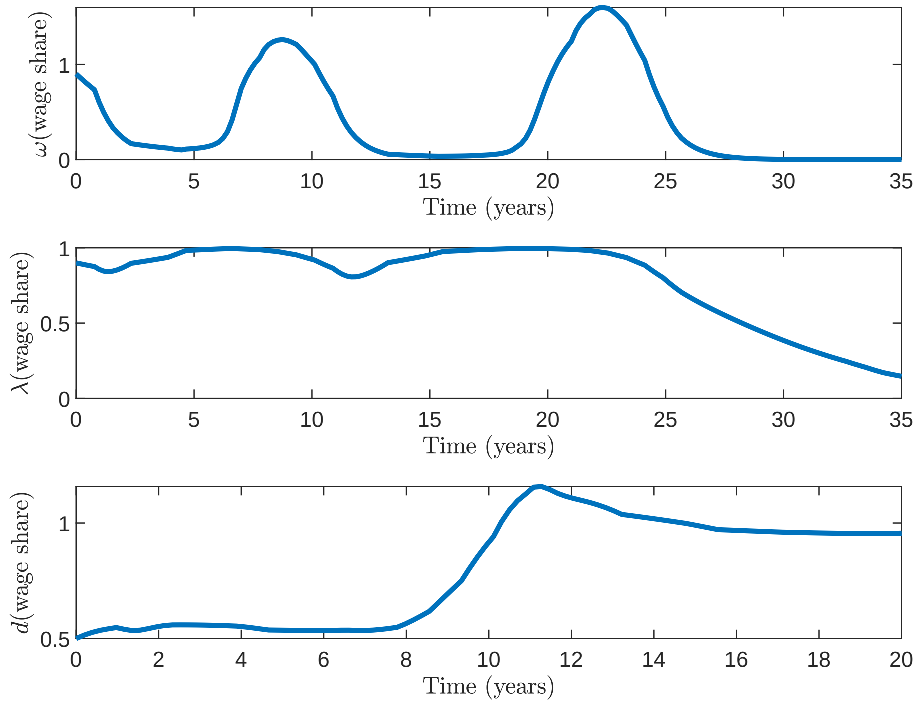

Figure 5.2: The graph shows the trajectory of the simulation for $x_0 = (0.9, 0.9, 0.5)$ for each variable of the model. It is observed that in the time lapse of 10 years debt is about the double of the original value and it continues increasing.

model 5.1. This is model I in section 5.3 where a detailed explanation of the linearization is provided. As seen in figure 5.4, the controlled trajectory does not reach the targeted reference point B which here is the equilibrium point of the Keen model (equation 4.16), but the obatined policy does not stop from reaching a point (C) in the stability region. Once the trajectory reaches its final point, system 5.1 stops working and the original Keen model 3.2 is set to function with the original constant parameters. After this, Debt share decreases until the system reaches the stable equilibrium point and the economy is stabilized. The control policies applied to the model to stabilize the economy are shown in figure 5.5. The time to perform this was set to 5 years with policies applied monthly. According to this result, labor productivity needs to decrease, while total labor force and real interest rate must

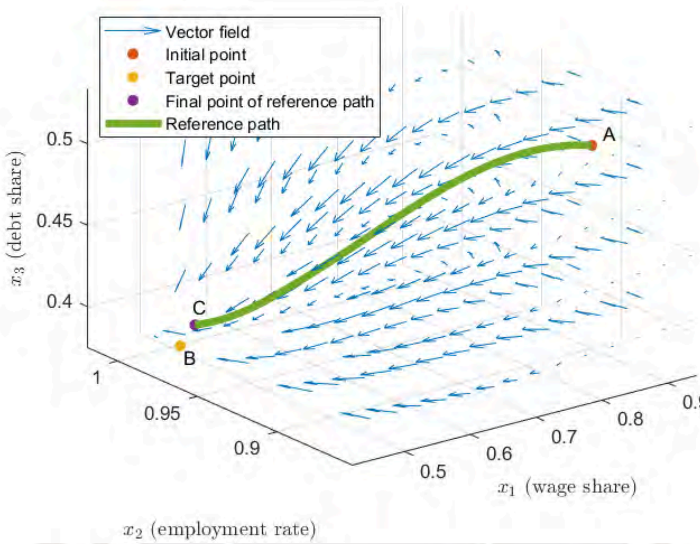

Figure 5.3: Reference trajectory from the initial point $A = x_0$ in the unstable region E_u to a final point B in the stable region E_s. Here B is set equal to the equilibrium in equation 4.16. This path is obtained with the Gradient Based Path Planning algorithm. The algorithm starts with a target point B which is the desired point to achieve, then the trajectory is obtained by modifying the vector field of the dynamical system. It is possible, as it is the case here, that the final point C of the built path is not the same as the target point, but lands on a neighborhood of it in the stability region. The trajectory depends on the force of the attraction field of the target point, so by increasing this parameter a more direct trajectory is obtained, but it would make it less natural with respect to the field of the system.

increase. A serious drawback of this technique is that for the specified control parameters, a big abrupt change in the real interest growth u_3 at $t = 4.5$ is generated making the control policy look unreal as it is seen in figure 5.5. This policy input is smoothed out by assuming a longer controlling time or by increasing the frequency in which the policies are set. Figure 5.7 shows the results of the stabilization assuming $t = 9$ years and figure 5.8 shows a smoother version of the inputs for this case. A better tracking of the output is seen in figure 5.9.

The decreasing of labor productivity might seem counter-intuitive to get an economy out of a future burst in debt share which might lead to a crisis, but as the Keen model inherits the trade-off behavior between firms and households from the Goodwin model, what it is truly meant is that either resources from production must be reduced so more of the utility goes to households or that more labor should be hired. In either case this translates into more of the share going to the households. From the financial sector side, the decreasing of the interest rate fosters less saving and promotes more investment. Figure 5.6 shows a considerably good tracking of the reference and certain parts of negative real interest rate which by the Fisher equation (Blanchard, 2017) it means that inflation is higher than the nominal interest rate in the corresponding parts.

To overcome the fact of real interest rate growth changing abruptly, a saturation function is imposed to the policies u as expressed in equation 5.3.2. In figure 5.12 the control policies are constrained to be between ± 0.09 and the time is $t = 3$ years. It is observed that the policy functions are less smooth than in figure 5.5 without the saturation. Figure 5.10 shows that the tracking of the reference path gets worse, but a point (C) in the stable region is still reached. Pushing the saturation of the control policies further, does not result in an stabilizable set of policies, this is the case when policies are bounded by ± 0.08. To tackle this problem, model II from 5.3 is provided as an alternative. The extra feature with respect to model I is that the variation of the control policiy dynamics are included. These are equations 5.11e and 5.11f of system 5.11 where a decay of the variation of each control policy u_1 and u_2 is set to 10. This represents a delay in the variables labor productivity and total labor force as they are not inmediately changeable as nominal interest rate is, for example. In this case optimal control is performed to obtain the policies. The cost function is the error of the tracking problem described in section 5.3.2 and the policies are limited to vary between ± 1. The time of the stabilization is kept to three years. Figure 5.13 shows that the tracking gets compromised compared to the case without optimization, but with saturation of the policies. The trajectory looks smooth and the control policies in figure 5.14 show an erratic behavior with smaller variation. The real interest rate in figure 5.15 is negative in part of the controlling time.

In order to avoid regimes with greater inflation rate than nominal interest rate that result in a negative real interest rate, the previous optimization is enhanced with a constraint based on Taylor's approximation polynomial of first degree $r(t+h) = r(t) - \dot{r}h$ where $r(t+h)$ which is the real interest rate in the next step $x_4(t+h)$ and $r(t)$ is $x_4(t)$ and h is the sampling time. The imposed constraint is $r_{min} \leq x_4(t+h) \leq r_{max}$ where the maximum real interest is set to be $r_{max} = 0.05$ and the minimum $r_{min} = 0.01$. Figure 5.16 shows a a small swinging at the beginning. The auxiliary inputs in figure 5.17

show a relative small range of variation and the outputs in figure 5.18 show the same trade-off between labor productivity and total labor force growth and the compliance of the constraints on the real interest rate which, now, is positive. The variation of the real interest rate can be shrunk even more to be less than 0.05, by allowing the control to be performed in a longer period, for example, five years. Another modification that helps with this goal is the increasing of the decaying factor of the labor productivity and total labor force which are τ_1 and τ_2 in equations 5.11e and 5.11f.

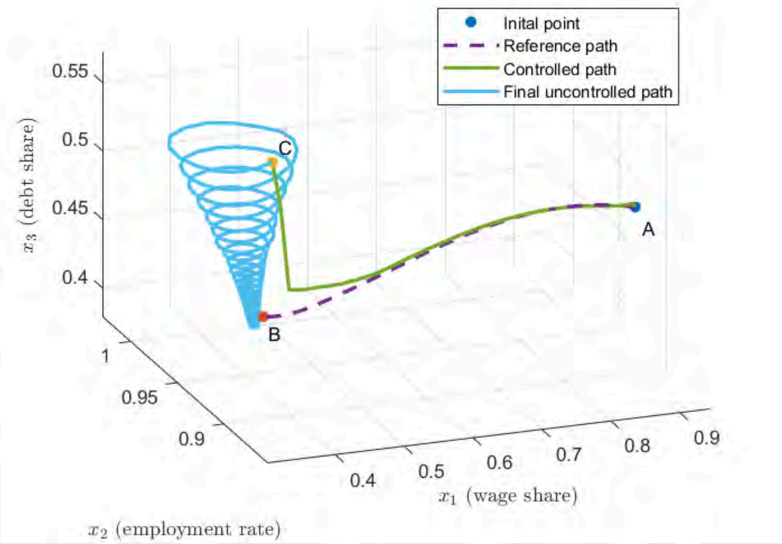

Figure 5.4: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 5 years. The initial point A is in the unstable region of the system. The controlled path does not reach the targeted point B, instead reaches point C which is also in the stable region. The tracking is good and after the system reaches point C the system goes back to the original setting with original constant parameters.

5.3 Control of the Keen model dynamics

In this section the control of debt share in the model of Keen is performed. First, the model is linearized by feedback linearization (Isidori, 2002). This

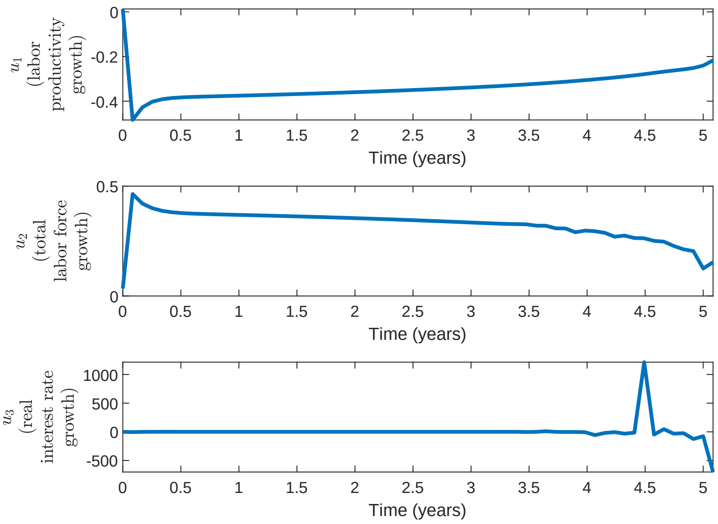

Figure 5.5: The graph shows the inputs of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 5 years. The growth of labor productivity and total labor force show a trade-off behavior, while the first shows a decreasing tendency the latter has an increasing tendency. This translates into a trade-off between firms and households. Real interest growth has a big peak at $t = 4.5$. This is a drawback which is overcome by setting a saturation on the inputs.

procedure is described, then pole placement and optimal control are applied to obtain the economic policy that takes debt share to an stable state. A diagram of this is depicted in Figure 5.19.

Keen model linearization

The real interest rate r in system (3.2) is a parameter of the investment function $\kappa(.)$. As this is composed by the inverse of a trigonometric function, then r is difficult to appear linearly after taking derivatives of the outputs y_2 and y_3. Because of this r is not considered as an input, but instead $\dot{r}$ is. The model is dynamically extende a new artificial state $x_4 := r$, this is

$$x_4 := r$$
$$u_3 := \dot{r}$$

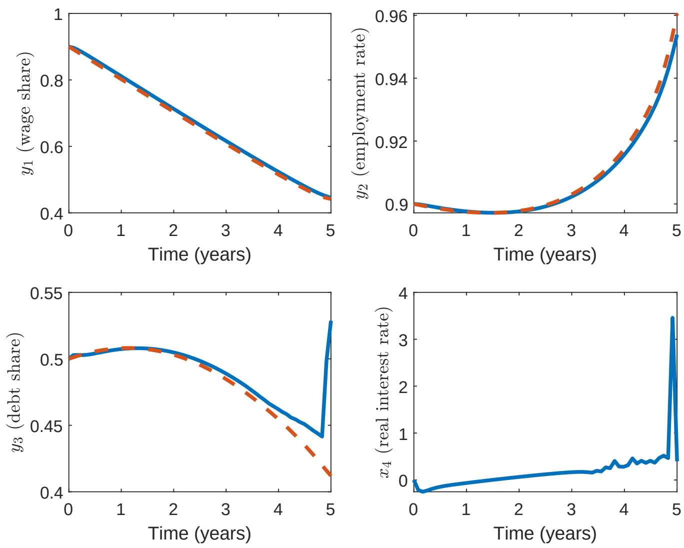

Figure 5.6: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 5 years. The control policy applied generates an steady reduction in wage share and increment in the employment rate. Debt share is reduced during most of time. A jump is observed in the last part of control. Real interest rate is negative in some parts indicating that the inflation rate is higher than the nominal interest rate. Also, a peak is observed in the real interest rate during last part of the control which is later improved by applying a saturation on the inputs or by setting a longer time to perform the control.

with u_3 the input that can be manipulated at will. With this, the system is easily written in the form

$$\dot{x} = f(x) + g_1(x)u_1 + g_1(x)u_2 + g_3(x)u_3. \tag{5.5}$$

Two models are handled in this manuscript. These are model I and model II described in the next sections. The first model is used with the pole placement control technique after the linearization. The saturation of the controls is also applied to this model. The second model extends the first by adding a set of two equations which represents the dynamics of labor productivity growth and total labor force growth by adding a delay in their response. It is used with optimal control which allows the handling of constraints, in

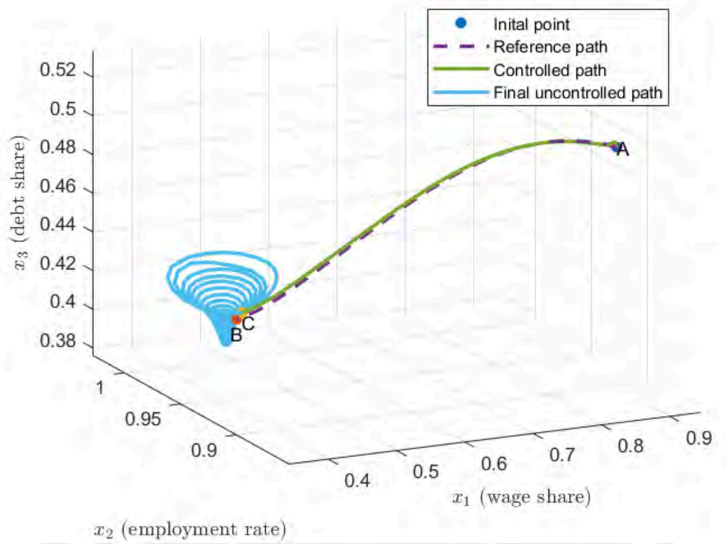

Figure 5.7: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 9 years. The initial point A is in the unstable region of the system. The controlled path does not reach the targeted point B, instead reaches point C which is also in the stable region. The tracking is good and after the system reaches point C the system goes back to the original setting with original constant parameters.

particular the positiveness of the real interest rate is targeted.

Model I

System (5.1) is taken as a base and the derivatives of the output $y = [y_1, y_2, y_3]^\top := [x_1, x_2, x_3]^\top$ are computed in order to find the vector relative degree of the augmented system. In a nutshell, the relative degree is the time an ouput function is derivated until the controls appear linearly. More

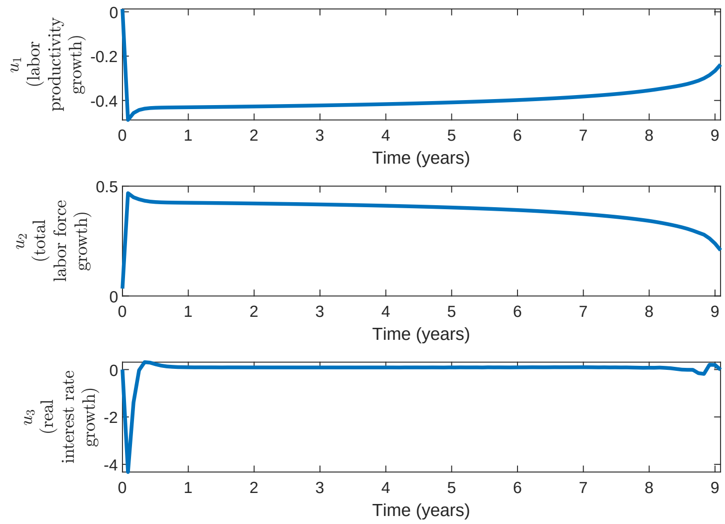

Figure 5.8: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 9 years. The growth of labor productivity and total labor force show a trade-off, while the first shows a decreasing tendency the latter has an increasing tendency.

details on this technique are found in section B.2 . The following is obtained:

$$\begin{aligned}
\dot{y}_1 &= x_1 \left(\Phi(x) - u_1\right), \\
\dot{y}_2 &= x_2 \left[\frac{\kappa(x)}{\nu} - u_1 - u_2 - \delta\right], \\
\dot{y}_3 &= x_3 \left[x_4 - \frac{\kappa(x)}{\nu} + \delta\right] + \left[\kappa(x) - (1 - x_1)\right], \\
\ddot{y}_3 &= \dot{x}_3 \left[x_4 - \frac{\kappa(x)}{\nu} + \delta\right] + x_3 \left[u_3 - \frac{(\kappa_x(x))(-\dot{x}_1 - x_4\dot{x}_3 - u_3 x_3)}{\nu}\right] + \\
&\quad + (\kappa_x(x))(-\dot{x}_1 - x_4\dot{x}_3 - u_3 x_3) + \dot{x}_1
\end{aligned}$$

so the vector relative degree is $p = (1, 1, 2)$. As described in section B.2, the linearized system has the form of an affine non-linear system $\dot{y} = f + g \cdot u$. Denoting by f_i the ith component of f in equation (5.5) for $i \in \{1, 2, 3, 4\}$,

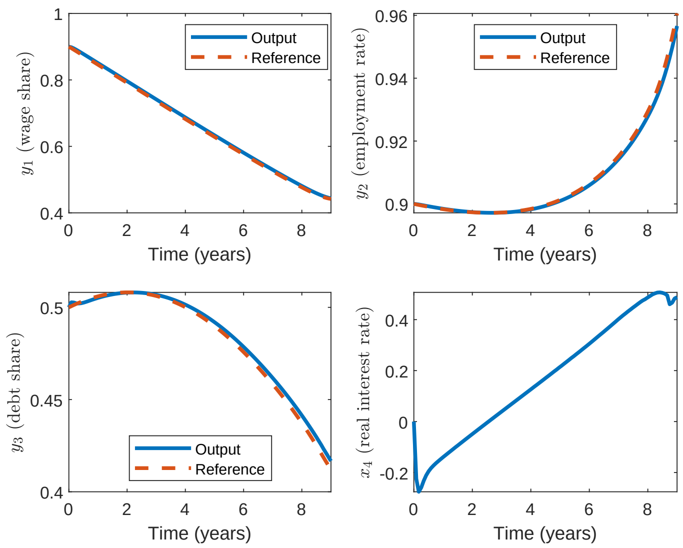

Figure 5.9: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique without saturation. The time of the control is set to 9 years. The control policy applied generates an steady reduction in wage and debt share, employment rate increases. Real interest rate is negative in some parts indicating that the inflation rate is higher than the nominal interest rate.

the decoupling matrix (equation B.14) is

$$D_k(x) = \begin{bmatrix} -x_1 & 0 & 0 \\ -x_2 & -x_2 & 0 \\ L_{g_1} f_3 & 0 & L_{g_3} f_3 \end{bmatrix}, \tag{5.6}$$

where

$$\begin{aligned} L_{g_1} f_3 = -x_1 \Bigg(& \frac{0.3202 x_3}{(4.734 x_1 + 4.734 x_3 x_4 - 2.489)^2 + 1} \\ & - \frac{1.507}{(4.734 x_1 + 4.734 x_3 x_4 - 2.489)^2 + 1} + 1 \Bigg), \end{aligned} \tag{5.7}$$

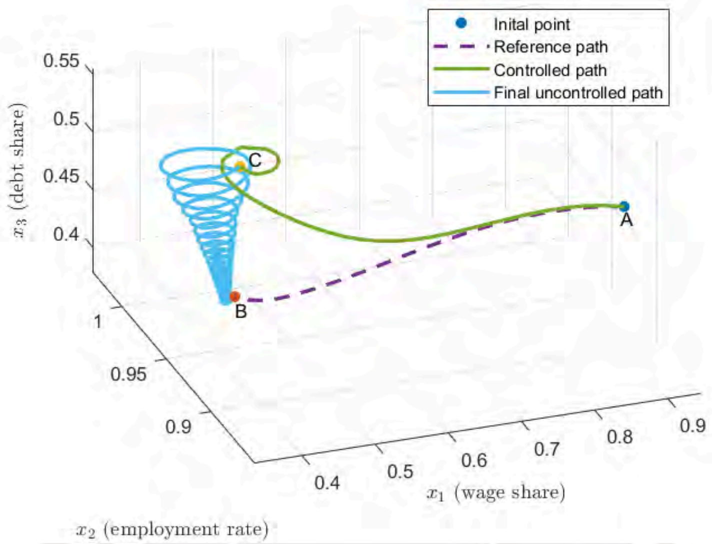

Figure 5.10: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique where a saturation of ± 0.09 on the inputs is imposed. The time of the control is set to 3 years. The initial point A is in the unstable region of the system. The controlled path does not reach the targeted point B, instead reaches point C which is also in the stable region. Tracking is not as good as without constraints, but it satisfies the goal of stabilizing the economy.

$$\begin{aligned} L_{g_3} f_3 = & x_3 \left(\frac{0.3202 x_3}{(4.734 x_1 + 4.734 x_3 x_4 - 2.489)^2 + 1} + 1 \right) \\ & - \frac{1.507 x_3}{(4.734 x_1 + 4.734 x_3 x_4 - 2.489)^2 + 1} \end{aligned} \tag{5.8}$$

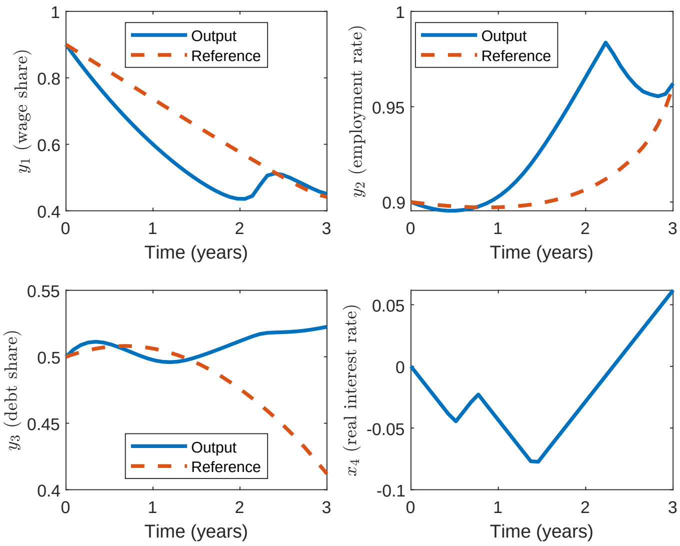

Figure 5.11: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique where a saturation of ± 0.09 on the inputs is imposed. The time of the control is set to 3 years. This does not show a good tracking of the output, but this is not important as the last point is in the stable region. It is also observed that real interest rate has reaches positives and negative values.

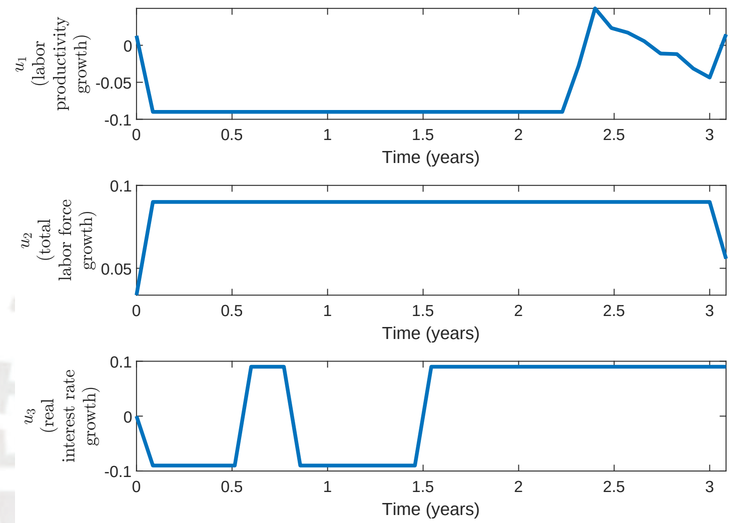

Figure 5.12: The graph shows the results of the control of the outputs performed to model I with the poleplacement technique where a saturation of ± 0.09 on the inputs is imposed. The time of the control is set to 3 years. The growth of labor productivity and total labor force show a trade-off behavior, while the first has a decreasing tendency the latter has an increasing tendency. This translates into a trade-off between firms and households.

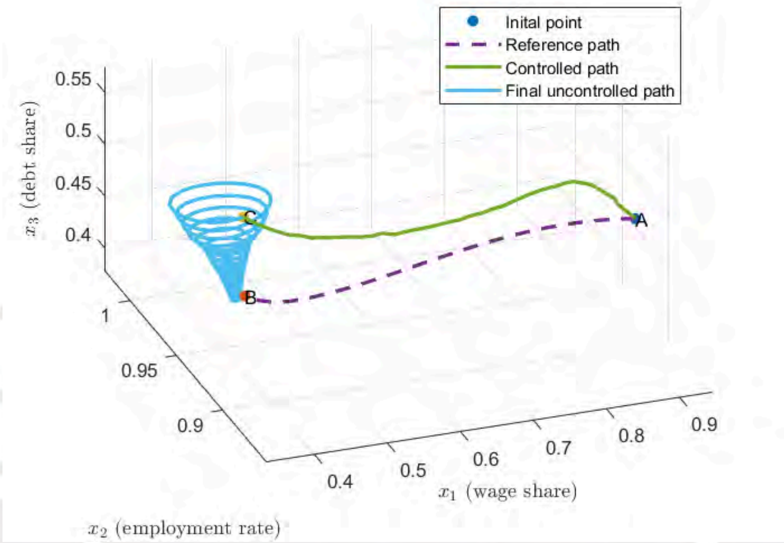

Figure 5.13: The graph shows the results of the control of the outputs performed to model II with optimal control and a constraint on the input of ± 1. The time of the control is set to 3 years. The initial point A is in the unstable region of the system. The controlled path does not reach the targeted point B, instead reaches point C which is also in the stable region. The generated controlled path is smooth, and after the system reaches point C the system goes back to the original setting with original constant parameters.

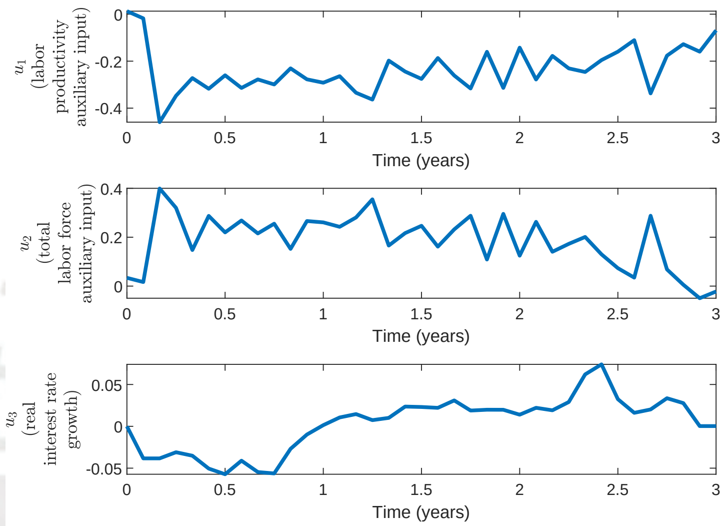

Figure 5.14: The graph shows the results of the control of the outputs performed to model II with optimal control and a constraint on the inputs of ±1. The growth of labor productivity and total labor force show a trade-off, while the first shows a decreasing tendency the latter has an increasing tendency.

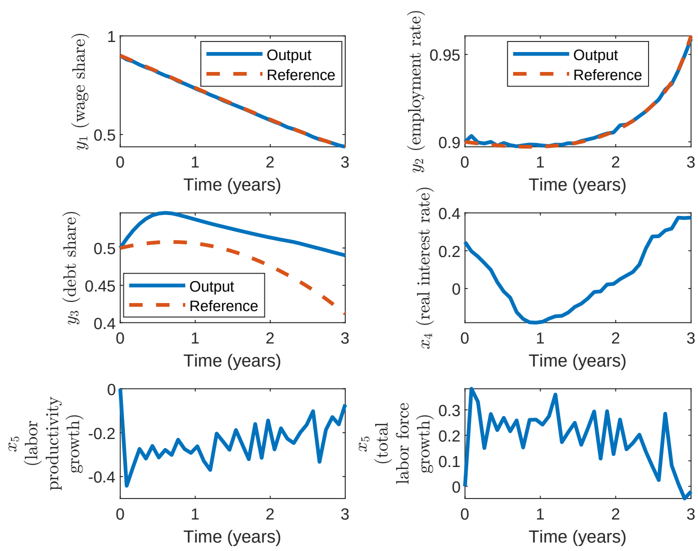

Figure 5.15: The graph shows the results of the control of the outputs performed to model II with optimal control and a constraint on the inputs of ± 1. The control policy applied generates an steady reduction in wage share, increment on employment rate and a small fluctuation on debt share. Real interest rate is negative in some parts indicating that the inflation rate is higher than the nominal interest rate.

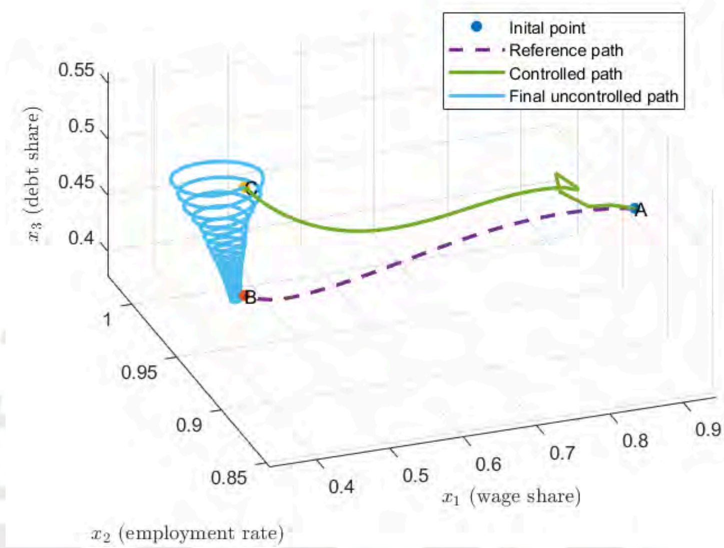

Figure 5.16: The graph shows the results of the control of the outputs performed to model II with optimal control with a constraint on the inputs of ±1 and a constraint on the growth of the real interest rate bounded by 0.01 and 0.05. The initial point A is in the unstable region of the system. The controlled path does not reach the targeted point B, instead reaches point C which is also in the stable region and closer than without optimization. The generated controlled path is smooth, and after the system reaches point C the system goes back to the original setting with original constant parameters.

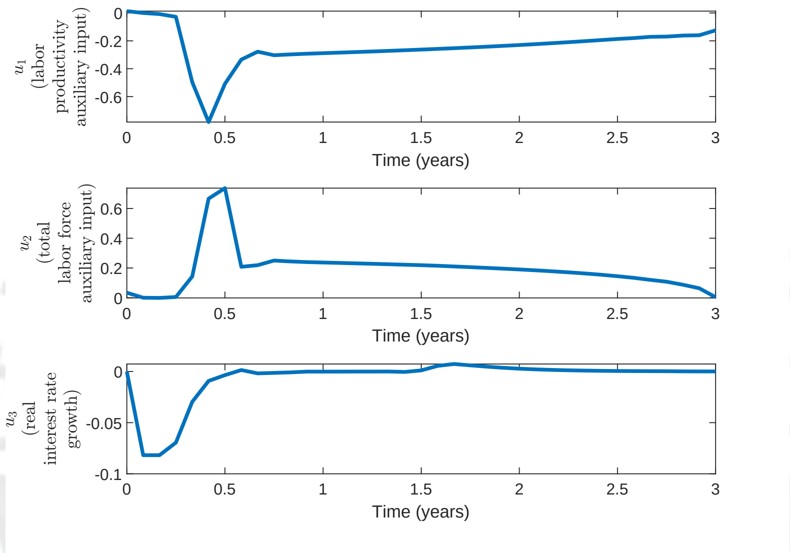

Figure 5.17: The graph shows the results of the control of the outputs performed to model II with optimal control with a constraint on the inputs of ±1 and a constraint on the growth of the real interest rate bounded by 0.01 and 0.05. The inputs have associated with labor productivity and total labor force show a trade-off behavior.

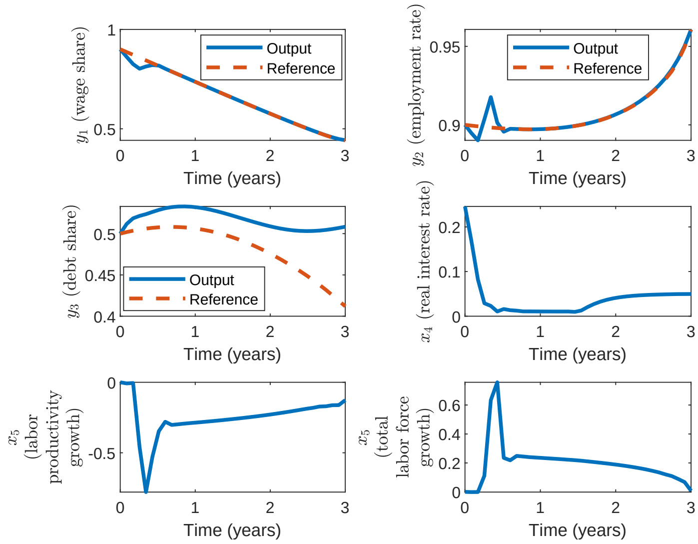

Figure 5.18: The graph shows the results of the control of the outputs performed to model II with optimal control with a constraint on the inputs of ±1 and a constraint on the growth of the real interest rate bounded by 0.01 and 0.05. The growth of labor productivity and total labor force show a trade-off in the speed of change. There is a tendency of faster increasing of labor productivity than total labor force, but both increase. Real interest growth is no longer negative.

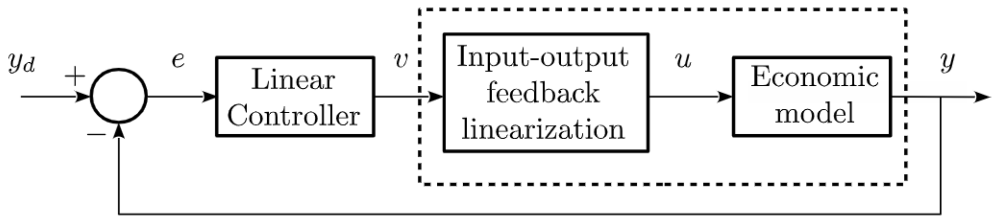

Figure 5.19: Diagram representation of the problem.

$$\alpha_1(x) = x_1 \left(\frac{0.0004886}{(x_2 - 1)^2} - 0.5419 \right) \tag{5.9a}$$

$$\alpha_2(x) = -\, x_2(0.06764 \arctan(4.734x_1 + 4.734x_3x_4 - 2.489) - 0.05126) \tag{5.9b}$$

$$\alpha_3(x) = \Bigg(x_1 - 0.3183 \arctan(4.734x_1 + 4.734x_3x_4 - 2.489) + \tag{5.9c}$$

$$+\, x_3(x_4 + 0.06764 \arctan(4.734x_1 + 4.734x_3x_4 - 2.489) - 0.05126) - 0.5 \Bigg) \times \tag{5.9d}$$

$$\times \Bigg(x_4 + 0.06764 \arctan(4.734x_1 + 4.734x_3x_4 - 2.489) - \frac{1.507x_4}{(4.734x_1 + 4.734x_3x_4 - 2.489)^2 + 1} + \tag{5.9e}$$

$$\frac{0.3202x_3x_4}{(4.734x_1 + 4.734x_3x_4 - 2.489)^2 + 1} - 0.05126 \Bigg) + \tag{5.9f}$$

$$+ \left(\frac{0.3202x_3}{(4.734x_1 + 4.734x_3x_4 - 2.489)^2 + 1} - \frac{1.507}{(4.734x_1 + 4.734x_3x_4 - 2.489)^2 + 1} + 1 \right) \times \tag{5.9g}$$

$$\times\, x_1 \left(\frac{0.0004886}{(x_2 - 1)^2} - 0.5419 \right) \tag{5.9h}$$

and the determinant of the decouple matrix is $\det D_k(x) = x_1 x_2 L_{g_3} f_3$. The system losses vector relative degree whenever $x_1 x_2 L_{g_3} f_3 = 0$ or equivalently when

$$x_1 = 0 \text{ or } x_2 = 0 \text{ or } x_1 \left[\kappa_x(x)\left(\frac{x_3}{\nu} - 1\right) + 1\right]\left[x_3\kappa_x(x)\left(-\frac{x_3}{\nu} + 1\right) - 1\right] = 0$$

with $\kappa_x(x) := \partial\kappa(x)/\partial x$. When this happens a change of coordinates (Byrnes-Isidori) cannot be performed to obatin a linear system. The feedback linearizing inputs are

$$\begin{aligned}
u_1 &= \Phi(x_2) - \frac{v_1}{x_1}, \\
u_2 &= -\frac{v_2}{x_2} + \frac{\kappa(x)}{\nu} - \delta - u_1, \\
u_3 &= \frac{1}{x_3(1 + (\kappa_x(x))(\frac{1}{\nu} + 1))}\left[v_3 - \dot{x}_3\left[x_4 - \frac{\kappa(x)}{\nu} + \delta\right]\right. \\
&\quad \left. + (\kappa(x))(-f_1 - x_4 f_3)(\frac{x_3}{\nu} - 1) - f_1\right]
\end{aligned}$$

where $v = [v_1, v_2, v_3]^T$ is the vector of input after exact linearization. The system in Byrnes-Isidori normal form is the following:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}, \tag{5.10a}$$

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix} \tag{5.10b}$$

now a matrix K can be designed to achieve some pre-defined objective by means of a linear input, this is, $v = Kz$.

Remark: Careful attention must be paid on the design of a feedback linearizing controller for the manipulation of dynamics of the form of the affine non-linear system (B.3) because of the so-called *zero dynamics*. For a system with relative degree $p < n$, there exist a subsystem $\tilde{z}_i = \tilde{f}(\tilde{z})$ for $p + 1 \leq i \leq n$, whose behavior does not affect the output and therefore it comprises an uncontrollable dynamics that if not stable can break the system even though the system looks linear and controllable from the input-output perspective. In fact, what this hidden dynamics represent is the nonlinear generalization of the well-known transmission-blocking property of zeros for linear-time invariant systems (Kailath, 1980). Model I has full relative degree as the sum of each relative degree for each output is 3 which is the same as the number of outputs.

Model II

To delay the response labor productivity growth and total labor force a dynamics is added to system 5.1:

$$\dot{x}_1 = x_1\left[\Phi(x) - \alpha_0 - x(5)\right], \tag{5.11a}$$

$$\dot{x}_2 = x_2\left[\frac{\kappa(x)}{\nu} - \alpha_0 - x(5) - \beta_0 - x(6)\right], \tag{5.11b}$$

$$\dot{x}_3 = x_3\left[x_4 - \frac{\kappa(x)}{\nu} + \delta\right] + \left[\kappa(x) - (1 - x_1)\right], \tag{5.11c}$$

$$\dot{x}_4 = u_3, \tag{5.11d}$$

$$\dot{x}_5 = -\tau_1(x(5) - u_1), \tag{5.11e}$$

$$\dot{x}_6 = -\tau_2(x(6) - u_1), \tag{5.11f}$$

$$y = \left[x_1\ x_2\ x_3\right]^T. \tag{5.11g}$$

where $x(5)$ and $x(6)$ are now the rate of growth of the original states which are the rate of growth of the productivity and the labor respectively. The linear dynamics consisting of equations 5.11e and 5.11f is augmented. This model has relative degree vector $(2, 2, 2)^T$ which sum to the number of states in the system, therefore there is no zero dynamics. The decoupling matrix is the following:

$$D_k(x) = \begin{bmatrix} -x_1 & -x_2 & 0 \\ 0 & -x_2 & 0 \\ 0 & L_{g_2}f_3 & L_{g_3}f_3 \end{bmatrix}, \tag{5.12}$$

$$L_{g_2}f_3 = -\frac{6.8x_2x_3}{(64x_1 + 64x_3x_4 - 52)^2 + 1} \tag{5.13}$$

$$L_{g_3}f_3 = \frac{6.8x_3^2 - 20x_3}{(64x_1 + 64x_3x_4 - 52)^2 + 1} + x_3 \tag{5.14}$$

and the function alpha function as in equation B.16 which along with the decoupling matrix build the linearizing controller (equation B.17) is especified in equation 5.15.

$$\alpha_1(x) = 10x_1x_5 + x_1(x_5 - \frac{0.0004886}{(x_2-1)^2} + 0.5548)^2 + \frac{0.0009771x_1x_2(x_5+x_6-0.004556)}{(x_2-1)^3} \tag{5.15a}$$

$$+ \frac{0.0000660x_1x_2\arctan(4.734x_1+4.734x_3x_4-2.489)}{(x_2-1)^3} \tag{5.15b}$$

$$\alpha_2(x) = x_2(x_5+x_6+0.06764\arctan(4.734x_1+4.734x_3x_4-2.489)-0.004556)^2 + 10x_2x_5 + 10x_2x_6 \tag{5.15c}$$

$$- \frac{0.3202x_2x_4(x_1+x_3(x_4+0.06764\arctan(4.734x_1+4.734x_3x_4-2.489)-0.05126)-0.5)}{(4.734x_1+4.734x_3x_4-2.489)^2+1} \tag{5.15d}$$

$$+ \frac{0.3202x_1x_2(x_5 - \frac{0.0004886}{(x_2-1)^2} + 0.5548) - 0.1019x_2x_4\arctan(4.734x_1+4.734x_3x_4-2.489)}{(4.734x_1+4.734x_3x_4-2.489)^2+1} \tag{5.15e}$$

$$\alpha_3(x) = -x_1\left(\frac{0.3202x_3}{(4.734x_1+4.734x_3x_4-2.489)^2+1} - \frac{1.507}{(4.734x_1+4.734x_3x_4-2.489)^2+1} + 1\right)\times \tag{5.15f}$$

$$\times\left(x_5 - \frac{0.0004886}{(x_2-1)^2} + 0.5548\right) + \tag{5.15g}$$

$$+\left(x_1 - 0.3183\arctan(4.734x_1+4.734x_3x_4-2.489) + \right. \tag{5.15h}$$

$$\left. + x_3(x_4+0.06764\arctan(4.734x_1+4.734x_3x_4-2.489)-0.05126)-0.5\right)\times \tag{5.15i}$$

$$\times\left(x_4 + 0.06764\arctan(4.734x_1+4.734x_3x_4-2.489) - \frac{1.507x_4}{(4.734x_1+4.734x_3x_4-2.489)^2+1} + \right. \tag{5.15j}$$

$$\left. + \frac{0.3202x_3x_4}{(4.734x_1+4.734x_3x_4-2.489)^2+1} - 0.05126\right) \tag{5.15k}$$

As described in the previous model and in section B.2 of the appendix, the input $u(x)$ which transforms to the Byrnes-Isidori normal form is the following:

$$u(x) = D_k(x)^{-1}(-\alpha(x) + v) \tag{5.16}$$

the system in Byrnes-Isidori normal form is the following:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{5.17a}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \tag{5.17b}$$

Once linearized the control techniques are applied to this system with control in v, then to obtain the control in u the decoupling matrix is used to transform coordinates from the first to the latter. Next, the control techniques are explained.

5.3.1 Pole placement

For Model I, once the system is in the Byrne-Isidori normal form, this is, system 5.10, the input v used is the following:

$$v(i) = y(i) - k_i^T e(i), \tag{5.18a}$$

$$e(1) = z(1) - y_r(1), \tag{5.18b}$$

$$e(2) = z(2) - y_r(2), \tag{5.18c}$$

$$e(3) = \begin{bmatrix} z(3) - y_r(3) \\ z(4) - \dot{y}_r(3) \end{bmatrix}, \tag{5.18d}$$

$$k_1 \in \mathbb{R},\ k_2 \in \mathbb{R},\ k_3 \in \mathbb{R}^2 \tag{5.18e}$$

The gains used are $k_1 = 10$, $k_2 = 10$ and $k_3 = (100, 20)^T$.

5.3.2 Restrictions and optimization

For Model I and pole placement by saturation of the control policies it is meant the following:

$$u^*[k] = \begin{cases} u_{\max}, & \text{if } u[k] > u_{\max} \\ u[k], & \text{if } u_{\min} \leq u[k] \leq u_{\max} \\ u_{\min}, & \text{if } u[k] < u_{\min} \end{cases} \tag{5.19}$$

or equivalently

$$u^*[k] = \max(-u_{\min}, \min(u_{\max}, u[k])) \tag{5.20}$$

where $u[k]$ is the control policy obtained by pole placement and $u^*[k]$ is the policy after the saturation. This sets $u^*[k]$ between $u_{\max}$ and $-u_{\min}$. This is, $-u_{\min} \leq u^*[k] \leq u_{\max}$ The parameters used are $u_{\min} = u_{\max} = (0.09, 0.09, 0.09)^T$. Saturation poses a challenge. As mentioned in section 5.2, trying to restrict more the policies to $u_{\min} = u_{\max} = (0.08, 0.08, 0.08)^T$ results in the controller not being able to steer to a point in the stability region, therefore debt share growing fast. The results of pole placement do

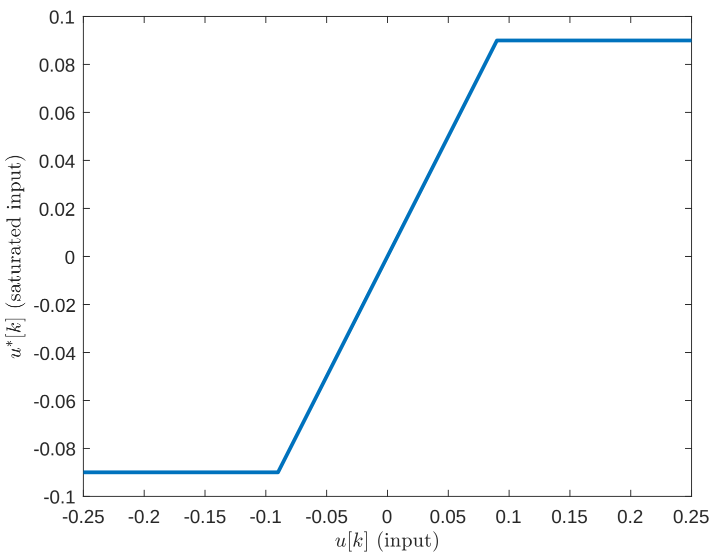

Figure 5.20: Saturation function with limits equal to ±0.09.

not guarantee the compliance of the positiveness of the real interest rate.

To enforce this condition, optimal control is used. For this, one-step ahead prediction of the state is used. This is, consider the discretized version of system 5.17 with sample time τ_S equal to the sample time of the algorithm of the differential equation solver (ODE45 is used):

$$z[k+1] = M[k]z[k] + Nv[k] \tag{5.21a}$$
$$y[k] = Wz[k] \tag{5.21b}$$

the error to minimize is

$$e[k] = y[k+1] - y_r[k+\tau_S] \tag{5.22}$$

define a matrix of weights Q for the error and the objective function is

$$J(v[k]) = e[k]^T Q e[k] \tag{5.23}$$

from 5.16, the objective function can be expressed in terms of the original discretized control $u[k]$ as follows

$$J(u[k]) = e_u[k]^T Q e_u[k], \tag{5.24}$$
$$e_u[k] = W(M[k]z[k] + N(D_k(z[k])u[k] + \alpha(z[k]))) - y_r[k+\tau_S] \tag{5.25}$$

an upper bound of 1 and a lower bound of -1 is imposed for $u(1)$, $u(2)$ and $u(3)$. This is

$$L : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} u[k] \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{5.26}$$

in both models $x(4)$ is the growth rate of the real interest rate. To enforce the condition of positiveness of it, the following constraint based on the polynomial of Taylor is added to L:

$$(r_{min} - x(4))/\tau_S \leq u(3) \leq (r_{max} - x(4))/\tau_S \tag{5.27}$$

in matrix terms, this is,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} u[k] \leq \begin{bmatrix} (r_{max} - x(4))/\tau_S \\ (x(4) - r_{min})/\tau_S \end{bmatrix} \tag{5.28}$$

the optimal input is then defined as

$$u^*[k] := \arg\min J(u[k]) \tag{5.29}$$
$$\text{s.t} \quad L \tag{5.30}$$

here, $r_{min} = 0.01$ and $r_{max} = 0.05$ are used. Constraining $u[k]$ to a narrower band may lead to unfeasibility of the optimization an strategies to compensate this could be a longer period of control as well as the modification of the weight matrix Q which here is taken as the identity matrix.

5.4 Reference signal

To obtain a reference signal in a natural way, the Gradient Based Planner algorithm is used. This uses same dynamics of the model to generate a path from two desired points under the vector field of a system. Call x_0 a initial point in the unstable region and x_f a final point in the stable region. Call F_K the vector field of the model and F_x an attractor field generated by the point x_f with attractive force K. The reference signal is trajectory of the point x_0 following the sum F_s of the fields F_K and F_x. The algorithm is the following:

Algorithm 1 Gradient Based Planner

procedure GBP($x_0, x_f, F_K, F_x, T, \delta$) ▷ Path between x_0 and x_f
  $F_s \leftarrow F_K + F_x$
  while $i \neq T \vee ||x_f - position|| < \delta$ **do**
    $position \leftarrow x_0$
    $Delta \leftarrow interpolate(F_s, position)$
    $position \leftarrow position + Delta/(T||Delta||)$
    $route \leftarrow stack(route, position)$
    $i \leftarrow i + 1$
  end while
  return *route*
end procedure

A greater attractive force K could deform the vector field in a way that although the generated path might be more straight forward and faster reaching, it might not be the more appropriate for control as a path like this one represents more control effort. If additionally the there are constraints this might lead to unfeasibility of the control problem. On the other side, a small attractive force K might not be able to build a path with a final point inside the desirable region. It could be also the case as in the present work that there is a singularity at $x_2 = 1$, because of the Phillips curve and if the attractive force is not strong enough the path might cross this singularity as the building mechanism relies on the mixture of both vector fields and not only on the vector field of the system. Then the control implementation is necessary have problems with this ill-behaved path. For the present work the attractive force of the targeted point in the stable region is $K = 7$.

Chapter 6

Conclusions and further research

In this manuscript, the calibration of the Goodwin model (equation (2.18)) and Keen model (equation (3.2)) for Peru for the period of 1991 to 2014 is performed. The sources of the data used are presented in table 4.2. After an adjustment of the parameters obtained from the regression, figure 4.2 shows a good fitting of the Goodwin model for Peru. With these parameters, the Keen model is calibrated. A hurdle is observed at first, because the significant equilibrium point has a negative coordinate which does not make sense in reality. This is overcome by taking the mean of the utility share in equation 4.3 to be a value in the confidence interval of it. The calibrated model shows that data for Peru lies inside the stability region which means that if no alterations of the parameters occur, then the Peruvian economy is being driven to a healthy state of finite debt share. Living in an small open economy, depending on the price of copper and with an unstable political situation this is likely not to happen. In the present work it is shown that if the economy of Peru gets out of the stability region and faces a financial crisis, then it is possible to stabilize it. For this, the stability region of the Keen system (3.2) for Peru is computed. It is shown in figure 4.11. Then, the system is dynamically extended and then feedback linearized as described in section 5.3 and section B of the appendix.

Two techniques are used to perform control: pole placement and optimal control. The first one shows a clear trade-off between labor productivity growth and the growth of total labor force and a decreasing real interest rate policy. The results obtained suggest that more resources should be placed on generating employment and a tendency to promote spending. This technique is applied on model I and satisfies the goal of reaching a stable point, despite the non-perfect tracking. Two scenarios are provided for this model, one with unsaturated and the other saturating the policy controls. As it is to expect, the saturated case performs better. The real interest rate is negative for this model and technique. To be able to impose the condition of positiveness of real interest rate along with the assumption of a delay on the of labor productivity growth and total labor force growth as they are not directly controlled, optimal control is applied to Model II with one step ahead prediction. In this case, the performance depends on the growth rate (τ) of the acceleration of labor productivity and total labor force. The smaller these parameters are, better performance is obtained. The smaller these parameters means that the labor productivity and total labor force are easier to be affected. Labor productivity and total labor force are not as smooth as with pole placement, but they still show trade-off between them and there is less variation as shown in figure 5.15. First, no constraint on real interest rate is considered showing no positiveness, then this is enforced. Although the trajectory of it is not that smooth as shown in figure 5.18. This is improved by changing the parameters such as making the control be performed in longer than three years or decreasing the delay of labor productivity and total labor force growth.

There are topics of further research for the present work such as formalizing the parameter adjustment in the process of calibration of the Goodwin model. The system is non-linear in the states, but linear in the parameters so a linear regression must be enough, but data for Peru shows that this is not the best calibration or there is not enough data. The calibration of the Keen model presents challenges, also, such as formalizing the choice of the investment function which for the present work was assume the same as in (Costa Lima, 2013). This function, as presented here, makes the the system non-linear in terms of the estimation so approaches such as neural networks might be better to calibrate this function. Another way of further research is found by making the optimization more complex by using, for example, the well-known MPC (Model Predictive Control) technique. This allows to make a more general framework by taking more than only one step ahead of prediction. Input terms can be added to the objective function as to put a weight on the controllers, a bigger penalty could be imposed on one of them, for example. This provides to policy making process with consciousness about which are more complex and which are simpler and more desirable policies to implement.

Finally, other algorithms for the generation of the reference path could be used. Depending on how they modify the dynamics a different path is obtained. A comparison between these algorithms is desirable to know which would be the optimal in the sense that less variation in the economic policies and economic variables is obtained.

Appendices

This section addresses theoretical aspects of non-linear control. The focus is on Feedback Linearization and the preliminaries to understand the technique. An outline of linear control is provided and zero dynamics are also explained. This section is based on the non-linear control work of (Gray, 1995), (Isidori, 2002) these are geometric based techniques and the linear part is succinctly summarized from (Kailath, 1980).

Appendix A

Ordinary differential equations

Consider the initial point $x_0 \in \mathbb{R}^n$ and the Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$. The initial value problem is the following:

Definition 1 (Initial value problem)**.**

$$\dot{x} = f(x), \tag{A.1a}$$

$$x(0) = x_0 \tag{A.1b}$$

Let the system have an stable equilibrium at $x = p$. The stability region of p is defined as

Definition 2 (Stability region)**.**

$$E(p) = \{x_0 \in \mathbb{R}^n : \lim_{t \to \infty} X(t) = p\} \tag{A.2}$$

where $X(t)$ is the integral path of the initial value problem A.1.

Appendix B

Control

The underlying assumption in this section is that the initial point of the Keen model (3.2) is in the unstable region, therefore, it is in a financial crisis scenario, which is outside the stability region descried in Figure 4.11. The idea is then to come up with framework under which one can devise a policy that manipulates the dynamics of the system to bring the state inside stability region shown in Figure 4.11. It is here where control theory provides systematic procedures to create policies for the positioning of the relevant states into the desired stable region (Kailath, 1980; Isidori, 2002).

B.1 Linear Control Systems

Consider a linear dynamical control system of the form

$$\dot{x} = Ax + Bu, \tag{B.1a}$$

$$y = Cx, \tag{B.1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$ and A, B and C are matrices of appropriate dimension. Moreover, system (B.1) is said to be controllable if one can manipulate x in (B.1) from some $x_f, \in \mathbb{R}^n$ back to the origin by providing an appropriate control policy u. It turns out that the only condition for this to be possible is that the rank of $\mathcal{C} = (B \;\; AB \;\; \cdots \;\; A^{n-1}B) \in \mathbb{R}^{nm\times n}$ is n, which implies that there are n directional vectors that can be employed to steer x to any desired place in $\mathbb{R}^n$. A control policy $u = Kx$, $K \in \mathbb{R}^{m\times n}$ is known as state feedback, and it provides the desired policy by selecting the components of K appropriately. If it exists, system (B.1) becomes

$$\dot{x} = (A + BK)x, \qquad y = Cx. \tag{B.2}$$

Trivially, the solution of (B.2) is $x = C\exp^{(A+BK)t} x_0$, which can be made zero by choosing K in a way that the eigenvalues of $A + BK$ are negative and large enough. Thus, the system is guaranteed to reach the origin after some time commanded by how fast the exponential decays, which is a tuning parameter for the control policy.

Unfortunately, (3.2) is not in the form (B.1). There are however techniques that allow to manipulate the states of a system towards desired locations in the phase protrait. In particular, differential geometry concepts allow for a transformation of nonlinear dynamics into an exact linear system when certain conditions are satisfied. This manuscript utilizes one of such techniques known as *feedback linearization*, which after applied allows the design of a vector K as in (B.2) for tracking closely a desired pre-defined path for the variables of (3.2). The next section, presents the tools necessary for feedback linearizing (2.18) and (3.2).

B.2 Feedback linearization

It is a technique used to obtain a linear system from an affine system. To accomplish this a diffeomorphism that works as change of coordinates must be found. In what follows the procedure to obtain this function is stated.

Definition 3 (Lie Derivative)**.** *Let f and h be smooth functions. Denote by*

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

the Lie derivative *of h with respect to f, and*

$$L_f^k h(x) = L_f(L_f^{k-1} h(x)) = \frac{\partial L_f^{k-1} h(x)}{\partial x} f(x)$$

is the kth iterated Lie derivative of h with respect to f.

The relative degree of a nonlinear system is then provided by the next definition.

Definition 4 (Relative degree)**.** *Consider the system*

$$\dot{x} = f(x) + g(x)\,u, \qquad x(0) = x_0, \tag{B.3a}$$

$$y = h(x) \tag{B.3b}$$

with $x \in \mathbb{R}^n$ and f, g and h sufficiently smooth in a domain of $R \subset \mathbb{R}^n$. (B.3) *is said to have* relative degree $0 \le p \le n$ *in the region $R_{x_0} \in \mathbb{R}^n$ if for all*

$x_0 \in R_{x_0}$

$$L_g L_f^{i-1} h(x) = 0 \qquad \textit{for} \quad i = 1, 2, \ldots, p-1,$$
$$L_g L_f^{p-1} h(x) \neq 0,$$

It is easy to show that if the relative degree of a system is $p > k$, then it follows that the k-th derivative of y in (B.3) is

$$y^{(k)} = L_f^k h(x)$$

in a neighborhood of t_0 and

$$y^{(p)}(t_0) = L_f^p h(x_0) + L_g L_f^{p-1} h(x_0) u(t_0). \tag{B.4}$$

Therefore, the relative degree p of a system is equal to the number of times the output must be differentiated to have the input appear explicitly. The functions $h(x), L_f h(x), \ldots, L_f^{p-1} h(x)$ define a key local coordinate transformation for the system about x_0 for this define the new set of coordinates $z = T(x)$ where $T : \mathbb{R}^n \to \mathbb{R}^p$

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{p-1} h(x) \end{bmatrix} \tag{B.5}$$

after this change, the system is the following:

$$\dot{h} = z_1 \tag{B.6}$$
$$\dot{z}_1 = z_2 \tag{B.7}$$
$$\vdots \tag{B.8}$$
$$\dot{z}_{p-2} = z_{p-1} \tag{B.9}$$
$$\dot{z}_{p-1} = L_f^p h(x_0) + L_g L_f^{p-1} h(x_0) u(t_0) \tag{B.10}$$

defining $D := L_g L_f^{p-1} h(x_0)$ and setting

$$u = D^{-1}(v - L_f^p h(x_0)) \tag{B.11}$$

the following linear system is obtained:

$$\begin{bmatrix} \dot{h} \\ \dot{z}_1 \\ \vdots \\ \dot{z}_{p-2} \\ \dot{z}_{p-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} h \\ z_1 \\ \vdots \\ z_{p-2} \\ z_{p-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \tag{B.12}$$

The controllability matrix

$$[B\ AB\ \cdots\ A^{p-1}B] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{B.13}$$

has full rank, therefore the system is controllable.

For a system with ℓ outputs and m inputs, one can compute the relative degree for each output, say p_i for $i = 1, \dots, \ell$, by simply computing derivatives until any of the inputs appear. This defines the vector relative degree $(p_1, \dots, p_\ell)$, which is well-defined if

$$D(x) = \begin{bmatrix} L_{g_1} L_f^{p_1-1} h_1(x) & \cdots & L_{g_m} L_f^{p_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{p_\ell-1} h_\ell(x) & \cdots & L_{g_m} L_f^{p_\ell-1} h_\ell(x) \end{bmatrix} \tag{B.14}$$

is invertible. If such is the case, then D is known as the system's *decoupling matrix*. Even in the case of a non-square matrix which means that the number of outputs and inputs are not equal ($\ell \neq m$), the pseudo inverse can be taken (Kolavennu, Palanki, & Cockburn, 2001). Analogous to the case of only one output, this creates ℓ decoupled linear systems in *normal form*. That is, let $z_{i,j}$ be the j-th component of the i-th system, the linearized system in normal form is given by

$$\dot{z}_{i,1} = \dot{z}_{i,2},\ \dot{z}_{i,2} = \dot{z}_{i,3},\ \cdots,\ \dot{z}_{i,p_i} = v_i \text{ with } y_i = z_{i,1} \tag{B.15}$$

for $i = 1, \dots, \ell$. As before, this system was obtained from (B.4), where one can define a chain of integrator as above. Defining

$$\alpha(x) := \begin{bmatrix} L_f^{p_1} h_1(x) \\ \vdots \\ L_f^{p_\ell} h_\ell(x) \end{bmatrix} \tag{B.16}$$

and putting together (B.4) for all outputs gives the control law

$$u(x) = D^{-1}(x)\left[-\alpha(x) + v\right]. \tag{B.17}$$

Note that (B.15) is also a linear system that is controllable. A standard state feedback law can now be applied as in (B.2) so that the overall system is stable and can be made to follow a desired path. The state feedback law is applied on $v_i = K_i z_i$, where $z_i = \begin{pmatrix} z_{1,i} & \cdots & z_{1,p_i} \end{pmatrix}^\top$ can be obtained from the derivatives of each output and K_i is a user-defined vector of gains. A block diagram of the overall system is shown in Figure 5.19.

Appendix C

Estimations

Table C.1: Results

	Dependent variable:	
	log of labor productivity	log of total labor force
	(1)	(2)
Year	0.013***	0.034***
	(0.002)	(0.001)
Constant	−16.807***	−51.274***
	(4.308)	(2.139)
Observations	24	24
R^2	0.620	0.978
Adjusted R^2	0.603	0.977
Residual Std. Error (df = 22)	0.073	0.036
F Statistic (df = 1; 22)	35.968***	999.661***
Note:		*p<0.1; **p<0.05; ***p<0.01

Table C.2: Results

	Dependent variable:			
	VWS	NLFL	VWS	NLFL
	(1)	(2)	(3)	(4)
lambda	0.431	2.557		
	(0.565)	(6.747)		
$1/(1-\text{lambda})^2$			0.00002	0.0001
			(0.00002)	(0.0003)
Constant	-0.411	-2.494	-0.010	-0.108
	(0.538)	(6.429)	(0.013)	(0.155)
Observations	23	22	23	22
R^2	0.027	0.007	0.034	0.007
Adjusted R^2	-0.019	-0.043	-0.012	-0.043
Residual Std. Error	0.028 (df = 21)	0.324 (df = 20)	0.028 (df = 21)	0.324 (df = 20)
F Statistic	0.582 (df = 1; 21)	0.144 (df = 1; 20)	0.743 (df = 1; 21)	0.138 (df = 1; 20)

Note: *p<0.1; **p<0.05; ***p<0.01

VWS = variation of wage share as in the left side of equation 4.2.

NLFL =Non-linear function of lambda as in the left side of equation 4.3.

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